Cosmological probes of Gravity

Lecture 3: Testing the perturbative regimen of gravity with the Cosmic Microwave Background

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Outline

1. Theoretical Introduction
2. Cosmic Microwave Background
3. Constraints on MG models from CMB observations
4. Introduction to weak lensing
5. CMB lensing
6. Constraints from CMB and CMB lensing observations
7. Relevant papers for this lecture
8. Appendix: Weak Lensing derivations

The perturbative sector

- Let us now consider the FRW space-time with perturbations:

$$ds^2 = -(1+2\Psi_N)dt^2 + a^2(t)(1-2\Phi_N)\delta_{ij}dx^i dx^j \tag{1}$$

where Ψ_N and Φ_N are the Newtonian potential and the spatial curvature perturbation, respectively.

- In the early universe, the Boltzmann equations of all particle species (photons, neutrinos, baryons, dark matter) are coupled to the perturbed Einstein equations through Ψ_N and Φ_N.
- For baryons and CDM, the boltzmann equations are given by:

$$\dot{\delta}_b = -kv_b + 3\dot{\phi}_N, \quad \dot{v}_b = -\mathcal{H}v_b + k\psi_N + \frac{\dot{\tau}}{R}(v_b - \Theta_1), \tag{2}$$

$$\dot{\delta}_c = -kv_c + 3\dot{\phi}_N, \quad \dot{v}_c = -\mathcal{H}v_c + k\psi_N, \tag{3}$$

where $\mathcal{H} := aH$ and $R = \frac{\rho_b}{\rho_\gamma}$.

- These equations should be modified if the matter content non-minimally couples to gravity (Bettoni 2011, 2015).

The perturbative sector in MG

- In GR and in the absence of anisotropic stress, the two scalar potentials are equal: $\Phi_N = \Psi_N$. However, in modified gravity theories, this equality may not hold, leading to a gravitational slip between the two potentials : $\eta = \frac{\Phi_N}{\Psi_N}$.
- The newtonian potentials Φ_N and Ψ_N obey the modified Poisson equations:

$$\begin{aligned} k^2\Psi_N &= -4\pi G a^2(1+\mu(a,k))\rho\delta \\ k^2(\Phi_N+\Psi_N) &= -8\pi G a^2(1+\Sigma(a,k))\rho\delta \end{aligned} \tag{4}$$

- The functions $\mu(a,k)$ and $\Sigma(a,k)$ parametrize deviations from GR at the perturbative level.

Summary statistics

- Since $\langle X \rangle = 0$ for an overdensity field, the lowest-order non-trivial statistic is the two-point correlation function. In Fourier space:

$$\langle X(\mathbf{k_1}) Y(\mathbf{k_2}) \rangle = (2\pi)^3 \delta^3(\mathbf{k_1} + \mathbf{k_2}) P_{XY}(k_1) \tag{5}$$

which is related to the power spectrum, while in real space

$$\langle X(\mathbf{r_1}) Y(\mathbf{r_2}) \rangle = \xi_{XY}(|\mathbf{r_1} - \mathbf{r_2}|) = \int \frac{d^3k}{(2\pi)^3} P_{XY}(k)\, e^{i\mathbf{k}\cdot(\mathbf{r_1}-\mathbf{r_2})} \tag{6}$$

- The next statistic is the three-point correlation function, which measures the clustering of triplets of points. In Fourier space, it is related to the bispectrum:

$$\langle X(\mathbf{k_1}) Y(\mathbf{k_2}) Z(\mathbf{k_3}) \rangle = (2\pi)^3 \delta^3(\mathbf{k_1} + \mathbf{k_2} + \mathbf{k_3}) B_{XYZ}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \tag{7}$$

while in real space

$$\begin{aligned} \langle X(\mathbf{r_1}) Y(\mathbf{r_2}) Z(\mathbf{r_3}) \rangle &= \zeta_{XYZ}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) \\ &= \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} B_{XYZ}(\mathbf{k}_1, \mathbf{k}_2, -\mathbf{k}_1 - \mathbf{k}_2)\, e^{i(\mathbf{k_1}\cdot\mathbf{r_{13}} + \mathbf{k_2}\cdot\mathbf{r_{23}})} \end{aligned} \tag{8}$$

with $\mathbf{r}_{ij} \equiv \mathbf{r}_i - \mathbf{r}_j$.

