# Lecture 4 of Eduardo Guendelman at the

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# Branes and their ¨cosmological constant problem ¨ which is resolved by

## The Modified Measure Brane Theory

The standard world sheet string sigma-model action using a world sheet metric is [27]:

$$S_{sigma-model} = -T \int d^d\sigma \frac{1}{2}\sqrt{-\gamma}(\gamma^{ab}\partial_a X^\mu \partial_b X^\nu g_{\mu\nu} + 2\Lambda). \quad (15)$$

Notice that now a cosmological term has been added as well, that was not needed in the usual formulation of the string theory or in the modified measure formulation of the string theory. In the modified measure formulation of the brane theory such term will not be required. As it is well known, and as we will review, in the standard formulation , this cosmological term needs to be fine tuned. Here again $\gamma^{ab}$ is the intrinsic Riemannian metric on the d-dimensional brane worldsheet and $\gamma = det(\gamma_{ab})$; $g_{\mu\nu}$ denotes the Riemannian metric on the embedding spacetime. $T$ is a brane tension, a dimension full scale introduced into the theory by hand.

From the variations of the action with respect to $\gamma^{ab}$ and $X^\mu$ we get the following equations of motion:

$$T_{ab} = (\partial_a X^\mu \partial_b X^\nu - \frac{1}{2}\gamma_{ab}\gamma^{cd}\partial_c X^\mu \partial_d X^\nu) g_{\mu\nu} - \gamma_{ab}\Lambda = 0, \tag{16}$$

$$\frac{1}{\sqrt{-\gamma}}\partial_a(\sqrt{-\gamma}\gamma^{ab}\partial_b X^\mu) + \gamma^{ab}\partial_a X^\nu \partial_b X^\lambda \Gamma^\mu_{\nu\lambda} = 0, \tag{17}$$

where $\Gamma^\mu_{\nu\lambda}$ is the Levi Civita affine connection for the external metric $g_{\mu\nu}$

From (16) we see that without the cosmological constant term, or if this term is not fine tuned, the equations of motion are inconsistent upon considering the trace for example. This feature is absent in the modified measure brane theory, where an integration constant, that takes the role of the cosmological term is determined dynamically by the consistency of the equations of motion.

By taking the trace of eq, 16, we see that, the first two terms do not combine to zero, as opposed to the string case, so the Lambda term is needed for formulating the brane without modified measure!

To show this, again , as in the string case, we consider other measure of integration different than $\sqrt{-\gamma}$. The only restriction is that it must be a density under arbitrary diffeomorphisms (reparametrizations) on the underlying spacetime manifold, as we have seen before while reviewing the modified measure string theory, a metric independent measure build along the lines of what we did in the previous section is an example for that

In the framework of a $d$ dimensional brane $d$ additional worldsheet scalar fields $\varphi^i (i = 1, 2, ..., d)$ are introduced. The new measure density is defined now as

$$\Phi(\varphi) = \epsilon_{ijk...m} \epsilon^{abc...d} \partial_a \varphi^i \partial_b \varphi^j ..... \partial_d \varphi^m \tag{18}$$

Then the modified bosonic brane action is

$$S = -\int d^d\sigma \Phi(\varphi)(\frac{1}{2}\gamma^{ab}\partial_a X^\mu \partial_b X^\nu g_{\mu\nu} - \frac{\epsilon^{abcd...}}{2\sqrt{-\gamma}} F_{abcd...}(A)), \tag{19}$$

where $F_{abcd...}$ is the rank $d$ totally antisymmetric field-strength derived from an auxiliary abelian gauge field $A_{bcd...}$ of rank $d-1$: $F_{ab...cd} = \partial_a A_{bcd...} + ....$ , the totally antisymmetrized curl of a $d-1$ potential $A_{bcd...}$

In the brane modified theory there is a scale invariance, although it is only a global one, as opposed to the string case, where it is local,

$$\gamma_{ab} \to J\gamma_{ab}, \tag{20}$$

$$\varphi^i \to \varphi'^i = \lambda^{ij}(\varphi^j) \tag{21}$$

such that

$$\Phi \to \Phi' = J\Phi \tag{22}$$

That is if $det(\lambda^{ij}) = J$, where $J, \lambda^{ij}$ are just a constant number and a constant matrix now, and finally the antisymmetric tensor field $A_{bcd\ldots}$ must transform as

$$A_{bcd\ldots} \to J^{\frac{d-2}{2}} A_{bcd\ldots} \tag{23}$$

To check that the new action is consistent with the sigma-model one, let us derive the equations of motion of the action (15). To start, the variation with respect to $\varphi^i$ leads to the following equations of motion:

$$K^a_b \partial_a (\gamma^{cd} \partial_c X^\mu \partial_d X^\nu g_{\mu\nu} - \frac{\epsilon^{cdef\dots}}{\sqrt{-\gamma}} F_{cdef\dots}) = 0. \tag{24}$$

Where the matrix $K^a_b$ is just like the measure, except that one factor of a derivative of a measure field is missing, so that there are two free indices therefore the determinant of $K^a_b$ is a power of the measure, so if the measure is non vanishing, this implies

$$\gamma^{cd} \partial_c X^\mu \partial_d X^\nu g_{\mu\nu} - \frac{\epsilon^{cdef\dots}}{\sqrt{-\gamma}} F_{cdef\dots} = M = const. \tag{25}$$

The equations of motion with respect to $\gamma^{ab}$ are

$$T_{ab} - \partial_a X^\mu \partial_b X^\nu g_{\mu\nu} - \frac{1}{2} \gamma_{ab} \frac{\epsilon^{cdef\dots}}{\sqrt{-\gamma}} F_{cdef\dots} = 0. \tag{26}$$

We will see now that the final equations equations are the same as in the sigma-model formulation (2), (3). Namely, taking the trace of (26), using also (25), and then reinserting into (26), we get that $M \neq 0$ now. By solving $\frac{\epsilon^{a_1 a_2 \ldots}}{\sqrt{-\gamma}} F_{a_1 a_2 \ldots}$ from (25) we obtain a relation between the intrinsic metric and the induced metric if $p \neq 1$, (where $p$ is the number of spacelike world sheet coordinates ($p = d - 1$) ) that if we do not deal with the string case ($p = 1$),

$$\gamma_{ab} = \frac{1-p}{M} \partial_a X^\mu \partial_b X^\nu g_{\mu\nu} \tag{27}$$

Using the global scale symmetry of the modified measure brane theory we can set $\frac{1-p}{M} = 1$ if $M$ is negative if we wish, for $p \neq 1$. That is, the intrinsic metric in the world sheet becomes equal to the induced metric from the embedding metric, while in the string case there is only a conformal equivalence between the intrinsic metric in the world sheet and the induced metric.

varying the action with respect to $A_{abc...}$:

$$\epsilon^{abc...d}\partial_d\left(\frac{\Phi(\varphi)}{\sqrt{-\gamma}}\right) = 0. \tag{28}$$

Then by integrating and comparing it with the standard action it is seen that

$$\frac{\Phi(\varphi)}{\sqrt{-\gamma}} = T. \tag{29}$$

That is how the brane tension $T$ is derived as a world sheet constant of integration, opposite to the standard equation (1) where the tension is put in from the start. One can see indeed that the string energy excitations will be proportional to $T$ in exactly the same way as it is the case in the standard case. The variation with respect to $X^\mu$ leads to the second sigma-model-type equation. The idea of modifying the measure of integration proved itself effective and profitable.

# Introducing world sheet currents

If to the action of the brane (19) we add a coupling to a world-sheet current $j^{a_2 \cdots a_{p+2}}$, $p = d - 1$ (the case $p = 1$ represents a string) i.e. a term

$$S_{\text{current}} = \int d^{p+1}\sigma A_{a_2 \cdots a_{p+2}} j^{a_2 \cdots a_{p+2}}, \tag{30}$$

see for example at [31] , [32], [33] for different applications of this. Then the variation of the total action with respect to $A_{a_2 \cdots a_{p+2}}$ gives

$$\epsilon^{a_1 \cdots a_{p+2}} \partial_{a_1} \left( \frac{\Phi}{\sqrt{-\gamma}} \right) = j^{a_2 \cdots a_{p+2}}. \tag{31}$$

We thus see indeed that, in this case, the dynamical character of the brane is crucial here.

