Lectures on Quantum Cosmology

Lecture 3: Loop Quantum Cosmology and Modified Quantizations

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Outline

1. From ADM to loop variables
2. Loop Quantum Cosmology
3. Modified Loop Quantum Cosmologies
4. Perturbations and observations

From ADM to loop variables

Lecture 1: ADM canonical data

ADM action:

$$S_{\rm ADM} = \int dt\, d^3x \left[\pi^{ab}\dot{h}_{ab} - N\mathcal{H} - N^a\mathcal{H}_a\right].$$

$$\boxed{(h_{ab}, \pi^{ab})} \quad \Rightarrow \quad \boxed{\{h_{ab}(\boldsymbol{x}), \pi^{cd}(\boldsymbol{y})\} = \delta^c_{(a}\delta^d_{b)}\delta^3(\boldsymbol{x}-\boldsymbol{y})}.$$

$$\begin{aligned} N \quad &: \quad \text{lapse,} \\ N^a \quad &: \quad \text{shift,} \\ \mathcal{H} \approx 0 \quad &: \quad \text{Hamiltonian constraint,} \\ \mathcal{H}_a \approx 0 \quad &: \quad \text{spatial diffeomorphism constraint.} \end{aligned}$$

Loop quantum gravity (LQG):

$$\boxed{(h_{ab}, \pi^{ab}) \Rightarrow (A_a{}^i, E_i{}^a) \Rightarrow \text{holonomies and fluxes.}}$$

$(h_{ab}, \pi^{ab}) \Rightarrow (E_i{}^a, K_a{}^i)$

Introduce a local orthonormal triad:

$$\boxed{h_{ab} = e_a{}^i e_b{}^j \delta_{ij}}, \qquad a, b: \text{ spatial}, \quad i, j: \text{ internal}.$$

Densitized triad:

$$\boxed{E_i{}^a \equiv \sqrt{h}\, e_i{}^a}, .$$

For flat FLRW, $h_{ab} = a^2 \delta_{ab}$:

$$\boxed{E_i{}^a = a^2 \delta_i{}^a} \qquad \Rightarrow \qquad E_i{}^a \Rightarrow \textbf{spatial area scale.}$$

Extrinsic curvature:

$$\boxed{\pi^{ab} \Leftrightarrow K_{ab}} \qquad \Rightarrow \boxed{K_a{}^i \equiv K_{ab} e^{bi}}.$$

New canonical pair:

$$(h_{ab}, \pi^{ab}) \Rightarrow (E_i{}^a, K_a{}^i), \qquad \{K_a{}^i(\boldsymbol{x}), E_j{}^b(\boldsymbol{y})\} = \kappa \delta_a^b \delta_j^i \delta^3(\boldsymbol{x} - \boldsymbol{y}).$$

Ashtekar–Barbero connection

Spin connection ${\Gamma_a}^j \Rightarrow$ ensures compatibility with the triad:

$$\boxed{D_a {e_b}^i = \partial_a {e_b}^i - {}^{(3)}{\Gamma^c}_{ab} {e_c}^i + {\epsilon^i}_{jk} {\Gamma_a}^j {e_b}^k = 0.}$$

But

$${\Gamma_a}^i \Rightarrow \text{intrinsic spatial geometry/parallel transport,}$$

$${K_a}^i \Rightarrow \text{change of spatial geometry} \Rightarrow \text{one slice to another.}$$

New connection:

$$\boxed{{A_a}^i \equiv {\Gamma_a}^i + \gamma {K_a}^i},$$

where γ is the Barbero–Immirzi parameter.
Then,

$$\boxed{D_a {e_b}^i = \gamma {\epsilon^i}_{jk} {K_a}^j {e_b}^k.}$$

A. Ashtekar, Phys. Rev. Lett. **57**, 2244 (1986).

J. F. Barbero, Phys. Rev. D **51**, 5507 (1995); G. Immirzi, Class. Quant. Grav. **14**, L177 (1997).

Gauss constraint and final canonical pair

Additional rotation:

$$e_a{}^i \Rightarrow R^i{}_j e_a{}^j \qquad \Rightarrow \qquad h_{ab} \text{ unchanged.}$$

Gauss constraint $\Rightarrow$ solves this issue:

$$\boxed{G_i \equiv D_a E_i{}^a = \partial_a E_i{}^a + \epsilon_{ij}{}^k A_a{}^j E_k{}^a \approx 0.}$$

$$\boxed{H_{\rm tot} = H_{\rm ADM}[N, N^a] + G[\Lambda], \qquad G[\Lambda] = \int d^3x\, \Lambda^i G_i.}$$

For a closed slice or suitable falloff conditions and vanishing of boundary terms:

$$\int d^3x\, \pi^{ab} \dot{h}_{ab} = \frac{1}{\kappa\gamma} \int d^3x\, E_i{}^a \dot{A}_a{}^i.$$

Therefore,

$$\boxed{\{A_a{}^i(\boldsymbol{x}), E_j{}^b(\boldsymbol{y})\} = \kappa\gamma \delta_a^b \delta_j^i \delta^3(\boldsymbol{x} - \boldsymbol{y}).}$$

LQG variables: holonomies and fluxes

Local fields ${A_a}^i$ and ${E_i}^a$ $\Rightarrow$ not elementary in LQG.

$$\boxed{{A_a}^i \Rightarrow h_\ell[A] \equiv \mathcal{P}\exp\left(\int_\ell A\right)} \quad , \quad \boxed{{E_i}^a \Rightarrow E_i(S) \equiv \int_S n_a {E_i}^a \, d^2\sigma.}$$

$h_\ell[A]$: holonomy: finite internal parallel transport along ℓ,

$E_i(S)$: flux of the densitized triad through S.

Loop Quantum Gravity (LQG):

Curves/surfaces intersect $\Rightarrow$ holonomies/fluxes $\Rightarrow$ closed Poisson algebra.

Loop Quantum Cosmology (LQC):

LQC: **reduce** (A, E) to (c, p) **first** $\Rightarrow$ quantize reduced algebra.

A. Ashtekar, M. Bojowald and J. Lewandowski, Adv. Theor. Math. Phys. **7**, 233 (2003).

Homogeneity and isotropy: $(A, E) \Rightarrow (c, p)$

Flat FLRW:

$$ds^2 = -N^2 dt^2 + a^2(t)\, {}^o q_{ab} dx^a dx^b.$$

$\Rightarrow$ Define fiducial cell $\mathcal{V}$, co-triad ${}^o\omega_a{}^i$, its dual ${}^o e_i{}^a$ and volume $V_0 \equiv \int_{\mathcal{V}} d^3x \sqrt{{}^o q}$.

$$\boxed{A_a{}^i = c\, V_0^{-1/3}\, {}^o\omega_a{}^i, \qquad E_i{}^a = p\, V_0^{-2/3} \sqrt{{}^o q}\, {}^o e_i{}^a.}$$

From $E_i{}^a = \sqrt{h}\, e_i{}^a$ and $A_a{}^i = \gamma K_a{}^i$:

$$\boxed{|p| = a^2 V_0^{2/3}} \quad , \quad \boxed{c = \gamma \frac{\dot{a}}{N} V_0^{1/3}}.$$

$$\boxed{\{c, p\} = \frac{\kappa\gamma}{3}}.$$

A. Ashtekar, T. Pawlowski and P. Singh, Phys. Rev. D **74**, 084003 (2006), arXiv:gr-qc/0607039.

Classical FLRW dynamics in (c, p)

Gravitational and matter Hamiltonians:

$$\boxed{\mathcal{H}^{\rm GR}_{\rm grav} = -\frac{3N}{\kappa\gamma^2}c^2\sqrt{|p|}}, \qquad \boxed{\mathcal{H}_{\rm matter} = N|p|^{3/2}\rho}.$$

Total constraint:

$$\boxed{\mathcal{H}^{\rm GR} = N\left[-\frac{3}{\kappa\gamma^2}c^2\sqrt{|p|} + |p|^{3/2}\rho\right] \approx 0.}$$

Using

$$|p| = a^2 V_0^{2/3}, \qquad c = \gamma\frac{\dot{a}}{N}V_0^{1/3},$$

the constraint gives

$$\boxed{H^2 \equiv \left(\frac{\dot{a}}{Na}\right)^2 = \frac{\kappa}{3}\rho}.$$

$$\boxed{p \Rightarrow a^2, \qquad c \Rightarrow \dot{a}, \qquad (c,p) \text{ contains the FLRW phase space.}}$$

Loop Quantum Cosmology

Holonomy

Write: $A_a = A_a{}^i \tau_i \Rightarrow \tau_i$ $SU(2)$ generators.

Along a curve $x^a = \ell^a(s) \Rightarrow A_s = A_a(\ell(s)) \frac{d\ell^a}{ds}$.

$h(s) \Rightarrow$ accumulated transport from 0 to s:

$$\boxed{h(s+ds) = (\mathbb{I} + A_s ds)\, h(s)} \Rightarrow \boxed{\frac{dh}{ds} = A_s h} \Rightarrow \boxed{h_\ell[A] = \mathcal{P} \exp\left(\int_\ell A_a \, dx^a\right)}$$

Straight FLRW edge of fiducial length $\boldsymbol{\mu} V_0^{1/3}$ in direction i,

$$\boxed{h_i^{(\mu)} = \exp(\mu c \tau_i)} = \cos\left(\frac{\mu c}{2}\right) \mathbb{I} + 2 \sin\left(\frac{\mu c}{2}\right) \tau_i.$$

Curvature from a loop

Curvature of connection:

$$\boxed{F_{ab} = \partial_a A_b - \partial_b A_a + [A_a, A_b]}.$$

Closed loop measures failure to return unchanged:

$$\boxed{h_{\square_{ij}} = h_i^{(\mu)} h_j^{(\mu)} \left(h_i^{(\mu)}\right)^{-1} \left(h_j^{(\mu)}\right)^{-1}}.$$

For a small square,

$$h_{\square_{ij}} = \mathbb{I} + \mu^2 F_{ij} + O(\mu^3).$$

Therefore,

$$\boxed{F_{ij} = \lim_{\mu \to 0} \frac{h_{\square_{ij}} - \mathbb{I}}{\mu^2}}.$$

For flat FLRW:

$$F_{ab}{}^{k} = \epsilon^{k}{}_{ij} A_a{}^{i} A_b{}^{j} \qquad \Rightarrow \qquad \boxed{F \propto c^2}.$$

Area gap and improved dynamics

Loop $\Rightarrow$ surrounds surface S.