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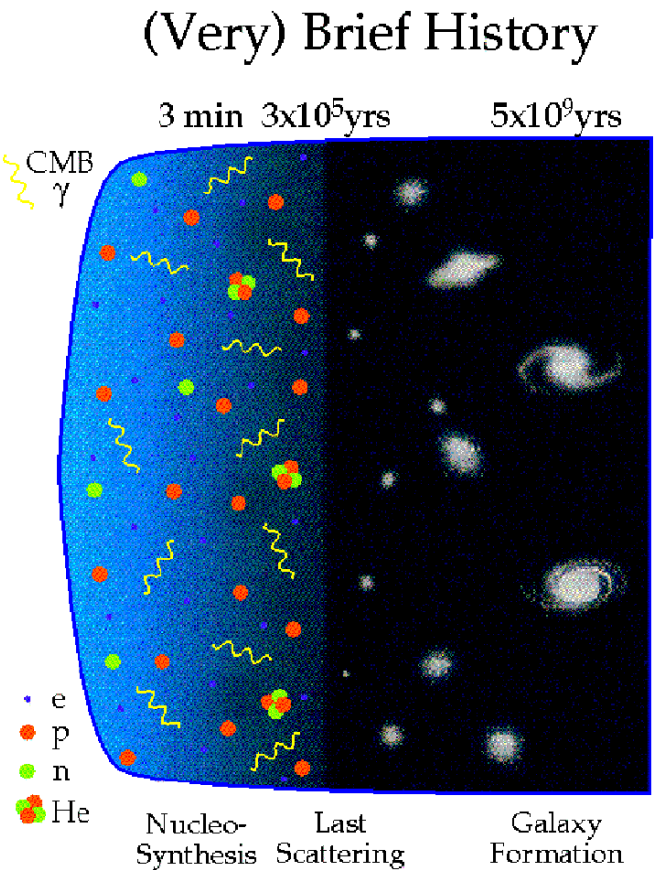
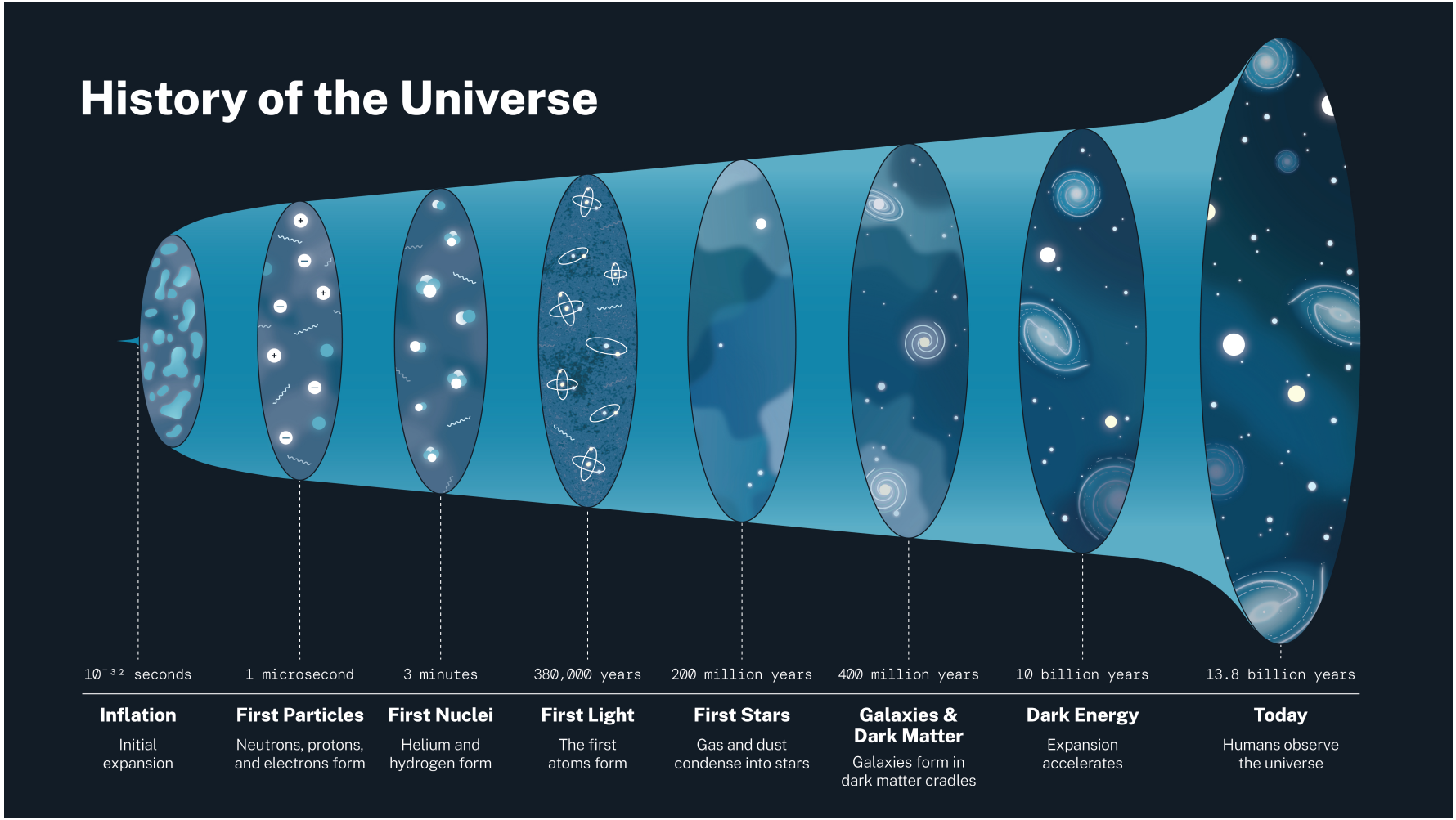
Formation of neutral hydrogen and the CMB