Suppose that we have an external scalar field $\phi(x^\mu)$ defined in the bulk. From this field we can define the induced conserved world-sheet current

$$j^{a_2\cdots a_{p+1}} = e\partial_\mu\phi\frac{\partial X^\mu}{\partial\sigma^a}\epsilon^{aa_2\cdots a_{p+1}} = e\partial_a\phi\epsilon^{aa_2\cdots a_{p+1}}, \tag{32}$$

Notice that for the string case, i.e, for $p = 1$ which is the most interesting, the current $j^{a_2\cdots a_{p+1}}$ has only one index and the current is $j^{a_2}$ where $e$ is some coupling constant.

Then (31) can be integrated to obtain

$$T = \frac{\Phi}{\sqrt{-\gamma}} = e\phi + T_i, \tag{33}$$

The constant of integration $T_i$ may vary from one brane to the other.

# Now 3 branes could represent universes, like ours indeed, then these different universes could have each generated their own cosmological constant.A Multiverse scenario could be formulated!

Notice that forthe branes the constant of integration M is of fundamental importance, M cannot be zero, in contrast , for the strings,

M= 0, this suggests that for the strings the measure can be replaced by a lagrange multiplier, that is we can have

**A Lagrange multiplier measure formulation for strings**

**Then the modified bosonic string action (before we introduce the tension field) is**

$$S = -\int d^2\sigma\Phi\left(\frac{1}{2}\gamma^{ab}\partial_a X^\mu \partial_b X^\nu g_{\mu\nu} - \frac{\epsilon^{ab}}{2\sqrt{-\gamma}}F_{ab}(A)\right), \qquad (34)$$

**where $F_{ab}(A) = \partial_a A_b - \partial_b A_a$ is the field-strength defined in a conventional way in terms of an auxiliary Abelian gauge field $A_a$ and now $\Phi$ being an independent density.**

It is important to notice that the action (34), is not the same as that discussed in (6), since there we wrote $\Phi = \Phi(\varphi)$, while here $\Phi$ is an independent field

is invariant under conformal transformations of the intrinsic measure combined with a diffeomorphism of the measure fields,

$$\gamma_{ab} \to J\gamma_{ab}, \tag{35}$$

and

$$\Phi \to \Phi' = J\Phi \tag{36}$$

The variation with respect to $\Phi$ in (34) leads to the following equation :

$$\gamma^{ab}\partial_a X^\mu \partial_b X^\nu g_{\mu\nu} - \frac{\epsilon^{ab}}{\sqrt{-\gamma}} F_{ab} = 0. \tag{37}$$

The equations of motion with respect to $\gamma^{ab}$ are the same, for $\Phi \neq 0$,

$$T_{ab} = \partial_a X^\mu \partial_b X^\nu g_{\mu\nu} - \frac{1}{2}\gamma_{ab}\frac{\epsilon^{cd}}{\sqrt{-\gamma}}F_{cd} = 0. \tag{38}$$

We see that these equations are the same as in the sigma-model formulation (2), (3). By solving $\frac{\epsilon^{cd}}{\sqrt{-\gamma}}F_{cd}$ from (37) we obtain (2).

A most significant result is obtained by varying the action with respect to $A_a$:

$$\epsilon^{ab}\partial_b\left(\frac{\Phi(\varphi)}{\sqrt{-\gamma}}\right) = 0. \tag{39}$$

Then by integrating and comparing it with the standard action it is seen that

$$\frac{\Phi}{\sqrt{-\gamma}} = T. \tag{40}$$

## 14.1 Standard Dp-Branes

We reviewed models of Gravity coupled to strings in the Nambu-Goto formalism after using modified measures which acquire target space scale invariance in $D = 4$.

We have examined strings/branes with dynamical tensions. It would be desirable if one could extend this work to the study of $Dp$-branes whose tension are *constant* and given by

$$T_{Dp} = \frac{1}{g_s(2\pi)^p(\alpha')^{(p+1)/2}} \tag{111}$$

where $g_s$ is the closed string coupling that is related to the open-string coupling by $g_s = g^2_{open}$. $\alpha'$ is the Regge slope whose units are $(length)^2$. The type IIA string coupling constant is used if $p$ is even, and the type IIB coupling constant is used if $p$ is odd [53].

It has been known for some time that the low-energy effective $D$-brane actions is given by the Dirac-Born-Infeld (DBI)-action [53]

$$S = -T_{D_p} \int d^{p+1}\sigma \sqrt{-det(G_{ab} + 2\pi\alpha' F_{ab})}$$

where $G_{ab} = \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}$ is the induced world-volume metric as a result of embedding the $Dp$-brane in a flat background, and $F_{ab}$ is the Maxwell field strength.

## 14.2 Dp-Branes with modified measures and target space scale invariance

Similarly to the string case we now introduce a modified world-volume integration measure density in terms of $p+1$ auxiliary scalar fields $\varphi^i$ $(i = 1, \ldots, p+1)$ :

$$\Phi(\varphi) = \frac{1}{(p+1)!} \epsilon_{i_1 \ldots i_{p+1}} \epsilon^{a_1 \ldots a_{p+1}} \partial_{a_1} \varphi^{i_1} \ldots \partial_{a_{p+1}} \varphi^{i_{p+1}} , \qquad (112)$$

and use it to construct the following new p-brane-type action (the coupling to the Ramond-Ramond background field is omitted for simplicity). Some time ago a Polyakov

type action classically equivalent to the original DBI-action of the $Dp$-brane was proposed in [54] :

$$S = -T \int d^{p+1}\sigma \sqrt{-\zeta} \left( e^{-\beta U} \zeta^{ab} \left( G_{ba} - F_{ba} \right) - (p-1) \right)$$

In this last sub-section we discuss the modified measure formalism for the Dp-brane first discussed in [55] and show now that it has target space scale invariance :

$$S = -\int d^{p+1}\sigma \, \Phi(\varphi) \left( e^{-\beta U} \zeta^{ab} \left( G_{ba} - F_{ba} \right) + \frac{\varepsilon^{a_1 \cdots a_{p+1}}}{(p+1)\sqrt{-\zeta}} F_{a_1 \cdots a_{p+1}} \right) \tag{113}$$

where

$$F_{a_1 \cdots a_{p+1}} = \partial_{[a_1} A_{a_2 \cdots a_{p+1}]}$$

Splitting the auxiliary tensor variable $\zeta^{ab} = \zeta^{(ab)} + \zeta^{[ab]}$ into symmetric and anti-symmetric parts The advantage of the present

more general action (113) is that not only it yields variable dynamical brane tension and naturally introduces additional higher-rank world-volume gauge fields, but also it does not need a "cosmological" term as it was the case of the p brane eq. (19). This action has target space scale invariance

$$A_{a_1\ldots a_{p+1}} \to \omega A_{a_1\ldots a_{p+1}}$$

$$\Phi(\varphi) \to \omega^{-1}\Phi(\varphi)$$

,

$$G_{ba} \to \omega G_{ba}$$

,

$$F_{ba} \to \omega F_{ba}$$

The integration of the equations of motion breaks the target space scale invariance. For example, the integration of the equation of motion of the degrees of freedom that define the measure leads to a constant Lagrangian density of the form

$$e^{-\beta U}\zeta^{ab}\left(G_{ba} - F_{ba}\right) + \frac{\varepsilon^{a_1\ldots a_{p+1}}}{(p+1)\sqrt{-\zeta}}F_{a_1\ldots a_{p+1}} = M$$

Furthermore, the integration of the equation obtained from the variation of the world sheet gauge fields ensures that

$$\frac{\Phi}{\sqrt{-\zeta}} = C - constant$$

By considering the variation with respect to $\zeta^{ab}$, we can in fact solve for $\zeta^{ab}$ and its inverse, $\zeta_{ab}$, and upon inserting back these solutions into the action we get the Nambu Goto type of action of the form

$$S^{DBI} = -T \int d^{p+1}\sigma\, e^{-\beta U} \sqrt{-\det ||G_{ab} - F_{ab}||}\,, \tag{114}$$

$$T = C(2M)^{-\frac{p-1}{2}}(p-1)^{\frac{p+1}{2}} \quad , \quad \beta' = \frac{p+1}{2}\beta\,, \tag{115}$$

If we wish, we could also include some contribution to the tension of the brane from the expectation value of the dilaton field $U$. It is interesting to establish a correspondence between the tension obtained from the standard formulation of the Dp-brane and the result above based on the modified measure of Dp-branes.