Area $\Rightarrow$ densitized triad $E_i{}^a$ flux through S.

Quantized fluxes $\Rightarrow$ $SU(2)$ angular-momentum generators:

$$\widehat{E}_i(S) = 8\pi\gamma\ell_{\rm Pl}^2\widehat{J}_i, \qquad \widehat{\boldsymbol{J}}^2|j,m\rangle = j(j+1)|j,m\rangle.$$

Elementary contribution:

$$\boxed{\Delta = A_{1/2} = 4\sqrt{3}\,\pi\gamma\ell_{\rm Pl}^2.}$$

Physical area:

$$L_{\rm phys} = a\mu V_0^{1/3} = \mu\sqrt{|p|} \quad \Rightarrow \quad \boxed{A_{\rm phys} = \mu^2|p|}.$$

Loop holonomies:

$$-2\,{\rm Tr}\left[\tau_k(h_{\Box_{ij}} - \mathbb{I})\right] \propto \sin^2(\bar{\mu}c).$$

Thus

$$\boxed{\text{FLRW: } F \propto c^2} \quad \Rightarrow \quad \boxed{c^2 \longrightarrow \frac{\sin^2(\bar{\mu}c)}{\bar{\mu}^2}}, \qquad \bar{\mu}c \ll 1 \Rightarrow \frac{\sin(\bar{\mu}c)}{\bar{\mu}} \simeq c.$$

A. Ashtekar, T. Pawlowski and P. Singh, Phys. Rev. D **74**, 084003 (2006); A. Corichi and P. Singh, Phys. Rev. D

Polymer representation: finite shifts of geometry

Holonomies $\Rightarrow$ define basis states $|\mu\rangle$ $\Rightarrow$ shifts geometric label:

$$\boxed{N_\mu(c) \equiv e^{i\mu c/2}} \quad , \quad \boxed{\widehat{N}_\alpha|\mu\rangle = |\mu+\alpha\rangle.}$$

New variables: volume v and conjugate b.

$$\boxed{\nu \propto v \equiv |p|^{3/2}, \qquad b \equiv \frac{c}{\sqrt{|p|}}, \qquad \bar{\mu}c = \lambda b.}$$

Holonomy $\Rightarrow$ shift in volume:

$$\boxed{\widehat{e^{i\lambda b}}|\nu\rangle = |\nu + \Delta\nu\rangle}$$

Discrete volume steps $\Rightarrow$ quantum constraint $\Rightarrow$ difference equation.

A. Ashtekar and P. Singh, Class. Quantum Grav. **28**, 213001 (2011), Secs. II.D–II.E.

Quantum Hamiltonian constraint in LQC

$$\boxed{\mathcal{H}^{\rm LQC}_{\rm grav} + N\frac{p_\phi^2}{2v} \approx 0.}$$

Quantizing: $p_\phi \Rightarrow -i\hbar\partial_\phi$

$$\boxed{\partial_\phi^2 \Psi(\nu,\phi) = -\Theta_{\rm LQC}\Psi(\nu,\phi).}$$

From LQC:

$$\boxed{\mathcal{H}^{\rm GR}_{\rm grav} \propto c^2\sqrt{|p|}} \quad \Rightarrow \quad \boxed{\mathcal{H}^{\rm LQC}_{\rm grav} \propto \sqrt{|p|}\,\frac{\sin^2(\bar{\mu}c)}{\bar{\mu}^2}.}$$

$e^{\pm i\lambda b} \Rightarrow$ shift volume label:

$$\boxed{(\Theta_{\rm LQC}\Psi)(\nu) = C_+(\nu)\Psi(\nu+\Delta\nu) + C_0(\nu)\Psi(\nu) + C_-(\nu)\Psi(\nu-\Delta\nu).}$$

WDW: differential Θ

LQC: difference Θ

A. Ashtekar, T. Pawlowski and P. Singh, Phys. Rev. D **74**, 084003 (2006), arXiv:gr-qc/0607039.

From quantum dynamics to an effective Hamiltonian

Sharply peaked states $\Rightarrow$ $\begin{cases} \text{follow classical trajectories at low curvature,} \\ \text{quantum-corrected effective trajectories at Planck-scale [1].} \end{cases}$

Expectation values and controlled approximations:

$$\boxed{\widehat{\mathcal{H}} \longrightarrow \mathcal{H}_{\rm eff}.}$$

Same structure $\Rightarrow$ quantum-corrected Hamiltonian ($N = 1$):

$$\boxed{\dot{p} = \{p, \mathcal{H}_{\rm eff}\} = -\frac{\kappa\gamma}{3}\frac{\partial \mathcal{H}_{\rm eff}}{\partial c}, \quad , \quad \dot{c} = \{c, \mathcal{H}_{\rm eff}\} = \frac{\kappa\gamma}{3}\frac{\partial \mathcal{H}_{\rm eff}}{\partial p}.}$$

LQC, mLQC-I and mLQC-II $\Rightarrow$ different effective Hamiltonians, but same procedure to obtain their equations of motion [2].

[1] V. Taveras, Phys. Rev. D **78**, 064072 (2008). A. Ashtekar and P. Singh, Class. Quantum Grav. **28**, 213001 (2011), Sec. V.

[2] B.-F. Li, P. Singh and A. Wang, Phys. Rev. D **98**, 066016 (2018).

Effective LQC dynamics

Again

$$\boxed{b \equiv \frac{c}{\sqrt{|p|}}, \qquad v \equiv |p|^{3/2}, \qquad \bar{\mu} c = \lambda b}.$$

Effective constraint:

$$\boxed{\mathcal{H}_{\rm eff}^{\rm LQC} = -\frac{3v}{\kappa\gamma^2\lambda^2}\sin^2(\lambda b) + v\rho \approx 0}.$$

Critical energy density:

$$\boxed{\sin^2(\lambda b) = \frac{\rho}{\frac{3}{\kappa\gamma^2\lambda^2}}} \Rightarrow \boxed{\rho_{\rm c} \equiv \frac{3}{\kappa\gamma^2\lambda^2}}.$$

Hubble parameter:

$$\boxed{H = \frac{\dot{v}}{3v} = \frac{\sin(\lambda b)\cos(\lambda b)}{\gamma\lambda}.}.$$

Using $\mathcal{H}_{\rm eff} \approx 0$:

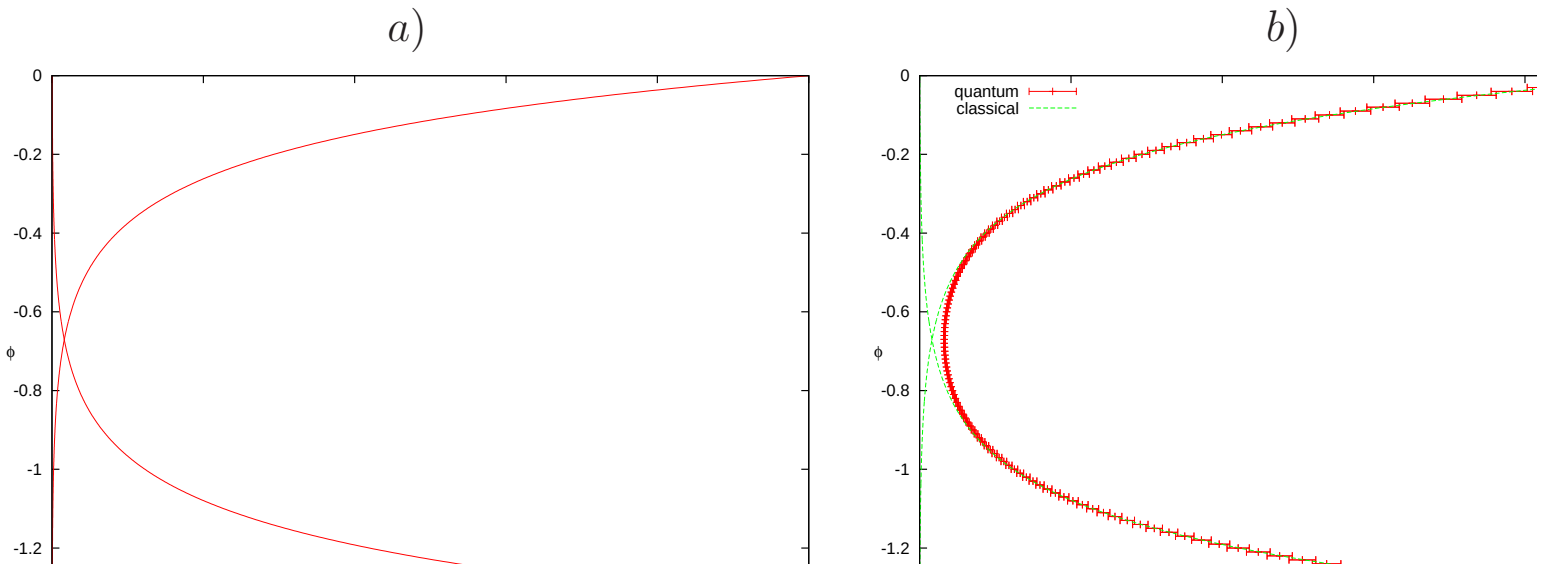
$$\boxed{H^2 = \frac{\kappa\rho}{3}\left(1 - \frac{\rho}{\rho_{\rm c}}\right)}.$$

$$\rho \ll \rho_{\rm c} \Rightarrow H^2 \simeq \frac{\kappa\rho}{3}, \qquad \rho = \rho_{\rm c} \Rightarrow H = 0.$$

V. Taveras, Phys. Rev. D **78**, 064072 (2008).

Bounce

$$\boxed{\text{contraction} \Rightarrow \text{bounce} \Rightarrow \text{expansion}}.$$



Source: A. Ashtekar and P. Singh, Class. Quantum Grav. **28**, 213001 (2011), Fig. 2.