$$p + e^- = H + \gamma$$



- After the Big Bang, the universe was a hot, dense plasma of photons, electrons, and protons. Eventually, the universe cooled to the point that the formation of neutral hydrogen was energetically favored. This process is known as recombination.
- Once photons decoupled from matter, they traveled freely through the universe without interacting with matter, and constitute what we observe today as Cosmic Microwave Background (CMB) radiation

Evolution of the Universe



Statistical Formalism

- The observational quantities are:

$$\frac{\Delta T(\hat{n})}{T_0} = \sum_{lm} a_{T,lm} Y_{lm}(\hat{n})$$

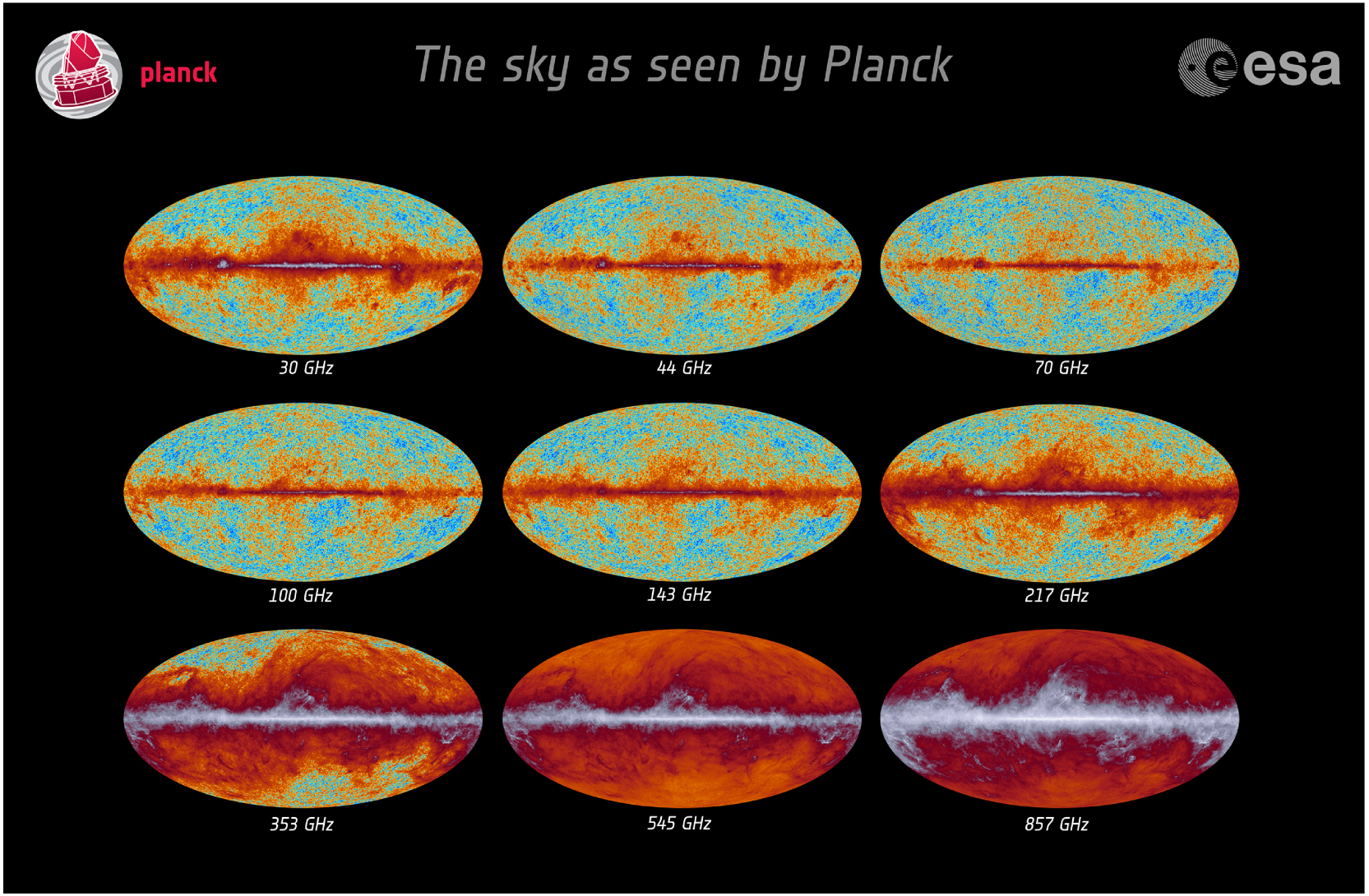
$$Q + iU(\hat{n}) = \sum_{lm} a_{2,lm} Y^2_{lm}(\hat{n})$$