# Creation of an inflationary epoch from an Emergent Universe through quantum tunneling

The question of the origin of the Universe, and of whether it had a beginning at all, remains one of the deepest open problems in cosmology. The inflationary paradigm [1–3] accounts for the homogeneity, isotropy, and flatness of the observable Universe, as well as for the spectrum of primordial perturbations, and is widely regarded as a successful description of its very early stages [4–6]. Inflation by itself, however, does not address what preceded it: under fairly general assumptions, classical inflationary cosmologies are geodesically past-incomplete [7–9], and the singularity theorems of general relativity imply that a classical expanding Universe is generically traced back to an initial singularity [10, 11], where the classical description of spacetime and matter breaks down.

A natural way to confront the initial singularity is to regard the birth of the Universe as a quantum event. In the canonical approach to quantum cosmology, the quantization of the Hamiltonian constraint of general relativity leads to the Wheeler–DeWitt (WDW) equation [12], $\mathcal{H}\Psi = 0$, whose solution $\Psi$, the wave function of the Universe, is defined on superspace. In practice one restricts to a minisuperspace spanned by the scale factor of a homogeneous, isotropic and closed Friedmann–Robertson–Walker (FRW) geometry together with a few matter variables, in which the WDW equation reduces to a one-dimensional Schrödinger-like problem governed by an effective potential. Unlike in ordinary quantum mechanics, the WDW equation must be supplemented by a boundary condition on $\Psi$, which plays the role of a law for the initial state and is not fixed by the dynamics itself. Two influential proposals have shaped the field: the no-boundary proposal of Hartle and Hawking [13] and the tunneling proposal of Vilenkin [14, 15]. Both describe the quantum creation of a

closed Universe "from nothing", and, although they have been the subject of a long standing debate as to which boundary condition is the correct one, they share the essential feature of describing a genuine quantum creation of spacetime itself.

In Vilenkin's tunneling scenario, a closed FRW Universe driven by a positive vacuum energy is created by quantum nucleation through the barrier of the effective potential $U(a)$. The process starts at vanishing scale factor, $a = 0$, where no classical spacetime exists, and ends at the classical turning point, beyond which the Universe expands and inflates. The expression "creation from nothing" thus refers to the absence of any classical spacetime or matter prior to nucleation: the Universe literally begins at zero size.

A conceptually different route to a non singular beginning is provided by the Emergent Universe (EU) scenario [16–18]. Here the closed Universe is past eternal: rather than originating in a singularity, it spends an indefinitely long time in a static Einstein state of finite radius before evolving towards an inflationary phase, so that the cosmological singularity is altogether absent and the "beginning of size" is replaced by a finite, non zero seed. Since the Einstein static solution is unstable in general relativity, a considerable effort has been devoted to identifying frameworks in which it becomes stable, including modified gravity, braneworld and quantum gravity corrections [19–37], as well as to the observational signatures of the emergent phase [38–46]. Concrete realizations of this idea have been constructed, in particular, within scale invariant TMT [47–53], where a static and classically stable Einstein Universe arises naturally. A recurrent difficulty of these scenarios concerns the stability of the static state against quantum decay, since the Einstein static Universe can

delicate to sustain [54, 55]. In addition, the question of whether inflationary spacetimes can be geodesically complete into the infinite past has been discussed in Ref.[56], see also Refs.[57–60].

These two lines of thought, quantum creation by tunneling and the emergent Universe, suggest a natural question. In the standard tunneling scenario the Universe is born from nothing, at $a = 0$, whereas in the emergent scenario it rests instead in a static configuration of finite radius. Is it then possible to formulate a quantum cosmological creation mechanism in which the Universe does not tunnel from $a = 0$, but rather from a pre existing classical state of finite, constant scale factor, namely the radius of the static Einstein (emergent) Universe? In the present work we address precisely this problem. Working within the framework of a scale invariant Two Measures Theory, we consider a Universe that is initially in the classical emergent (static Einstein) state, of finite radius, and show that it can be quantum mechanically destabilized, tunneling not from nothing but from this finite radius into an

mechanically destabilized, tunneling not from nothing but from this finite radius into an expanding, superinflationary phase that subsequently develops into a slow-roll inflationary stage. To this end, we reconstruct the effective potential as a function of $\dot{\phi}$, analyze the WKB tunneling action, and evaluate the corresponding tunneling probability within the TMT framework. A distinctive feature of this construction is that spacetime already exists prior to the tunneling event, so that, unlike in the Hartle–Hawking or Vilenkin scenarios, the process does not describe a quantum creation of spacetime.

As a direct consequence, the ambiguity in the choice of boundary conditions for the wave function of the Universe, which lies at the core of the long-standing debate between the followers of the Hartle–Hawking and the Vilenkin proposals, does not arise in our construction: the relevant initial state is known and is given by the classical Emergent Universe

## II. TWO MEASURES THEORY

In this section we will present a brief analysis of the TMT [62–64]. The action associated with the TMT can be written as [63, 64]

$$S_{TMT} = \int L_1 \Phi d^4x + \int L_2 \sqrt{-g} d^4x \,, \tag{1}$$

where $L_1$ and $L_2$ are two Lagrangians and the quantities $\sqrt{-g}$ and $\Phi$ are two measures of integration. Here, the quantity $\sqrt{-g}$ is the usual measure of integration in the 4 dimensional spacetime manifold related with the metric $g_{\mu\nu}$ and $\Phi$ is the new measure of integration in the same manifold. In addition, the scalar density measure $\Phi$ becomes a total derivative and it is defined by means of four scalar fields $\varphi_a$ with $a = 1, 2, 3, 4$, (see Refs. [65, 66])

$$\Phi = \varepsilon^{\mu\nu\alpha\beta}\varepsilon_{abcd}\partial_\mu\varphi_a\partial_\nu\varphi_b\partial_\alpha\varphi_c\partial_\beta\varphi_d.$$

The Lagrangian densities $L_1$ and $L_2$ are assumed to be functions of all matter fiel
the dilaton field, the metric, the connection, but independent of the measure fields
When the measure fields enter the theory exclusively through the measure $\Phi$, the acti
associated with the TMT given by Eq. (1) exhibits an infinite dimensional symmetry.

the specific realization defined by Eq. (2), this symmetry is generated by transformations of the form $\varphi_a \to \varphi_a + f_a(L_1)$, where $f_a(L_1)$ are arbitrary functions of $L_1$ [63]. Also, within this framework, we consider that all fields, including the metric, the connection, and the measure fields, constitute independent variables.

By varying the measure fields $\varphi_a$, one obtains $B^\mu_a \partial_\mu L_1 = 0$, where the quantity $B^\mu_a$ is defined as $B^\mu_a = e^{\mu\nu\alpha\beta}\varepsilon_{abcd}\partial_\nu\varphi_b\partial_\alpha\varphi_c\partial_\beta\varphi_d$. Hence, since $\det(B^\mu_a) = \frac{4^{-4}}{4!}\Phi^3$ it follows that, for $\Phi \neq 0$,

$$L_1 = sM^4 = \text{constant}, \tag{3}$$

where the parameter $s = \pm 1$ and the quantity $M$ is an integration constant with dimension of mass.