WDW: expanding and contracting sectors remain separate.

LQC: the difference equation evolves a semiclassical packet through a nonzero minimum volume and connects the two branches.

Pre-inflationary LQC background

Consider a scalar field ϕ with $V(\phi)$.

For a kinetic-dominated bounce,

$$\frac{1}{2}\dot{\phi}_{\rm B}^2 \gg V(\phi_{\rm B}), \qquad w_{\rm B} \simeq 1.$$

Near the bounce,

$$\boxed{a(t) = a_{\rm B}\left[1 + 3\kappa\rho_{\rm c}t^2\right]^{1/6}}.$$

The background sequence is

$$\boxed{\text{contraction} \Rightarrow \text{bounce} \Rightarrow \text{short super-inflation} \Rightarrow \text{transition} \Rightarrow \text{slow-roll inflation}}.$$

This will be our focus.

V. Taveras, Phys. Rev. D **78**, 064072 (2008). A. Ashtekar and D. Sloan, Gen. Relativ. Gravit. **43**, 3619 (2011).

Alternative isotropic scenarios in LQC

Matter bounce [1]: $w \simeq 0$: dust contraction⇒LQC bounce⇒expansion

Nearly scale-invariant spectrum ⇒ generated during contraction ⇒ slow-roll inflation is not required.

Excessive scalar power/ amplifies anisotropies/challange to satisfy tensor and non-Gaussianity bounds.

Ekpyrotic evolution [2,3]: $w \gg 1$: ekpyrotic contraction⇒LQC bounce⇒expansion

Negative potential driven contraction

Single ekpyrotic field does not generally produce the required curvature spectrum; entropy-to-curvature conversion is often invoked [3].

Cyclic evolution [2]:

expansion⇒recollapse⇒contraction⇒LQC bounce⇒new expansion

LQC supplies the bounce, but recollapse, scalar-field turnaround and repeated cycles depend on the potential and are not automatic [2].

[1] E. Wilson-Ewing, JCAP **03**, 026 (2013), arXiv:1211.6269.
[2] P. Singh, K. Vandersloot and G. V. Vereshchagin, Phys. Rev. D **74**, 043510 (2006), arXiv:gr-qc/0606032.
[3] J.-L. Lehners, P. McFadden, N. Turok and P. J. Steinhardt, Phys. Rev. D **76**, 103501 (2007), arXiv:hep-th/0702153.

Modified Loop Quantum Cosmologies

Euclidean and Lorentzian parts of the constraint

In $(A_a{}^i, E_i{}^a)$, the gravitational Hamiltonian is

$$\boxed{\mathcal{H}_{\rm grav}[N] = \mathcal{H}^E_{\rm grav}[N] - (1+\gamma^2)\mathcal{H}^L_{\rm grav}[N]}.$$

$$\boxed{\mathcal{H}^E_{\rm grav}[N] = \frac{1}{2\kappa}\int d^3x\, N \frac{E_i{}^a E_j{}^b}{\sqrt{q}} \epsilon^{ij}{}_k F_{ab}{}^k}$$

$$\boxed{\mathcal{H}^L_{\rm grav}[N] = \frac{1}{\kappa}\int d^3x\, N \frac{E_i{}^a E_j{}^b}{\sqrt{q}} K_{[a}{}^i K_{b]}{}^j}.$$

$\mathcal{H}^E_{\rm grav}$: curvature $F[A]$,

$\mathcal{H}^L_{\rm grav}$: extrinsic curvature K.

For flat FLRW, $\Gamma_a{}^i = 0$ and $A_a{}^i = \gamma K_a{}^i$:

$$\boxed{\mathcal{H}^L_{\rm grav} = \gamma^{-2}\mathcal{H}^E_{\rm grav}}.$$

Standard LQC $\Rightarrow$ uses this identity *before* quantization $\Rightarrow$ quantizes resulting expression.

T. Thiemann, Class. Quant. Grav. **15**, 839 (1998).

Quantization ambiguity: LQC, mLQC-I and mLQC-II

All three prescriptions start from

$$\mathcal{H}_{\rm grav} = \mathcal{H}^E_{\rm grav} - (1+\gamma^2)\mathcal{H}^L_{\rm grav}.$$

rewrite the classical constraint $\Rightarrow$ regularize with holonomies $\Rightarrow$ promote to an operator.

	LQC	**mLQC-I**	**mLQC-II**
$\mathcal{H}^L$ and $\mathcal{H}^E$ before quantization	$\mathcal{H}^L = \gamma^{-2}\mathcal{H}^E$	kept separate	kept separate
Representation of ${K_a}^i$	eliminated before quantization	Thiemann identity	${K_a}^i = {A_a}^i/\gamma$

different quantization prescriptions $\Rightarrow$ different quantum dynamics.

A. Dapor and K. Liegener, Phys. Lett. B **785**, 506 (2018), arXiv:1706.09833; J. Yang, Y. Ding and Y. Ma, Phys. Lett. B **682**, 1 (2009), arXiv:0904.4379; B.-F. Li, P. Singh and A. Wang, Phys. Rev. D **97**, 084029 (2018), arXiv:1801.07313.

mLQC-I: separate quantization of $\mathcal{H}^E$ and $\mathcal{H}^L$

From

$$\boxed{\mathcal{H}^{\rm I}_{\rm grav} = \mathcal{H}^{E}_{\rm grav} - (1+\gamma^2)\mathcal{H}^{L}_{\rm grav}}$$

Lorentzian term contains ${K_a}^i \Rightarrow$ rewrite with Thiemann identity:

$$\boxed{{K_a}^i = \frac{1}{\kappa\gamma^3}\left\{{A_a}^i, \{\mathcal{H}^{E}_{\rm grav}, V\}\right\}.}$$

Quantize:

$$\boxed{A \Rightarrow \text{holonomies}, \qquad V \Rightarrow \widehat{V}, \qquad \{\ ,\ \} \Rightarrow \frac{1}{i\hbar}[\ ,\ ].}$$

$\widehat{\mathcal{H}}^{E}_{\rm grav} - (1+\gamma^2)\widehat{\mathcal{H}}^{L}_{\rm grav} \Rightarrow$ difference equation with a larger shift range than LQC.

Two branches arise.

T. Thiemann, Class. Quant. Grav. **15**, 839 (1998).

M. Assanioussi et al., Phys. Rev. Lett. **121**, 081303 (2018).

mLQC-I: energy density and two branches

$$\boxed{\mathcal{H}_{\rm mLQC-I} = \frac{3v}{8\pi G\lambda^2}\left[\sin^2(\lambda b) - \frac{(\gamma^2+1)\sin^2(2\lambda b)}{4\gamma^2}\right] + \mathcal{H}_M \approx 0.} \quad , \quad \boxed{\rho \equiv \frac{\mathcal{H}_M}{v},}$$

Solving constraint for b:

$$\boxed{\sin^2(\lambda b_\pm) = \frac{1 \pm \sqrt{1-\rho/\rho_{\rm c}^{\rm I}}}{2(\gamma^2+1)}} \quad , \quad \boxed{\rho_{\rm c}^{\rm I} = \frac{\rho_c}{4(1+\gamma^2)}.}$$

For $0 < \rho < \rho_{\rm c}^{\rm I} \Rightarrow$ $\boxed{b = b_- \qquad \text{or} \qquad b = b_+.}$

$$\sin^2(\lambda b_-) = \begin{cases} 0, & \rho = 0, \\ \dfrac{1}{2(\gamma^2+1)}, & \rho = \rho_{\rm c}^{\rm I}, \end{cases} \quad , \quad \sin^2(\lambda b_+) = \begin{cases} \dfrac{\mathbf{1}}{\boldsymbol{\gamma^2+1}}, & \rho = 0, \\ \dfrac{1}{2(\gamma^2+1)}, & \rho = \rho_{\rm c}^{\rm I}. \end{cases}$$

B.-F. Li, P. Singh and A. Wang, Phys. Rev. D **97**, 084029 (2018), arXiv:1801.07313.

mLQC-I: Friedmann equations for the two branches

Branch $b = b_-$:

$$\boxed{H^2 = \frac{8\pi G\rho}{3}\left(1-\frac{\rho}{\rho_{\rm c}^{\rm I}}\right)\left[1+\frac{\gamma^2}{\gamma^2+1}\left(\frac{\sqrt{\rho/\rho_{\rm c}^{\rm I}}}{1+\sqrt{1-\rho/\rho_{\rm c}^{\rm I}}}\right)^2\right].}$$

$$\rho \ll \rho_{\rm c}^{\rm I} \quad \Rightarrow \quad \boxed{H^2 \simeq \frac{8\pi G}{3}\rho} \qquad \text{(classical GR)}.$$