$$Q - iU(\hat{n}) = \sum_{lm} a_{-2,lm} Y^{-2}_{lm}(\hat{n})$$

  where Q and U are the Stokes Parameters.

- The parameter Q can be defined operationally as the difference of the intensities transmitted by perfect polarizers oriented along axis x and y.
- The parameter U can be defined operationally as the difference of the intensities transmitted by perfect polarizers rotated 45 to the axes x and y.

Temperature maps-Planck



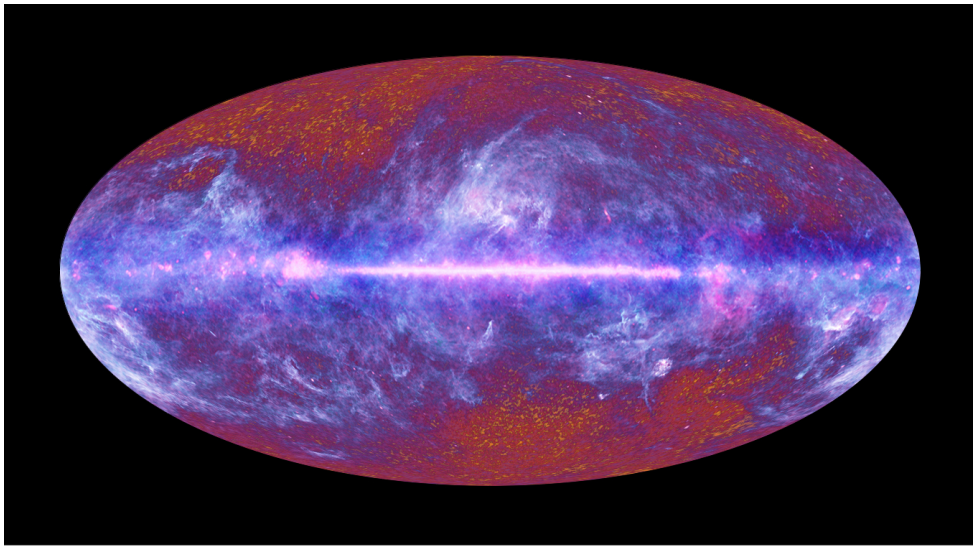
E and B modes

- The Stokes parameters are not invariant under rotations in the plane perpedicular to $\hat{n}$. Therefore, the following rotational invariant quantities are defined:

$$
\begin{aligned}
E(\hat{n}) &= \frac{1}{2}\left[\bar{\partial}^2 (Q + iU) - \partial^2 (Q - iU)\right] \\
B(\hat{n}) &= \frac{i}{2}\left[\bar{\partial}^2 (Q + iU) - \partial^2 (Q - iU)\right]
\end{aligned}
$$

  where ∂ and $\bar{\partial}$ are the spin raising and lowering operators, respectively.