Within the context of TMT, a dilaton field $\phi$ enables the spontaneous breaking of the global scale invariance. We assume that the theory is invariant under the global scale transformations given by

$$g_{\mu\nu} \to e^{\theta} g_{\mu\nu}, \quad \Gamma^\mu_{\alpha\beta} \to \Gamma^\mu_{\alpha\beta}, \quad \varphi_a \to \lambda_{ab}\varphi_b \quad \text{where} \quad \det(\lambda_{ab}) = e^{2\theta}, \quad \phi \to \phi - \frac{M_p}{\alpha}\theta. \tag{4}$$

Also, we adopt an action that, apart from the structural modifications implied by TMT, contains no exotic field or interactions and closely follows the conventional minimal coupled scalar gravity framework. In this way, following Ref. [67] we retain the general structure of the action defined by Eq. (1), obtaining

$$S_{TMT} = \int d^4x e^{\alpha\phi/M_p}\Big[-\frac{1}{2\kappa}R(\Gamma,g)(\Phi+b_g\sqrt{-g})+(\Phi+b_\phi\sqrt{-g})\frac{1}{2}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - e^{\alpha\phi/M_p}\left(\Phi V_1+\sqrt{-g}V_2\right)\Big], \tag{5}$$

where the parameter $\kappa = 8\pi/M_p^2$ with $M_p$ denoting the four dimensional Planck mass. In addition, the quantities $V_1$ and $V_2$ are two constants and $b_g$ and $b_\phi$ are two dimensionless real parameters [67].

addition, the quantities $V_1$ and $V_2$ are two constants and $b_g$ and $b_\phi$ are two dimensionless real parameters [67].

Considering a conformal metric $\bar{g}_{\mu\nu}$ in the "Einstein frame" given by

$$\bar{g}_{\mu\nu} = e^{\alpha\phi/M_P}(\zeta + b_g)g_{\mu\nu}, \tag{6}$$

where the quantity $\zeta$ is defined as $\zeta \equiv \frac{\Phi}{\sqrt{-g}}$, the action given by Eq. (5) can be reduced to the standard General Relativity (GR) action. Here we note that the conformal metric $\bar{g}_{\mu\nu}$ is invariant under the scale transformations given by Eq. (4). Thus, assuming the change of variables to the Einstein frame, the gravitational equations assume their standard GR form [67]

$$G_{\mu\nu}(\bar{g}_{\alpha\beta}) = \kappa T^{vII}_{\mu\nu}, \tag{7}$$

where $G_{\mu\nu}(\bar{g}_{\alpha\beta})$ is the Einstein tensor.

The effective energy-momentum tensor in the Einstein frame $T^{\rm Eff}_{\mu\nu}$ is given by

$$T^{\rm Eff}_{\mu\nu} = \frac{\zeta + b_\phi}{\zeta + b_g}\left(\phi_{,\mu}\phi_{,\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{g}^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta}\right) - \bar{g}_{\mu\nu}\frac{b_g - b_\phi}{2(\zeta + b_g)}\bar{g}^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} + \bar{g}_{\mu\nu}V_{\rm Eff}(\phi;\zeta, M), \quad (8)$$

where the quantity $V_{\rm Eff}(\phi;\zeta, M)$ is given by

$$V_{\rm Eff}(\phi;\zeta, M) = \frac{b_g\left[sM^4 e^{-2\alpha\phi/M_{\rm P}} + V_1\right] - V_2}{(\zeta + b_g)^2}. \quad (9)$$

In addition, the scalar field $\zeta$ is fixed by the consistency of Eq. (7) with Eq. (3), leading to the constraint

$$(b_g - \zeta)\left[sM^4 e^{-2\alpha\phi/M_{\rm P}} + V_1\right] - 2V_2 - \delta \cdot b_g(\zeta + b_g)\mathcal{Z} = 0, \quad (10)$$

where the quantities $\mathcal{Z}$ and $\delta$ are defined as $\mathcal{Z} \equiv \frac{1}{2}\bar{g}^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta}$ and $\delta = \frac{b_g - b_\phi}{b_g}$, respectively.

where the quantities $Z$ and $\delta$ are defined as $Z = \frac{1}{2}\bar{g}^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta}$ and $\delta = \frac{2s-2s'}{b_g}$, respectively.

The effective energy-momentum tensor given by Eq. (8) can be identified with that of a perfect fluid $T^{eff}_{\mu\nu} = (\rho + p)u_\mu u_\nu - p\bar{g}_{\mu\nu}$, where the four-velocity is given by $u_\mu = \frac{\phi_{,\mu}}{(2Z)^{1/2}}$. Here, $\rho$ and $p$ are the energy density and pressure associated with the perfect fluid. Thus, considering the solution $\zeta = \zeta(\phi, Z; M)$ given by Eq. (10), the energy density $\rho$ and pressure $p$ can be written as [67]

$$\rho(\phi, Z; M) = Z + \frac{(sM^4 e^{-2\alpha\phi/M_p} + V_1)^2 - 2\delta b_g(sM^4 e^{-2\alpha\phi/M_p} + V_1)Z - 3\delta^2 b_g^2 Z^2}{4[b_g(sM^4 e^{-2\alpha\phi/M_p} + V_1) - V_2]}, \tag{11}$$

and

$$p(\phi, Z; M) = Z - \frac{\left(sM^4 e^{-2\alpha\phi/M_p} + V_1 + \delta b_g Z\right)^2}{4[b_g(sM^4 e^{-2\alpha\phi/M_p} + V_1) - V_2]}. \tag{12}$$

The different constants associated with this model are subject to the observational constraints and stability conditions given by Eqs. (18)–(19), as studied in Ref. [48].

## III. EMERGENT UNIVERSE

The EU scenario is based on the assumption that the Universe originates from a past-eternal Einstein static (ES) state, subsequently evolving toward an inflationary phase and a hot Big Bang era. In a series of papers [47–52], we have studied a class of EU scenarios based on a spontaneously broken scale invariance induced by the dynamics of a Two Measures Theory.

In the present work we focus in particular on the model of Ref. [48], in which the ES solution is classically stable under homogeneous and isotropic perturbations and which, as shown in Ref. [50], does not suffer the semiclassical instability towards collapse reported by Vilenkin [54]. We note that the results obtained in this case can also be applied to the models studied in Refs. [47, 51], which present symmetries similar to those of the model of Ref. [48].

We begin by considering a metric describing a closed FRW spacetime, for which the line element $ds^2$ is defined as:

$$ds^2 = dt^2 - a(t)^2 \left( \frac{dr^2}{1 - r^2} + r^2 (d\theta^2 + \sin^2\theta d\phi^2) \right), \tag{13}$$

where $a(t)$ is the scale factor and $t$ is the cosmic time.

We further assume that the scalar field is homogeneous, i.e., $\phi$ depends only on the cosmic time $t$, such that $\phi(\mathbf{x}, t) = \phi(t)$.

For the dynamics, we will assume a stage in which the scalar field $\phi$ is moving in the extreme left region $\phi \to -\infty$ of the effective potential given by Eq. (9). In this context, the expressions for the energy density $\rho$ and pressure $p$ given by Eqs. (11) and (12) can be written as

$$\rho = \frac{A}{2}\dot{\phi}^2 + 3B\dot{\phi}^4 + C, \tag{14}$$

and

$$p = \frac{A}{2}\dot{\phi}^2 + B\dot{\phi}^4 - C, \tag{15}$$

respectively. Here, the values of the constants $A$, $B$ and $C$ are defined by

$$A = 1 - \frac{2\delta b_g V_1}{4(b_g V_1 - V_2)}, \quad B = -\frac{\delta^2 b_g^2}{4(b_g V_1 - V_2)}, \quad \text{and} \quad C = \frac{V_1^2}{4(b_g V_1 - V_2)}. \tag{16}$$

In the following, the dots denote derivatives with respect to cosmic time $t$.

In analyzing the stability of the model of Ref. [48], we noted that its Wheeler DeWitt potential leaves open the possibility of additional tunneling channels, corresponding to a transition from the ES solution to a different value of the scale factor, from which the Universe would subsequently begin to evolve classically. It is precisely this possibility that we explore in the present work.