Branch $b = b_+$:

$$\boxed{H^2 = \frac{8\pi G\alpha\rho_\Lambda}{3}\left(1-\frac{\rho}{\rho_{\rm c}^{\rm I}}\right)\left[1+\left(\frac{1-2\gamma^2+\sqrt{1-\rho/\rho_{\rm c}^{\rm I}}}{4\gamma^2\left(1+\sqrt{1-\rho/\rho_{\rm c}^{\rm I}}\right)}\right)\frac{\rho}{\rho_{\rm c}^{\rm I}}\right].}$$

where

$$\boxed{\alpha \equiv \frac{1-5\gamma^2}{1+\gamma^2}}\,, \qquad \boxed{\rho_\Lambda \equiv \frac{3}{8\pi G\alpha\lambda^2(1+\gamma^2)^2}}.$$

$$\rho \ll \rho_{\rm c}^{\rm I} \quad \Rightarrow \quad \boxed{H^2 \simeq \frac{8\pi G_\alpha}{3}\left(\rho+\rho_\Lambda\right), \qquad G_\alpha \equiv \alpha G} \qquad \text{(de Sitter branch)}.$$

B.-F. Li, P. Singh and A. Wang, Phys. Rev. D **97**, 084029 (2018), Eqs. (2.19), (2.26) and (2.30); arXiv:1801.07313.

mLQC-II: a different representation of $K_a{}^i$

From:

$$\mathcal{H}^{\rm II}_{\rm grav} = \mathcal{H}^{E}_{\rm grav} - (1+\gamma^2)\mathcal{H}^{L}_{\rm grav}.$$

Flat FLRW:

$$\Gamma_a{}^i = 0 \quad \Rightarrow \quad A_a{}^i = \gamma K_a{}^i \quad \Rightarrow \quad \boxed{K_a{}^i = \frac{A_a{}^i}{\gamma}}.$$

Use this identity inside the Lorentzian term before regularizing:

$$\boxed{\mathcal{H}^{E}[A,E],\ \mathcal{H}^{L}[\boldsymbol{A/\gamma},E] \Rightarrow \widehat{\mathcal{H}}^{\rm II}_{\rm grav}.}$$

The distinction from mLQC-I is precise:

$$\boxed{\begin{array}{lcl} \text{mLQC-I} & : & K \Rightarrow \{A,\{\mathcal{H}^{E},V\}\}, \\ \text{mLQC-II} & : & K \Rightarrow A/\gamma. \end{array}}$$

J. Yang, Y. Ding and Y. Ma, Phys. Lett. B **682**, 1 (2009).

mLQC-II: energy density and physical branch

$$\boxed{\mathcal{H} = -\frac{3v}{2\pi G\lambda^2\gamma^2}\sin^2\left(\frac{\lambda b}{2}\right)\left[1+\gamma^2\sin^2\left(\frac{\lambda b}{2}\right)\right] + \mathcal{H}_M \approx 0.} \quad , \quad \boxed{\rho \equiv \frac{\mathcal{H}_M}{v}.}$$

Solving the constraint for b:

$$\boxed{\sin^2\left(\frac{\lambda b_\pm}{2}\right) = \frac{-1 \pm \sqrt{1+\gamma^2\rho/\rho_c}}{2\gamma^2}} \quad , \quad \boxed{\rho_c^{\rm II} = 4(1+\gamma^2)\rho_c.}$$

For $\rho > 0$, b_- is not physical. For the physical root $\boxed{b = b_+}$:

$$\sin^2\left(\frac{\lambda b_+}{2}\right) = \begin{cases} 0, & \rho = 0, \\ 1, & \rho = \rho_c^{\rm II}. \end{cases}$$

$$\boxed{H^2 = \frac{8\pi G\rho}{3}\left(1+\gamma^2\frac{\rho}{\rho_c}\right)\left[1 - \frac{(1+\gamma^2)\rho/\rho_c}{\left(1+\sqrt{1+\gamma^2\rho/\rho_c}\right)^2}\right].}$$

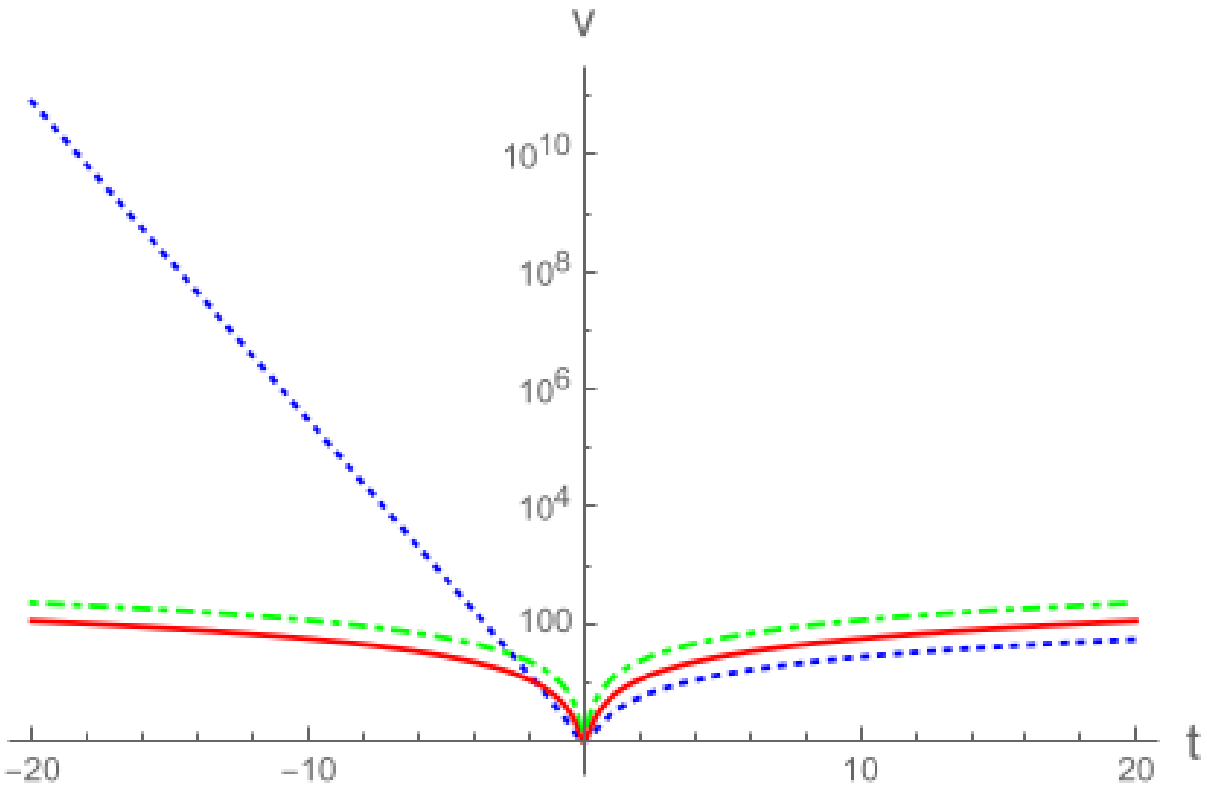
At low density,

$$\rho \ll \rho_c \quad \Rightarrow \quad \boxed{H^2 \simeq \frac{8\pi G}{3}\rho} \qquad \text{(classical GR)}.$$

$$\boxed{\text{GR contraction} \Rightarrow \text{mLQC-II bounce} \Rightarrow \text{GR expansion.}}$$

B.-F. Li, P. Singh and A. Wang, Phys. Rev. D **98**, 066016 (2018), Eqs. (2.5), (2.9), (2.10) and (2.14);

Background comparison



Physical volume $v(t)$ for a massless scalar field: LQC (solid), mLQC-I (dotted) and mLQC-II (dash-dotted). Source: B.-F. Li, P. Singh and A. Wang, Phys. Rev. D **98**, 066016 (2018), Fig. 2.

$$\rho_{\rm c}^{\rm I} < \rho_{\rm c}^{\rm LQC} < \rho_{\rm c}^{\rm II}$$

finite-density bounce is shared but Planck-regime dynamics is regularization dependent.

Perturbations and observations

Perturbations: the choice made in this lecture

Dressed metric [1] $\Rightarrow$ quantized homogeneous geometry + quantum perturbations propagating on its dressed metric.

LQC, mLQC-I, mLQC-II backgrounds + dressed-metric perturbations

Two other constructions:

hybrid [2] $\Rightarrow$ quantize truncated background–perturbation system,

deformed algebra [3] $\Rightarrow$ modify the effective constraint algebra.

Comparisons will be focused in dressed-metric approach, secondly in the hybrid approach.

[1] I. Agullo, A. Ashtekar and W. Nelson, Phys. Rev. Lett. **109**, 251301 (2012).
[2] M. Fernández-Méndez, G.A. Mena Marugán and J. Olmedo, Phys. Rev. D **86**, 024003 (2012).
[3] T. Cailleteau et al., Class. Quant. Grav. **29**, 095010 (2012).

Dressed metric approach

Background $\Rightarrow$ LQC state $\Psi_0(\nu, \phi)$.
Perturbations $\Rightarrow$ Fock state $\psi_{\rm pert}$ [1].

$$\boxed{\Psi \simeq \Psi_0 \otimes \psi_{\rm pert}}, \qquad \boxed{\langle \widehat{\rho}_{\rm pert} \rangle_{\rm ren} \ll \langle \widehat{\rho}_{\rm bg} \rangle}.$$

Neglect backreaction: assume perturbations do not significantly change background.
(This assumption can fail in dressed metric [2] and for hybrid [3].)

Expectation values in Ψ_0 define the geometry seen by the modes [1]:

$$d\widetilde{s}^2 = \widetilde{a}^2(-d\widetilde{\eta}^2 + d\boldsymbol{x}^2), \qquad \boxed{\widetilde{a}^4 = \frac{\langle \widehat{H}_0^{-1/2} \widehat{a}^4 \widehat{H}_0^{-1/2} \rangle}{\langle \widehat{H}_0^{-1} \rangle}}.$$

Sharply peaked approximation.

[1] I. Agullo, A. Ashtekar and W. Nelson, Phys. Rev. D **87**, 043507 (2013).
[2] L.L. Graef, R.O. Ramos and G.S. Vicente, Phys. Rev. D **102**, 043518 (2020), arXiv:2007.02395.
[3] G.S. Vicente, R.O. Ramos and L.L. Graef, Phys. Rev. D **106**, 043518 (2022), arXiv:2207.00435.

Dressed metric: scalar and tensor modes

Scalar and tensor modes [1]:

$$\boxed{\mu_k^{S\prime\prime} + \left(k^2 - \frac{a''}{a} + \mathcal{U}\right)\mu_k^S = 0}, \qquad \boxed{\mu_k^{T\prime\prime} + \left(k^2 - \frac{a''}{a}\right)\mu_k^T = 0}.$$

Scalar effective potential:

$$\boxed{\mathcal{U}(\phi) = a^2\left[f^2 V(\phi) + 2f V_{,\phi}(\phi) + V_{,\phi\phi}(\phi)\right]}, \qquad \boxed{f = \sqrt{24\pi G}\,\frac{\dot{\phi}}{\sqrt{\rho}}}.$$

LQC, mLQC-I and mLQC-II $\Rightarrow$ same mode equations, but different effective backgrounds [2].

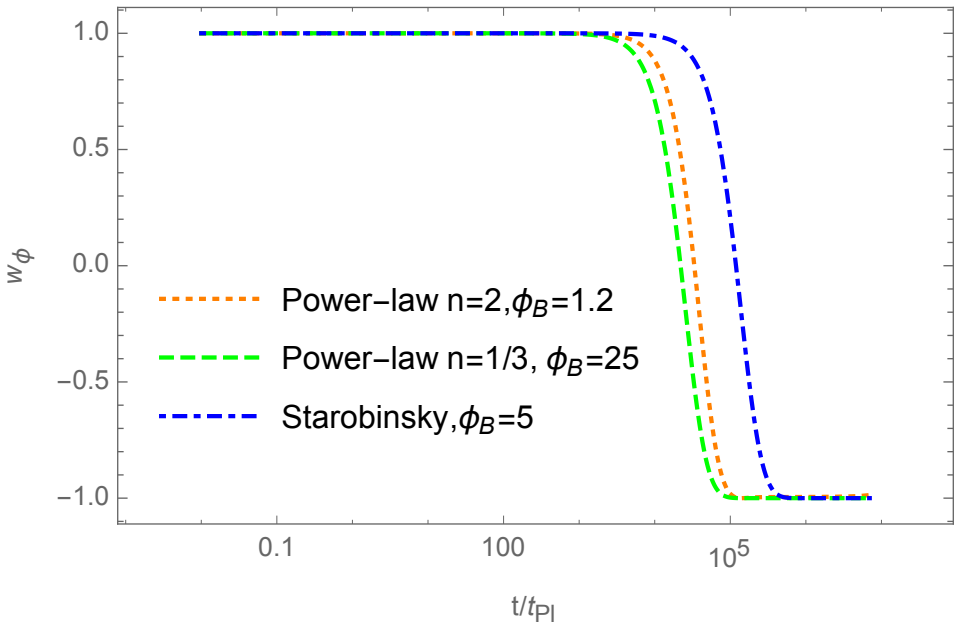
[1] I. Agullo, A. Ashtekar and W. Nelson, Phys. Rev. D **87**, 043507 (2013).

[2] B.-F. Li, P. Singh and A. Wang, Phys. Rev. D **101**, 086004 (2020), arXiv:1912.08225.

Pre-inflationary background

Kinetic domination at the bounce $\Rightarrow w_{\rm B} = P_{\rm B}/\rho_{\rm B} \simeq 1$ [1].

$$a(t) \simeq a_{\rm B}[1 + 3\kappa\rho_c t^2]^{1/6}.$$



Different potentials share early $w \simeq 1$ phase.

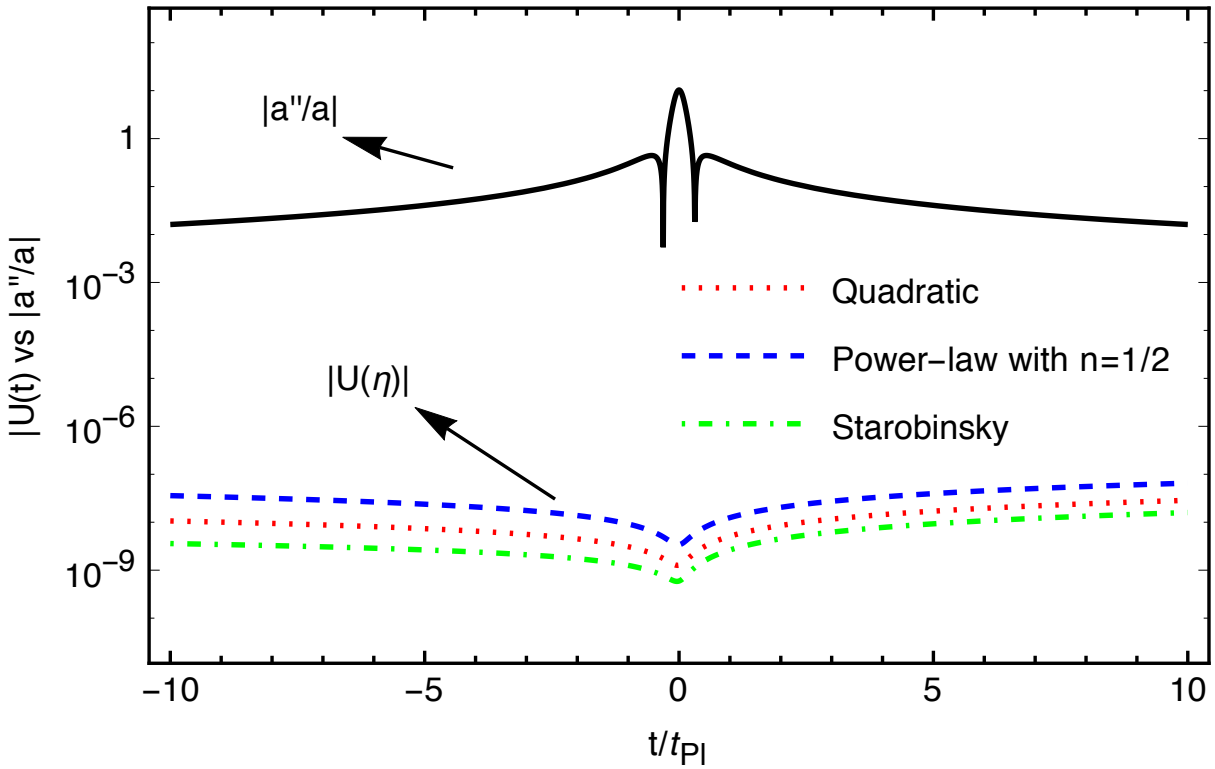
bounce $\Rightarrow$ short super-inflation $\Rightarrow$ transition $\Rightarrow$ slow roll

[1] T. Zhu, A. Wang, G. Cleaver, K. Kirsten and Q. Sheng, Phys. Rev. D **96**, 083520 (2017). Figure: same reference.

Bounce nearly universal

Kinetic-dominated bounce $\Rightarrow$ during the bounce phase [1]:

$$\boxed{|\mathcal{U}| \ll |a''/a|} \quad \Rightarrow \quad \boxed{\mu_k'' + \left(k^2 - \frac{a''}{a}\right)\mu_k \simeq 0}$$



Curvature dominates scalar potential.

[1] T. Zhu et al., arXiv:1705.07544 (2017). Figure: same article.

Initial perturbation states: prescriptions and models

1. **BD-like vacuum $\Rightarrow$ contracting phase.**
   LQC $\Rightarrow$ dressed metric [1].
2. **4th order WKB $\Rightarrow$ at the bounce.**
   LQC $\Rightarrow$ dressed metric [2].
   No remote-past vacuum required $\Rightarrow$ may fail for infrared modes.
3. **4th order WKB $\Rightarrow$ contracting phase.**
   LQC and mLQC-II $\Rightarrow$ dressed metric [3,4].
   LQC and mLQC's $\Rightarrow$ hybrid [4].
   Can modify spectrum at small and intermediate k.
4. **De Sitter state $\Rightarrow$ mLQC-I contracting phase.** mLQC-I $\Rightarrow$ dressed metric [3,4].

$$\mu_k'' + \left(k^2 - \frac{2}{\eta^2}\right)\mu_k \simeq 0 \qquad \Rightarrow \qquad \boxed{\mu_k^{\rm dS}(\eta) = \frac{e^{-ik\eta}}{\sqrt{2k}}\left(1 - \frac{i}{k\eta}\right)}.$$