We now continue with the review of Ref. [48], in which the constant value of the scale factor characterizing the Einstein static Universe, $a = a_+$, is given by

$$a_+ = \sqrt{\left(\frac{3}{8\pi G}\right)\frac{12B}{A^2 + 24BC - A\sqrt{A^2 + 12BC}}}\,. \tag{17}$$

In order for this static solution to be stable under homogeneous and isotropic perturbations, the following conditions must be satisfied, as discussed in Ref. [50]:

$$0.5 < y < 0.54\,, \qquad B < 0\,, \qquad \text{and} \tag{18}$$

$$-\frac{1}{64B} < C < -\sqrt{\frac{3}{B^2}} - \frac{7}{4B}\,, \tag{19}$$

where the constant $A$ is defined as $A = 1 - y$, with the quantity $y$ given by $y = \frac{2\beta_g C}{V_1}$.

In addition, as established in Ref. [48], there is a conserved quantity $\Pi_\phi$. Indeed, in the limit $\phi \to -\infty$, the theory acquires an additional symmetry under the shift $\phi \to \phi + c$, with $c$ a constant. In the Einstein frame, the action takes the form

$$S_E = \frac{1}{\kappa}\left[\int R\sqrt{-g}\,d^4x + \int p\sqrt{-g}\,d^4x\right], \tag{20}$$

and the symmetry $\phi \to \phi + c$ leads to the conservation law [48]

$$a^3(t)\left[A\dot{\phi} + 4\,B\dot{\phi}^3\right] = \Pi_\phi = \text{constant}\,. \tag{21}$$

Without loss of generality, we take $\Pi_\phi$ to be positive. From Eq. (21), we can obtain a relation between the scale factor and $\dot{\phi}$ given by

$$a(\dot{\phi}) = \left(\frac{\Pi_\phi}{A\dot{\phi} + 4\,B\dot{\phi}^3}\right)^{1/3}. \tag{22}$$

Because $\Pi_\phi > 0$, the range of $\dot{\phi}$ is given by $-\infty < \dot{\phi} < -\sqrt{\frac{A}{4|B|}}$ or alternatively $0 < \dot{\phi} < \sqrt{\frac{A}{4|B|}}$.

In the first case, in which $-\infty < \dot{\phi} < -\sqrt{\frac{A}{4|B|}}$, the scale factor as a function of $\dot{\phi}$, i.e., $a(\dot{\phi})$, goes to zero when $\dot{\phi} \to -\infty$ and diverges when $\dot{\phi} \to -\sqrt{\frac{A}{4|B|}}$. However, in this region, the energy density $\rho$ defined by Eq. (14) is negative, which is therefore not of physical interest here.

Alternatively, in the range $0 < \dot{\phi} < \sqrt{\frac{A}{4|B|}}$, the scale factor $a(\dot{\phi})$ has an extremum (minimum) at $\dot{\phi} = \dot{\phi}_0$, where the scale factor $a(\dot{\phi} = \dot{\phi}_0) = a_0$, and these values are given by

$$\dot{\phi}_0 = \sqrt{\frac{A}{12|B|}}, \quad \text{and} \quad a_0 = \left(\frac{12|B|}{A}\right)^{1/6} \left[\frac{3\Pi_\phi}{2A}\right]^{1/3}, \tag{23}$$

respectively. In addition, we note that from Eq. (22), the scale factor $a(\dot{\phi})$ diverges when $\dot{\phi} \to 0$ or when $\dot{\phi} \to \sqrt{\frac{A}{4|B|}}$.

Moreover, we note that values of the scale factor smaller than $a_0$, i.e., $a(\dot{\phi}) < a_0$, l
outside the domain where the physical solutions exist. To illustrate this, Fig. 1 shows th
scale factor as a function of $\dot{\phi}$, for the values $B = -1$, $C = 0.016$, $y = 0.505964$ an
$\Pi_\phi = 113.41$.

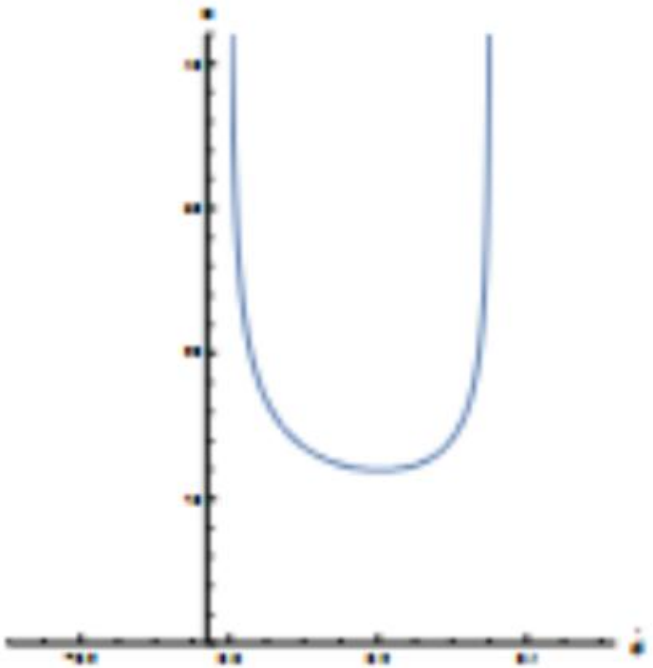

FIG. 1: Scale factor $a$ as a function of $\dot{\phi}$, obtained from Eq. (21). Here we have considered the values $B = -1$, $C = 0.016$, $y = 0.505964$ and $\Pi_\phi = 113.41$.

Friedmann equation. From Eqs. (13) and (20), for a closed Universe with energy density $\rho$, it can be written as

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2 = \frac{8\pi G}{3}\rho(a) - \frac{1}{a^2}\,, \tag{24}$$

where $H = \dot{a}/a$ is the Hubble parameter.

Additionally, the equation for $\dot{H}$ for a closed Universe is given by

$$\dot{H} = -4\pi G(\rho + p) + \frac{1}{a^2}\,. \tag{25}$$

Now, using the relation between the scale factor and $\dot{\phi}$ defined by Eq. (21), we can rewrite the Friedmann equation given by Eq. (24) in terms of $\dot{\phi}$, becoming

$$\frac{\dot{\phi}^2}{2} + V(\dot{\phi}) = 0 \tag{26}$$

where the quantity $V(\dot{\phi})$ can be interpreted as an effective potential, depending on $\dot{\phi}$ as

$$V(\dot{\phi}) = \frac{9(A\dot{\phi} + 4B\dot{\phi}^3)^2}{(A + 12B\dot{\phi}^2)^2} \left[ \frac{1}{\Pi_\phi^{2/3}} (A\dot{\phi} + 4B\dot{\phi}^3)^{2/3} - \frac{\kappa}{3} \left( \frac{A}{2}\dot{\phi}^2 + 3B\dot{\phi}^4 + C \right) \right] . \qquad (27)$$

As an example, Fig. 2 shows the potential $V(\dot{\phi})$, where we have considered the value of the constant $\Pi_\phi = 113.41$ together with the parameter values used in Ref. [48], $B = -1$, $C = 0.016$ and $y = 0.505964$. From Eq. (26) we have $\bar{\phi}^2 = -V(\dot{\phi})$, so that the classically allowed regions are those in which $V(\dot{\phi}) \leq 0$, whereas the regions with $V(\dot{\phi}) > 0$ are classically forbidden and can only be traversed by tunneling. In Fig. 2 we can identify a potential well with $V(\dot{\phi}) < 0$ extending from $\dot{\phi} = 0$ up to the divergent barrier located at $\dot{\phi} = \dot{\phi}_0 = \sqrt{A/(12|B|)} \simeq 0.20$. Immediately to the right of this barrier the potential vanishes again, at the value $\dot{\phi}_+ = \sqrt{(-A + \sqrt{A^2 + 12BC})/(6B)} \simeq 0.21$, which corresponds to the static (ES) Universe. Beyond it, the potential exhibits a second, finite barrier, on the

to the static (ES) Universe. Beyond it, the potential exhibits a second, finite barrier, on the far side of which it becomes negative again, defining a second classically allowed region that extends up to $\dot{\phi} = \sqrt{A/(4|B|)}$, see Fig. 3.