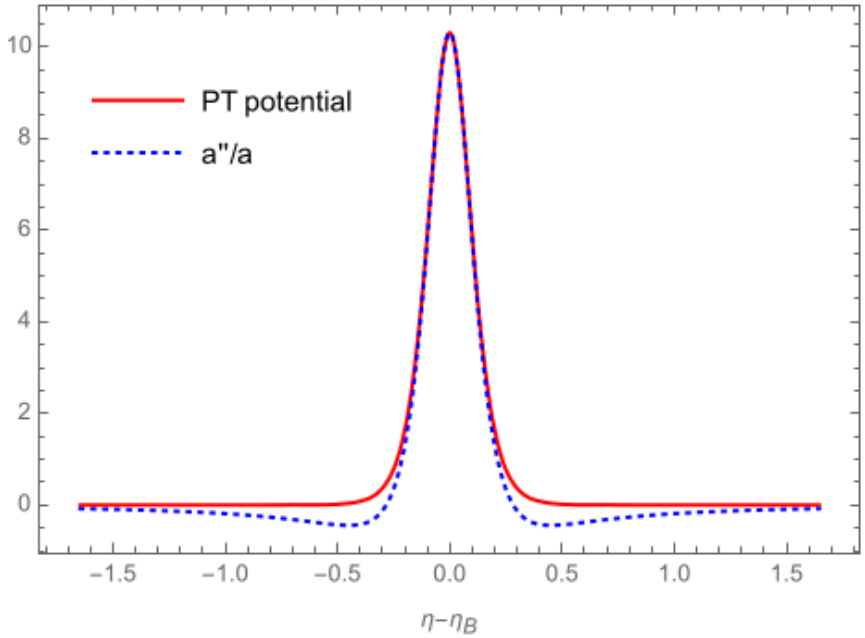
[1] T. Zhu et al., Phys. Lett. B **773**, 196 (2017), arXiv:1607.06329.
[2] I. Agullo, A. Ashtekar and W. Nelson, Class. Quant. Grav. **30**, 085014 (2013).
[3] B.-F. Li, P. Singh and A. Wang, Phys. Rev. D **101**, 086004 (2020).
[4] B.-F. Li, M. Motaharfar and P. Singh, Phys. Rev. D **110**, 066005 (2024).

Bounce evolution and primordial spectra

Near the bounce $\Rightarrow$ Pöschl–Teller approximation [6]:

$$\boxed{\frac{a''}{a} \simeq \frac{\mathcal{V}_0}{\cosh^2[\alpha(\eta-\eta_{\rm B})]}},$$

α and $\mathcal{V}_0$ to be matched with full $\frac{a''}{a}$.



After the bounce, in the low-curvature transition:

$$\boxed{\mu_k(\eta) \simeq \frac{1}{\sqrt{2k}}\left(\alpha_k e^{-ik\eta} + \beta_k e^{ik\eta}\right)}, \qquad |\alpha_k|^2 - |\beta_k|^2 = 1.$$

Non Bunch-Davies condition for inflation.
Primordial power spectra:

$$\boxed{\mathcal{P}_{\mathcal{R}}^{\rm LQC} = |\alpha_k + \beta_k|^2 \mathcal{P}_{\mathcal{R}}^{\rm GR}}, \qquad \boxed{\mathcal{P}_{h}^{\rm LQC} = |\alpha_k + \beta_k|^2 \mathcal{P}_{h}^{\rm GR}}.$$

[6] T. Zhu et al., arXiv:1705.07544 (2017), Secs. IV–V (PT figure).

The analytic correction to the spectrum

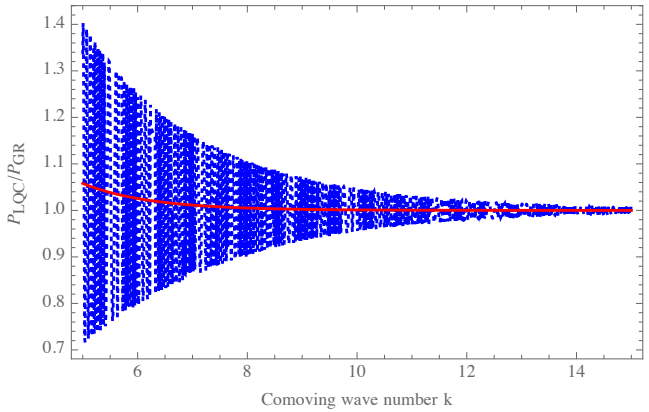
Kinetic bounce + contracting positive-frequency state + PT approximation [6]:

$$\boxed{|\alpha_k + \beta_k|^2 = 1 + 2|\beta_k|^2 + 2\,\mathrm{Re}(\alpha_k \beta_k^*)}.$$

$$\boxed{\delta_{\mathrm{Pl}}(k) = 2|\beta_k|^2 = \left[1 + \cos\left(\frac{\pi}{\sqrt{3}}\right)\right] \mathrm{csch}^2\left(\frac{\pi k}{\sqrt{6}\, k_{\mathrm{B}}}\right)}.$$

Averaging the rapid oscillations:

$$\boxed{\mathcal{P}_{\mathcal{R}}^{\mathrm{LQC}}(k) \simeq [1 + \delta_{\mathrm{Pl}}(k)]\mathcal{P}_{\mathcal{R}}^{\mathrm{GR}}(k)}\,, \quad \boxed{\mathcal{P}_{h}^{\mathrm{LQC}}(k) \simeq [1 + \delta_{\mathrm{Pl}}(k)]\mathcal{P}_{h}^{\mathrm{GR}}(k)}\,, \quad \boxed{r \equiv \frac{\mathcal{P}_{h}^{\mathrm{LQC}}}{\mathcal{P}_{\mathcal{R}}^{\mathrm{LQC}}} = r_{\mathrm{GR}}}.$$



Small $k \lesssim k_{\mathrm{B}}$ $\Rightarrow$ strongest bounce effects; large $k \gg k_{\mathrm{B}}$ $\Rightarrow$ standard slow-roll spectrum.

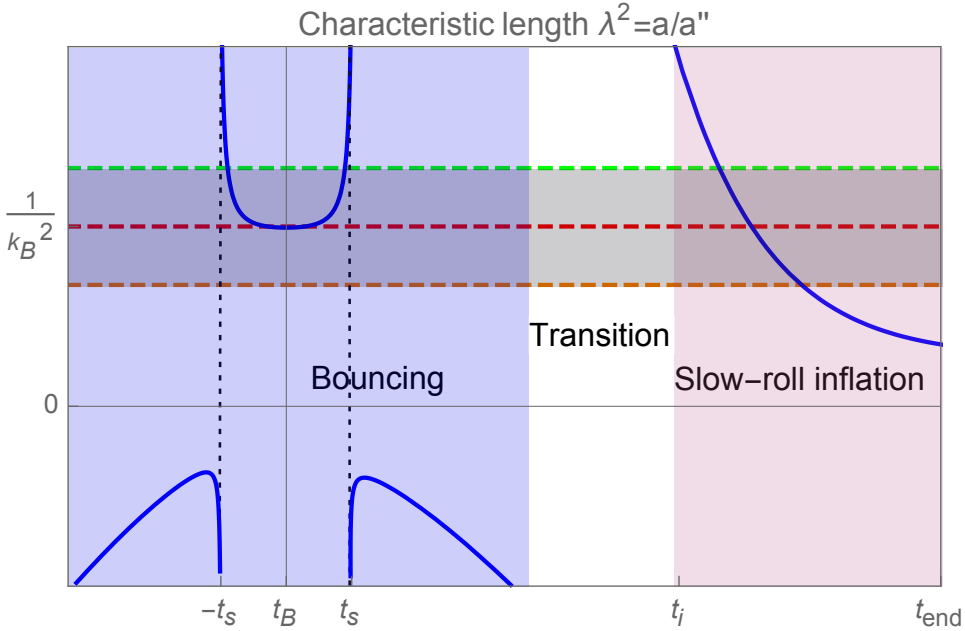
[6] T. Zhu et al., arXiv:1705.07544 (2017), Eqs. (5.17), (6.1)–(6.3). Source of the analytic correction and figure.

Which modes feel the bounce?

Mode equation $\Rightarrow$ compares k^2 with a''/a [1] $\Rightarrow$ kinetic-dominated bounce:

$$\boxed{k_{\rm B} \equiv \sqrt{\frac{a''}{a}}\bigg|_{\rm B} \sqrt{\frac{\gamma_{\rm B}}{3}}\, a_{\rm B} m_{\rm Pl}} \quad , \quad \boxed{\gamma_{\rm B} \equiv \frac{24\pi\rho_c}{m_{\rm Pl}^4} \simeq 30.9}.$$

Often denoted as $k_{\rm LQC}$.



Blue curve: a/a''.
Horizontal lines: $1/k^2$, fixed for a comoving mode [1].

$$k^2 \gg |a''/a| \Rightarrow \mu_k \sim e^{\pm ik\eta}.$$

$$k^2 \lesssim a''/a$$

$\boxed{k \sim k_{\rm B} \Rightarrow \text{potentially observable excitation}}$.

[1] T. Zhu et al., arXiv:1705.07544 (2017). Figure: same article.

Observable window for the bounce scale

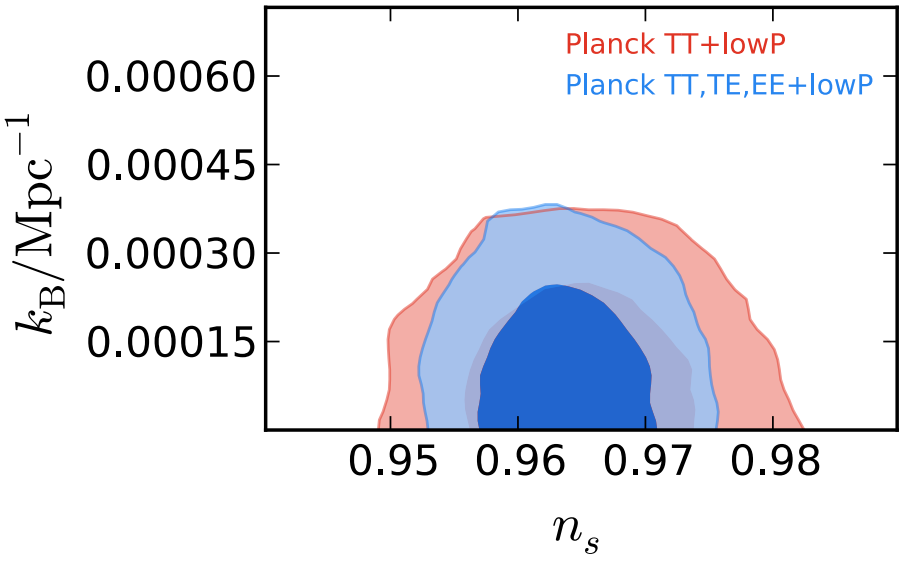
$$\boxed{\frac{k_{\rm B}}{a_0} = \sqrt{\frac{\gamma_{\rm B}}{3}}\frac{a_{\rm B}}{a_0}m_{\rm Pl} = \sqrt{\frac{\gamma_{\rm B}}{3}}m_{\rm Pl}e^{-N_{\rm tot}}}, \qquad N_{\rm tot} \equiv \ln\frac{a_0}{a_{\rm B}}.$$

Observations $\Rightarrow$ constrain $k_{\rm B}/a_0$.

Let $[k_{\rm min}, k_{\rm max}]$ $\Rightarrow$ prior range $\Rightarrow$ compare it with $k_{\rm B}$ [1]:

[1] I. Agullo, A. Ashtekar and W. Nelson, Class. Quant. Grav. **30**, 085014 (2013), arXiv:1302.0254.

Upper limit on k_B/a_0



Published analysis: Planck 2015 [6].

$$\frac{k_{\rm B}}{a_0} < 3.05 \times 10^{-4}\ {\rm Mpc}^{-1}$$

at 95% C.L., using TT,TE,EE+lowP in the scalar fit.
Contours: 68% and 95% allowed regions in the $(n_s, k_{\rm B}/a_0)$ plane.

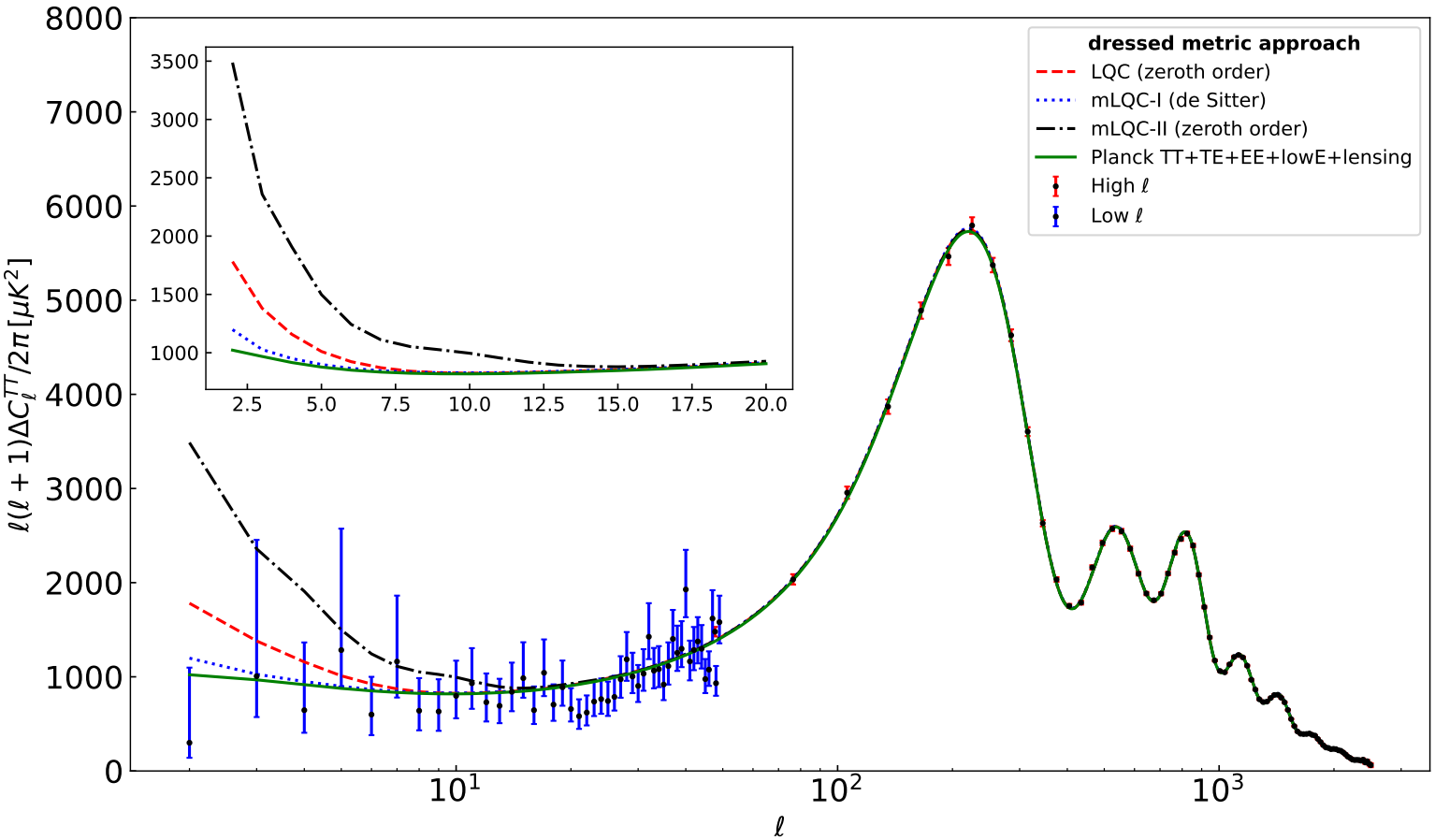
upper limit on the bounce scale, not a detection

For $\rho_c \simeq 0.41 m_{\rm Pl}^4$, this gives $N_{\rm tot} > 141$.

These are model-dependent Planck 2015 bounds, not universal or updated limits.

[6] T. Zhu et al., arXiv:1705.07544 (2017), Table I and Sec. VI. Source of the Planck 2015 bound and likelihood figure.

Dressed-metric results: LQC versus mLQCs



High $\ell \Rightarrow$ curves converge.
Low $\ell \Rightarrow$ mLQC-I "**closest**" to ΛCDM.
mLQC-II $\Rightarrow$ **largest excess.**

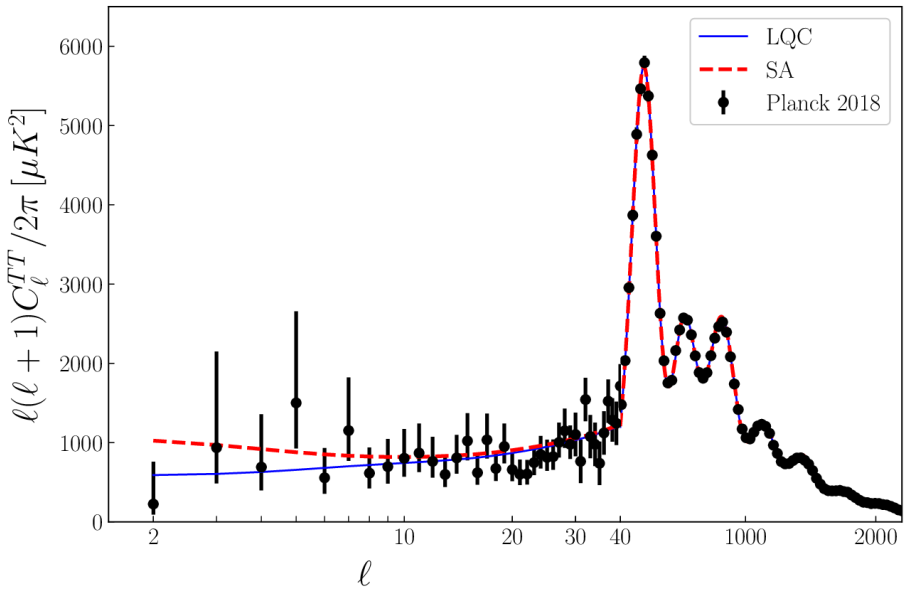
mLQC-II most disfavored in this comparison

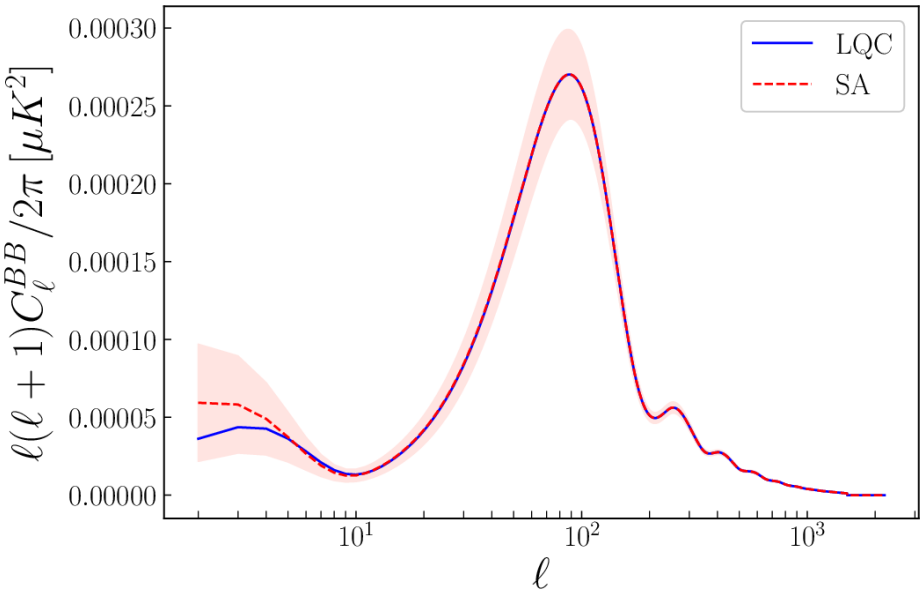
[1] B.-F. Li, M. Motaharfar and P. Singh, Phys. Rev. D **110**, 066005 (2024), Fig. 8 and Secs. IV–V.