From this potential, we note that the possibility of tunneling through the divergent barrier located at $\dot{\phi}_0 = \sqrt{A/(12|B|)}$ is allowed, see Ref. [68]. Thus, the tunneling process proceeds from the static value $\dot{\phi}_+$ to the value $\dot{\phi}_-$ situated to the left of the infinite barrier, where the potential also vanishes, i.e., $V(\dot{\phi}_-) = 0$, see Fig. 2.

As discussed in Ref. [48], this tunneling process does not represent a collapse of the Universe towards $a \to 0$, but rather a transition from the static configuration, with scale

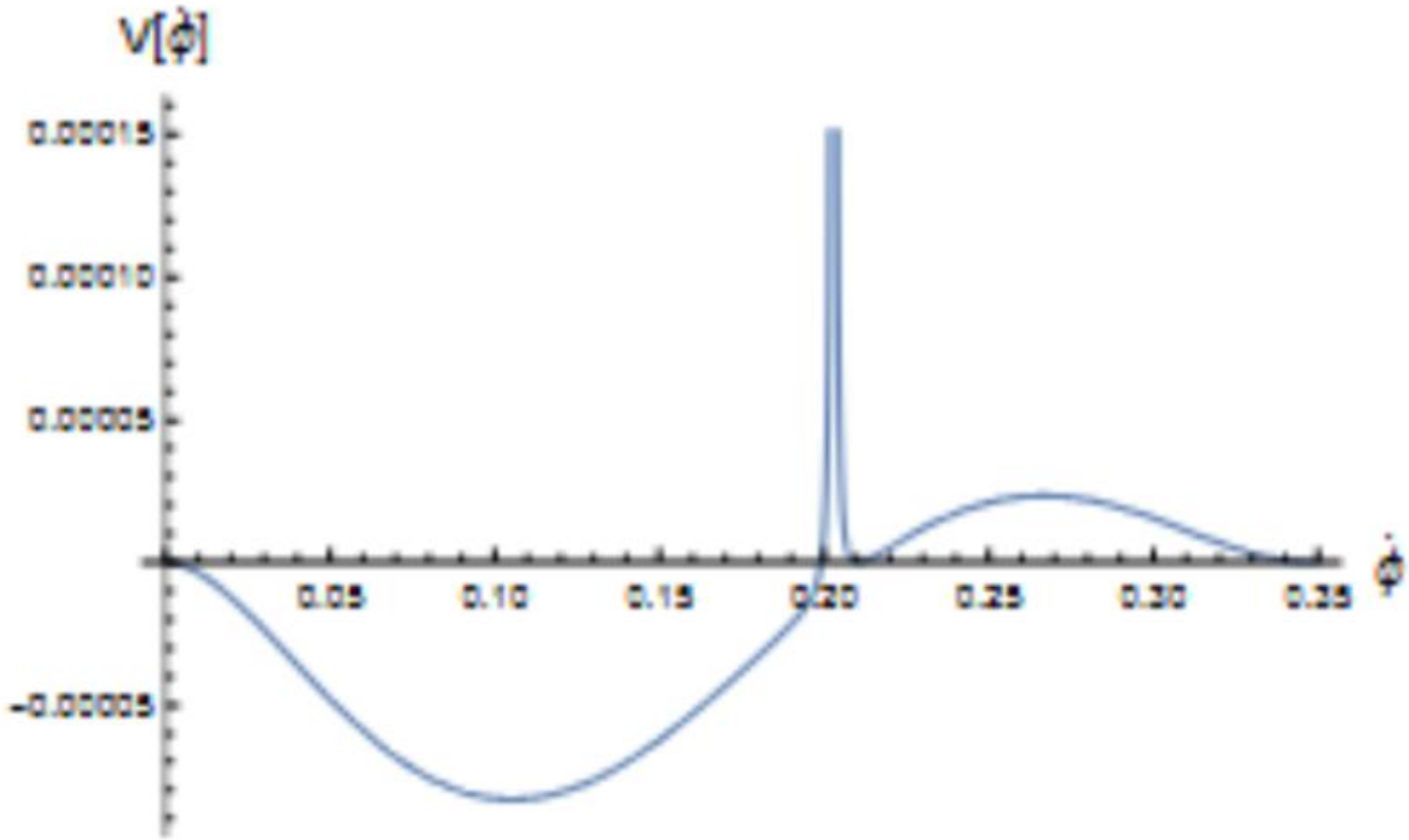


FIG. 2: Potential $V(\dot{\phi})$ as a function of $\dot{\phi}$, for $\dot{\phi} > 0$. Here, the infinite barrier is located at $\dot{\phi}_0 = 0.20$.

factor $a_+$ given by Eq. (17), to the configuration with $a_- = a(\dot{\phi}_-)$, from which the Universe subsequently begins to evolve. Indeed, after the tunneling event the Universe emerges with $a(\dot{\phi}_-) = a_-$ and $\dot{a}_- = 0$, see Eqs. (24) and (27). Then, as $\dot{\phi}$ goes asymptotically to zero after tunneling, see Fig. 2, the scale factor grows, see Fig. 1. During this process, the Hubble parameter goes from zero to a constant value $H_0$. This evolution will be discussed in more detail in Sec. IV.

We note that there is also the possibility of tunneling through the finite barrier located to the right of the potential $V(\dot{\phi})$, into the second classically allowed region discussed above, see Fig. 3. After tunneling through this branch, $\dot{\phi}$ evolves within this region towards $\sqrt{A/(4|B|)}$, where the Universe reaches a de Sitter solution with $H = \sqrt{\frac{\kappa(A^2+16BC)}{48B}}$, since in this limit $\rho + p = 0$, see Ref. [48]. However, this case does not correspond to a standard slow-roll inflationary regime, as the kinetic term in Eq. (14) is no longer negligible. In this paper, we restrict our analysis to the tunneling through the infinite barrier, and leave a detailed

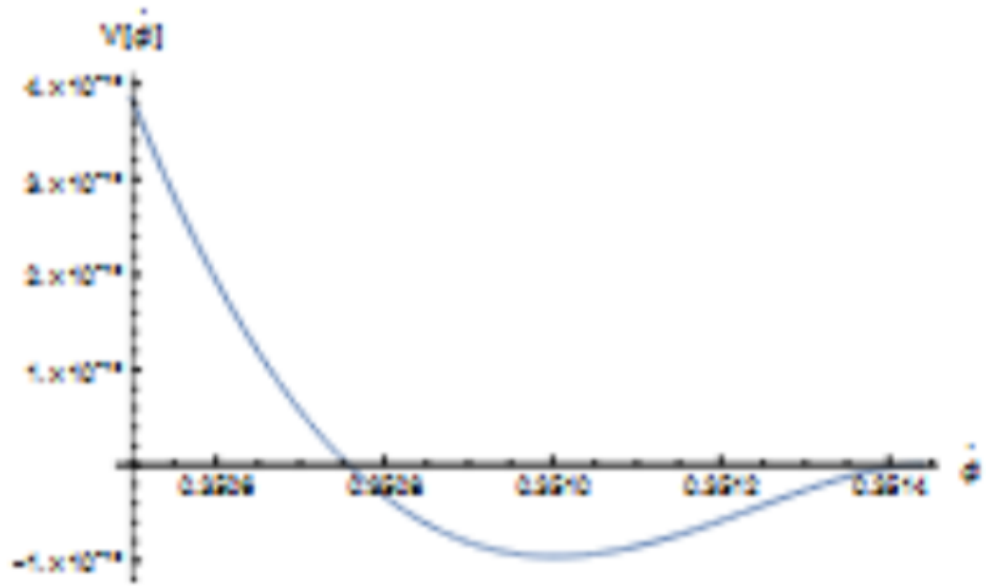


FIG. 3: Potential $V(\dot{\phi})$ as a function of $\dot{\phi}$, to the right of the finite barrier approaching the maximum value $\sqrt{\frac{A}{4|B|}}$.