Dressed metric: power suppression

A specific initial-state prescription [1]: **maximal quantum homogeneity and isotropy in the Planck regime + minimal field dispersion at the end of inflation.**

$$\boxed{\mathcal{P}_{\mathcal{R}}^{\mathrm{LQC}}(k) = f(k)\,\mathcal{P}_{\mathcal{R}}^{\mathrm{standard}}(k)}, \qquad f(k) < 1 \text{ at small } k, \quad f(k) \to 1 \text{ at large } k.$$





SA: inflation.

Alleviates anomalies for this prescription.

BB **curve is a prediction**, not a detection.

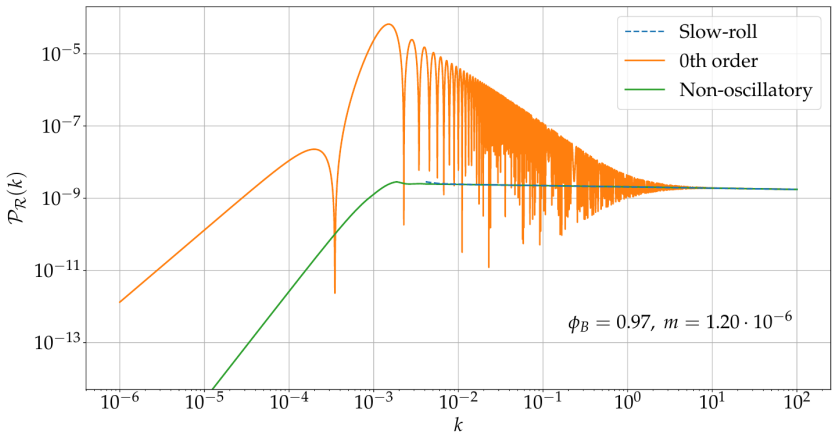
[1] A. Ashtekar, B. Gupt, D. Jeong and V. Sreenath, Phys. Rev. Lett. **125**, 051302 (2020), arXiv:2001.11689.
Figures 2 and 5.

Hybrid: suppression with a non-oscillating vacuum

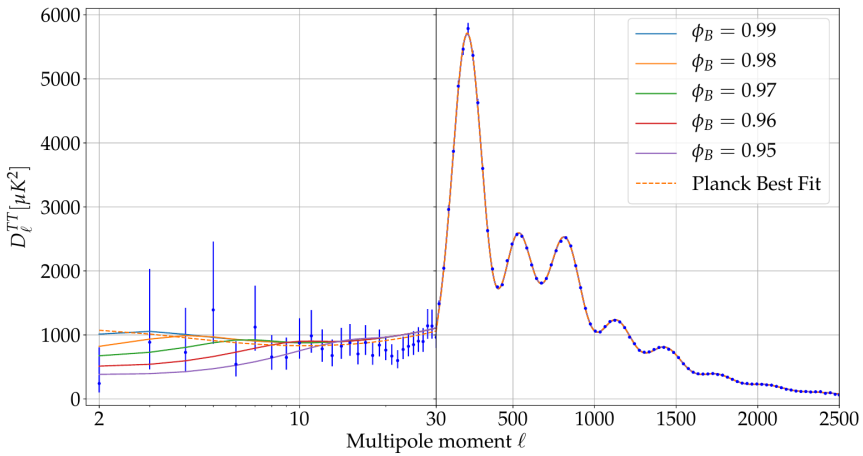
Non-oscillating (NO) vacuum $\Rightarrow$ selects modes by minimizing their amplitude variations during a pre-inflationary interval [1].

Minimize: $$I_k = \int_{\eta_i}^{\eta_f} \left| \frac{d|\mu_k(\eta)|^2}{d\eta} \right| d\eta.$$

Initial data at the bounce.



Green $\Rightarrow$ NO vacuum; orange $\Rightarrow$ 0th order adiabatic. Blue dashed $\Rightarrow$ slow-roll spectrum.



$\phi_\mathrm{B} \equiv \phi(t_\mathrm{B})$: value of the homogeneous inflaton field at the bounce.

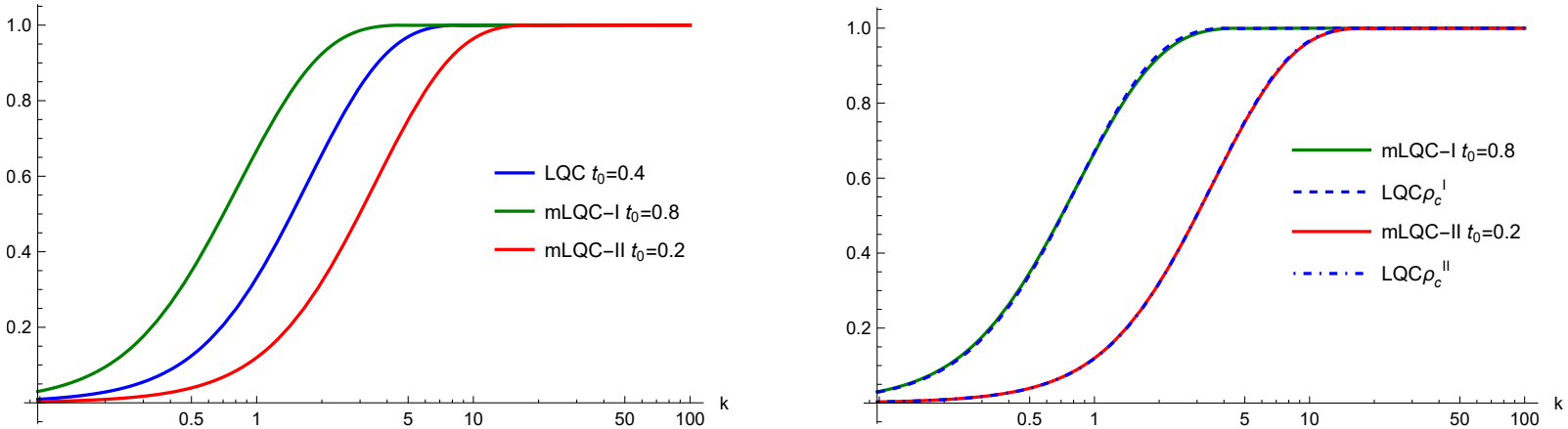
Suitable $\phi_\mathrm{B} \Rightarrow$ lowers the low-ℓ power while preserving the high-ℓ spectrum [1].

[1] B. Elizaga Navascués, D. Martín de Blas and G. A. Mena Marugán, Universe 4, 98 (2018), arXiv:1809.09874.

Figures 4 and 5.

Power suppression in LQC and mLQCs

Hybrid approach + same initial-state prescription $\Rightarrow$ compare LQC, mLQC-I and mLQC-II [1].
NO-AHD vacuum $\Rightarrow$ selects modes by asymptotically diagonalizing the perturbation Hamiltonian at large k [1].



Green: mLQC-I. Blue: LQC. Red: mLQC-II.
Normalized primordial power spectrum.

Small k $\Rightarrow$ all three suppress power.
Large k $\Rightarrow$ all three approach the same plateau.

Different critical densities (artificial varying) $\Rightarrow$ suppression ends at different values of k.

No Planck fit here.

[1] M. Kowalczyk, G. A. Mena Marugán and T. Pawłowski, Phys. Rev. D 111, 126013 (2025), arXiv:2503.16377.
Fig. 4.

Conclusions: spectra and observations

Large k $\Rightarrow$ the studied spectra recover the slow-roll regime.

Small k $\Rightarrow$ enhancement or suppression depends on the initial state.

Dressed metric $\Rightarrow$ suitable states suppress low-ℓ power and alleviate CMB anomalies [1].

Hybrid NO-AHD $\Rightarrow$ suppression in LQC and both mLQCs. The critical density mainly shifts its scale [2].

Recent NO-AHD results $\Rightarrow$ dressed and hybrid LQC: $N_{\rm tot} \sim 140$ gives high-ℓ agreement and possible low-ℓ improvement in the examples studied [3].

suppression is possible, but is not a universal LQC prediction

[1] A. Ashtekar et al., Phys. Rev. Lett. **125**, 051302 (2020).
[2] M. Kowalczyk, G. A. Mena Marugán and T. Pawłowski, Phys. Rev. D **111**, 126013 (2025).
[3] A. Guillén et al., arXiv:2605.14657 (2026).

Conclusions: status and open questions

No established observational detection of the LQC bounce

Initial state $\Rightarrow$ NO/NO-AHD are promising. Observations have not selected a unique vacuum [1].

Consistency $\Rightarrow$ check backreaction and trans-Planckian modes [1,2]. $\lambda_{\rm phys}(t_{\rm B}) < \ell_{\rm Pl}$.

Model comparison $\Rightarrow$ use explicit states and matched late-time cosmology. No universal Prescription is not universal [3].

Other scenarios $\Rightarrow$ anisotropic, matter-bounce, ekpyrotic and cyclic models need their own perturbation analysis [1].

better low-ℓ agreement $\not\Rightarrow$ evidence for a bounce

[1] I. Agullo, A. Wang and E. Wilson-Ewing, arXiv:2301.10215 (review).
[2] G. S. Vicente, R. O. Ramos and L. L. Graef, Phys. Rev. D **106**, 043518 (2022).
[3] B.-F. Li, M. Motaharfar and P. Singh, Phys. Rev. D **110**, 066005 (2024).

Thank you!