## IV. TUNNELING FROM EU TO INFLATION

In this section, we study the tunneling of $\dot{\phi}$ through the infinite barrier of the effective potential, from the static value $\dot{\phi}_+$ to the value $\dot{\phi}_-$ located to its left, see Fig. 2. To determine the tunneling probability, we first reconstruct the action associated with the effective potential defined by Eq. (27). Equation (26) can also be obtained from an effective action, as follows: first, we write the effective action associated to the scalar field in terms of the Lagrangian density $L$ as

$$S = \int L(\dot{\phi}, \bar{\phi})\, dt = \int L\, dt = \int dt\, [\bar{\phi}^2 - V(\dot{\phi})]. \tag{28}$$

By varying this action and integrating by parts, one obtains

$$\delta S = \int dt \left( \frac{\partial L}{\partial \dot{\phi}} - \frac{d}{dt} \left( \frac{\partial L}{\partial \bar{\phi}} \right) \right) \delta\dot{\phi}. \tag{29}$$

Integrating by parts once more, the variation of the action can be written in terms of $\delta\phi$ as

$$\delta S = -\int dt \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{\phi}}\right)\right)\delta\phi = 0, \tag{30}$$

which implies that

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{\phi}}\right)\right) = 0. \tag{31}$$

Thus, the first integral of Eq. (31) becomes

$$\frac{\partial L}{\partial \dot{\phi}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{\phi}}\right) = M = \text{const.}, \tag{32}$$

where the constant $M$ can be absorbed by defining the effective Lagrangian $L_{eff}(\dot{\phi}, \bar{\phi})$ as

$$L_{eff}(\dot{\phi}, \bar{\phi}) = L_{eff} = L - M\dot{\phi}, \tag{33}$$

and Eq. (32) then reduces to

$$\frac{\partial L_{eff}}{\partial \dot{\phi}} - \frac{d}{dt}\left(\frac{\partial L_{eff}}{\partial \bar{\phi}}\right) = 0. \tag{34}$$

From Eq. (26) we identify that the effective Lagrangian can be written as

$$L_{eff} = \bar{\phi}^2 - V_{eff}(\dot{\phi}), \tag{35}$$

where the new effective potential $V_{eff}(\dot{\phi})$ is defined as

$$V_{eff}(\dot{\phi}) = V(\dot{\phi}) + M\dot{\phi}. \tag{36}$$

Thus, from Eq. (34), the equation of motion becomes

$$2\dddot{\phi} + \frac{\partial V}{\partial \phi} + M = 0. \tag{37}$$

By imposing that the integration constant $M = 0$, we recover the Friedmann constraint given by Eq. (26). Under this condition, the Hamiltonian associated with our emergent Universe can be written as

$$\mathcal{H}(\dot{\phi}, p_{\dot{\phi}}) = \mathcal{H} = \frac{p_{\dot{\phi}}^2}{4} + V(\dot{\phi}), \tag{38}$$

where the momentum $p_{\dot{\phi}}$ is given by

$$p_{\dot{\phi}} = \frac{\partial L}{\partial \ddot{\phi}} = 2\ddot{\phi}, \tag{39}$$

where $V(\dot{\phi})$ is defined by Eq. (27).

Within the framework of quantum theory, the Universe can be described by a wave function $\psi(\tilde{\phi})$ which depends on the variable $\tilde{\phi}$. In this quantization formalism, the conjugate momentum $p_{\tilde{\phi}}$ corresponds to the differential operator $-id/d\tilde{\phi}$ and the Hamiltonian constraint $\mathcal{H}\psi(\tilde{\phi}) = 0$ is replaced by the Wheeler DeWitt (WDW) equation [12], such that

$$\mathcal{H}\psi(\tilde{\phi}) = 0, \quad \Rightarrow \quad \left(-\frac{1}{4}\frac{d^2}{d\tilde{\phi}^2} + V(\tilde{\phi})\right)\psi(\tilde{\phi}) = 0\,. \tag{40}$$

Here, we have adopted the minisuperspace approximation, which is appropriate for our model since the Universe is considered homogeneous, isotropic, and closed. This is possible

because the system has a single dynamical degree of freedom, namely $\dot{\phi}$, to which the scale factor is related through Eq. (22), see Ref. [15].

In this formalism, the tunneling probability $\mathcal{P}$ associated with $\dot{\phi}$ can be quantified as [54]

$$\mathcal{P} \sim e^{-2S}, \tag{41}$$

where $S$ denotes the WKB tunneling action defined as [54]

$$S = \int_{\dot{\phi}_-}^{\dot{\phi}_+} \sqrt{V(\dot{\phi})}\, d\dot{\phi}. \tag{42}$$

Here, Eq. (41) represents the probability that the Universe tunnels from the static value $\dot{\phi}_+$ to the value $\dot{\phi}_-$ situated to the left of the infinite barrier, where the potential vanishes, see Fig. 2. As discussed in Sec. III, this process does not correspond to a collapse towards $a \to 0$, but to a transition from the static configuration with $a_+$ to the configuration with

In order to obtain an analytical expression for the value of $\dot{\phi}_-$, we determine the zero of the potential by expanding the term enclosed in the square brackets of Eq. (27) around $\dot{\phi}_0$, where the potential diverges, obtaining

$$\left[\frac{1}{\Pi_\phi^{2/3}}(A\dot{\phi}+4B\dot{\phi}^3)^{2/3}-\frac{\kappa}{3}\left(\frac{A}{2}\dot{\phi}^2+3B\dot{\phi}^4+C\right)\right]\simeq\frac{1}{144}\left[\frac{A^2\kappa}{B}-48C\kappa+\frac{48}{\Pi_\phi^{2/3}}(A\sqrt{A/|B|})^{2/3}\right]$$

$$+\left[\frac{A\kappa}{3}+\frac{4B}{A\Pi_\phi^{2/3}}(A\sqrt{A/|B|})^{2/3}\right]\left(\dot{\phi}-\sqrt{\frac{A}{12|B|}}\right)^2+\mathcal{O}\left(\dot{\phi}-\sqrt{\frac{A}{12|B|}}\right)^3. \tag{43}$$

Thus, from Eq. (43), we find that the value of $\dot{\phi}_-$, situated to the left of the infinite barrier, becomes:

$$\dot{\phi}_-=\dot{\phi}_0-R_1, \tag{44}$$

where the quantities $R_1$ and $\dot{\phi}_0$ are defined as

where the quantities $R_1$ and $\dot{\phi}_0$ are defined as

$$R_1 = \sqrt{-\frac{A[48B([A\sqrt{A/|B|}]^{2/3} - \kappa C\Pi_\phi^{2/3}) + \kappa A^2\Pi_\phi^{2/3}]}{48B[12B(A\sqrt{A/|B|})^{2/3} + \kappa A^2\Pi_\phi^{2/3}]}}, \quad \text{and} \quad \dot{\phi}_0 = \sqrt{\frac{A}{12|B|}}. \tag{45}$$

Here, we have taken the negative branch of the root in Eq. (44) in order to ensure that the solution lies on the left side of the infinite barrier, where the zero of the potential is located.

In addition, from Ref. [48] the emergent value $\dot{\phi}_+$ introduced in Sec. III is given by

$$\dot{\phi}_+ = \sqrt{\frac{-A + \sqrt{A^2 + 12BC}}{6B}} = \dot{\phi}_0 + R_2, \tag{46}$$

where

$$R_2 = \dot{\phi}_+ - \dot{\phi}_0 = \sqrt{\frac{-A + \sqrt{A^2 + 12BC}}{6B}} - \sqrt{\frac{A}{12|B|}}. \tag{47}$$

Considering Eqs. (44) and (46), the WKB tunneling action given by Eq. (42) can be written as

$$S = \int_{\dot{\phi}_-}^{\dot{\phi}_+} \sqrt{V(\dot{\phi})}\, d\dot{\phi} \approx F(\dot{\phi}_0) \int_{\dot{\phi}_0 - R_2}^{\dot{\phi}_0 + R_2} P\left[\frac{1}{(1 - \dot{\phi}/\dot{\phi}_0)}\right] d\dot{\phi}, \tag{48}$$

where the constant $F(\dot{\phi}_0)$ is defined as

where the constant $F(\dot{\phi}_0)$ is defined as

$$F(\dot{\phi}_0) = \sqrt{\frac{9\left[\frac{1}{\Pi_\star^{5/3}}(A\dot{\phi}_0 + 4B\dot{\phi}_0^3)^{8/3} - \frac{6}{5}\left(\frac{A}{2}\dot{\phi}_0^2 + 3B\dot{\phi}_0^4 + C\right)(A\dot{\phi}_0 + 4B\dot{\phi}_0^3)^2\right]}{A(1+\dot{\phi}_0/\dot{\phi}_0)^2}}, \tag{49}$$

or equivalently

$$F(\dot{\phi}_0) = \left(\left[\frac{\left(A\sqrt{\frac{A}{|B|}}\right)^{2/3}}{3\Pi_\star^{2/3}} + \frac{\kappa(A^2 - 48BC)}{144B}\right]\frac{A^2}{12|B|}\right)^{1/2}. \tag{50}$$

Although both branches of the square root are allowed by analyticity in Eq. (48), the semiclassical interpretation of the tunneling probability, Eq. (41), requires $\mathcal{S} > 0$ in order to ensure $\mathcal{P} \leq 1$. This condition uniquely fixes the branch of the square root used in Eq. (48). With this choice, Eq. (48) follows, with $P$ denoting the Cauchy principal value;

In order to determine the value of the constant $\Pi_\phi$, we evaluate this quantity at the moment when both the scale factor and $\dot{\phi}$ take the values corresponding to the emergent Universe scenario. Thus, from Eq. (21), we find that the constant $\Pi_\phi$ can be written as:

$$\Pi_\phi = a_+^3 \left[A\dot{\phi}_+ + 4\,B\dot{\phi}_+^3\right] = 12\sqrt{6}\sqrt{\frac{\sqrt{A^2+12BC}-A}{B}} \times$$

$$\left(A+2\sqrt{A^2+12BC}\right)\left[\frac{B}{(A^2+24BC-A\sqrt{A^2+12BC})\kappa}\right]^{3/2}. \tag{51}$$

Here, we have utilized that the scale factor $a_+$ is given by Eq. (17) and the emergent value $\dot{\phi}_+$ is defined by Eq. (46).

Thus, integrating Eq. (48), we find that the WKB tunneling action related to the probability $\mathcal{P}$ becomes

$$\mathcal{S} \approx F(\dot{\phi}_0)\,\dot{\phi}_0 \log\left[\frac{R_2}{R_1}\right]. \tag{52}$$

As a result, we find that the tunneling probability exhibits a characteristic power law behavior given by

$$\mathcal{P} \sim \left[\frac{R_1}{R_2}\right]^{2\dot{\phi}_0 F(\dot{\phi}_0)} = \left[\frac{R_1}{R_2}\right]^{\gamma(\dot{\phi}_0)}, \quad \text{with} \quad \gamma(\dot{\phi}_0) = 2\dot{\phi}_0 F(\dot{\phi}_0), \tag{53}$$

instead of the standard exponential expression associated to semiclassical tunneling processes. Since $R_2 > R_1$ and the exponent $\gamma(\dot{\phi}_0)$ is positive, the tunneling probability decreases as the ratio $R_2/R_1$ increases. This implies that transitions with increasingly asymmetric turning points, i.e., with increasing $R_2/R_1$, yield progressively smaller probabilities. In addition, for large values of the exponent $\gamma(\dot{\phi}_0) \gg 1$, the tunneling probability is strongly suppressed. For the case in which the exponent $\gamma(\dot{\phi}_0) \ll 1$, the power law suppression becomes very weak and the probability decreases only slowly with the ratio, implying that quantum tunneling is enhanced. In particular, using the numerical values of the parameters

given in the caption of Fig. 1, we find that the exponent $\gamma(\phi_0) \simeq 3 \times 10^{-7} \ll 1$ and the tunneling probability $\mathcal{P}$ defined by Eq. (53) is very close to unity.

We now examine in more detail the evolution following the tunneling event. The Universe emerges with the conditions $a(\dot{\phi}_-) = a_-$ and $\dot{a}_- = 0$, see Eqs. (24) and (27). Then, as $\dot{\phi}$ goes asymptotically to zero after tunneling, see Fig. 2, we have that $H$ goes from zero to a constant value $H_0$, where $H_0 = \sqrt{\frac{8}{3}C}$. This process is known as the superinflation stage. During this period we have $\dot{H} > 0$. We note from Eq. (25) that this behavior can be achieved without violating the null energy condition $\rho + p > 0$, owing to the positive spatial curvature of the closed Universe. This is a distinctive feature of the EU scenario, which produces a suppression of the CMB anisotropies at large scales, see Ref. [38], and after the superinflationary period we arrive at the inflationary stage.

In this article, we have studied the quantum transition from a classically stable EU to a superinflationary phase and its subsequent evolution toward slow roll inflation within the framework of TMT with spontaneously broken scale invariance. In this scenario, the Universe starts from a classically stable Einstein static state, thereby avoiding the initial singularity and remaining in this phase until the quantum tunneling event takes place.

In contrast to the standard proposals of quantum cosmology, where quantum effects are responsible for the creation of spacetime itself, the tunneling process considered here takes place in a Universe that already exists as a classically stable EU. Quantum tunneling simply provides the mechanism that drives the transition from this static stage to a superinflationary phase, which subsequently evolves into the standard slow roll inflationary epoch, i.e., the tunneling from $\dot{\phi}_+$ to $\dot{\phi}_-$.

In this context, we have derived an effective potential in terms of $\dot{\phi}$, $V(\dot{\phi})$, which provides a description of the dynamics of the EU. Using the conserved quantity associated with the scalar field, we have expressed the scale factor as a function of $\dot{\phi}$ and rewritten the cosmological evolution in terms of this effective potential, see Eq. (26). This new formulation shows the existence of an infinite potential barrier that separates the static EU from the subsequent inflationary evolution. This barrier is what makes the quantum tunneling process possible. After reconstructing the corresponding effective action in terms of $\dot{\phi}$ and $\ddot{\phi}$, we quantized the minisuperspace Hamiltonian, leading to the Wheeler-DeWitt equation governing the quantum dynamics. Thus, the tunneling probability in terms of $\dot{\phi}$ was then evaluated using the WKB approximation, determining the probability that the Universe tunnels from the static value $\dot{\phi}_+$ to the value $\dot{\phi}_-$ situated to the left of the infinite barrier.

In our analysis we have found that the tunneling probability does not exhibit the conventional exponential suppression characteristic of semiclassical tunneling processes. Instead,

tional exponential suppression characteristic of semiclassical tunneling processes. Instead, due to the logarithmic contribution generated by the integration on the effective potential, the tunneling probability follows a power law behavior, see Eqs. (52) and (53). For the parameter values considered in this work, we have obtained that the exponent associated to the power law behavior is very small, implying that the tunneling probability is close to unity. Therefore, once the Universe reaches the emergent state, the transition to the superinflationary phase occurs with a high probability, followed by the standard slow roll

inflationary epoch.

After the tunneling process, the Universe emerges with a finite scale factor and a vanishing Hubble parameter, while $\dot{\phi}$ tunnels through the infinite barrier to the value $\dot{\phi}_-$, initiating the subsequent superinflationary phase. Subsequently, $\dot{\phi}$ evolves toward smaller values (to the left of the barrier), the Hubble parameter increases from zero to its asymptotic constant value $H_0 = \sqrt{\kappa C/3}$, and the Universe evolves into the standard slow-roll inflationary regime. The present analysis therefore provides a consistent quantum mechanism connecting the stable Einstein static Universe, quantum tunneling, superinflation, slow-roll inflation, and the subsequent standard cosmological evolution.

Several directions deserve further investigation. In particular, a detailed study of the scalar and tensor perturbations spectra generated during the superinflationary phase following the tunneling process is left for future work. Finally, the alternative tunneling process through the finite barrier located to the right of the effective potential $V(\tilde{\phi})$, towards the

second classically allowed region discussed in Sec. III, remains an interesting direction for future investigation, since it may lead to novel and distinct cosmological evolutions. In particular the non linear k-essence terms present in these directions, which we have seen contribute to the vacuum energy, can be shown to contribute, when considering perturbations, to an effective Dark Matter contribution, and in this way the Dark Matter may find an explanation this way, see [53].