- The E mode is a scalar under parity transformations, while the B mode is a pseudo-scalar.



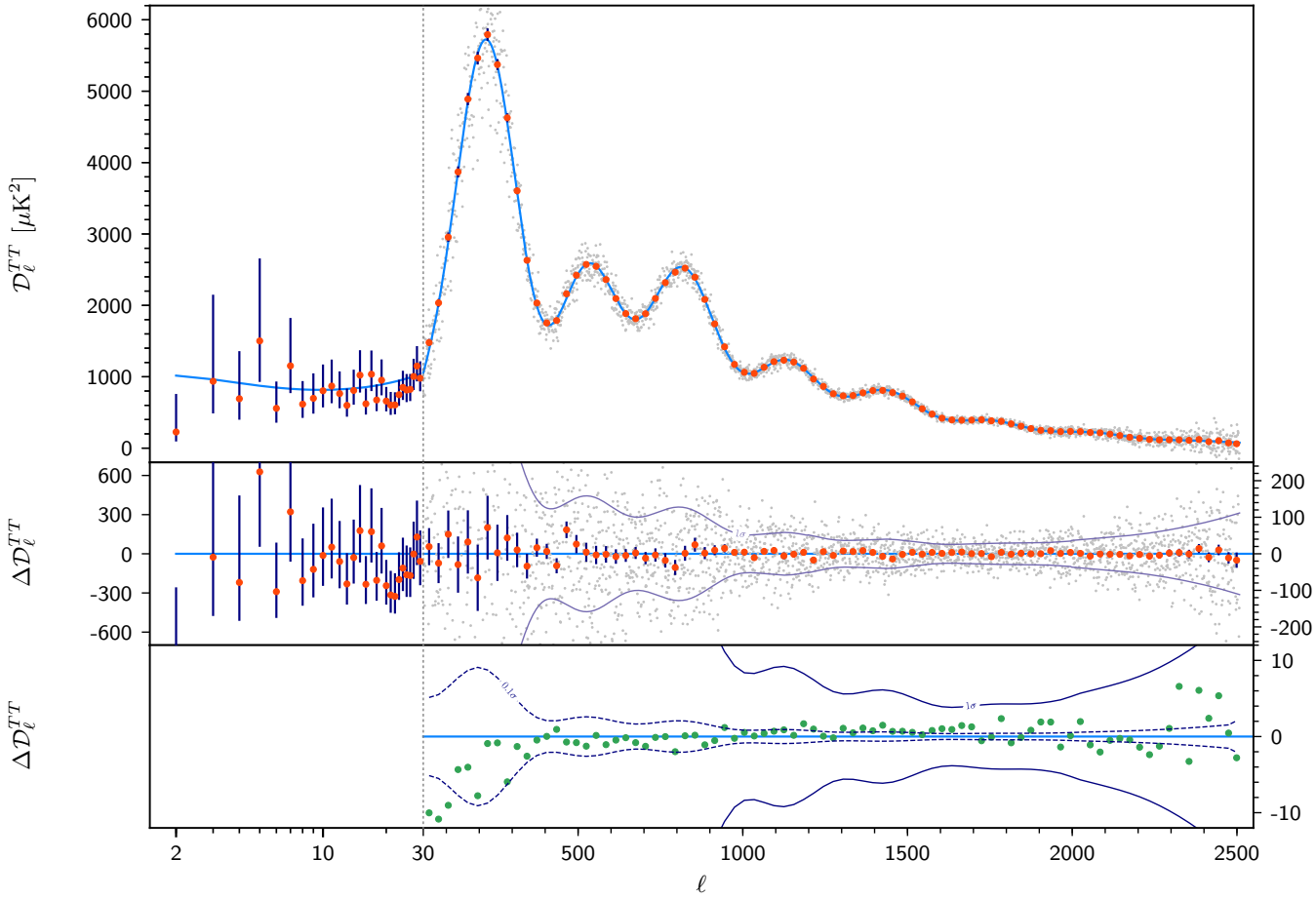
Correlations Functions

- The following correlation functions are defined:

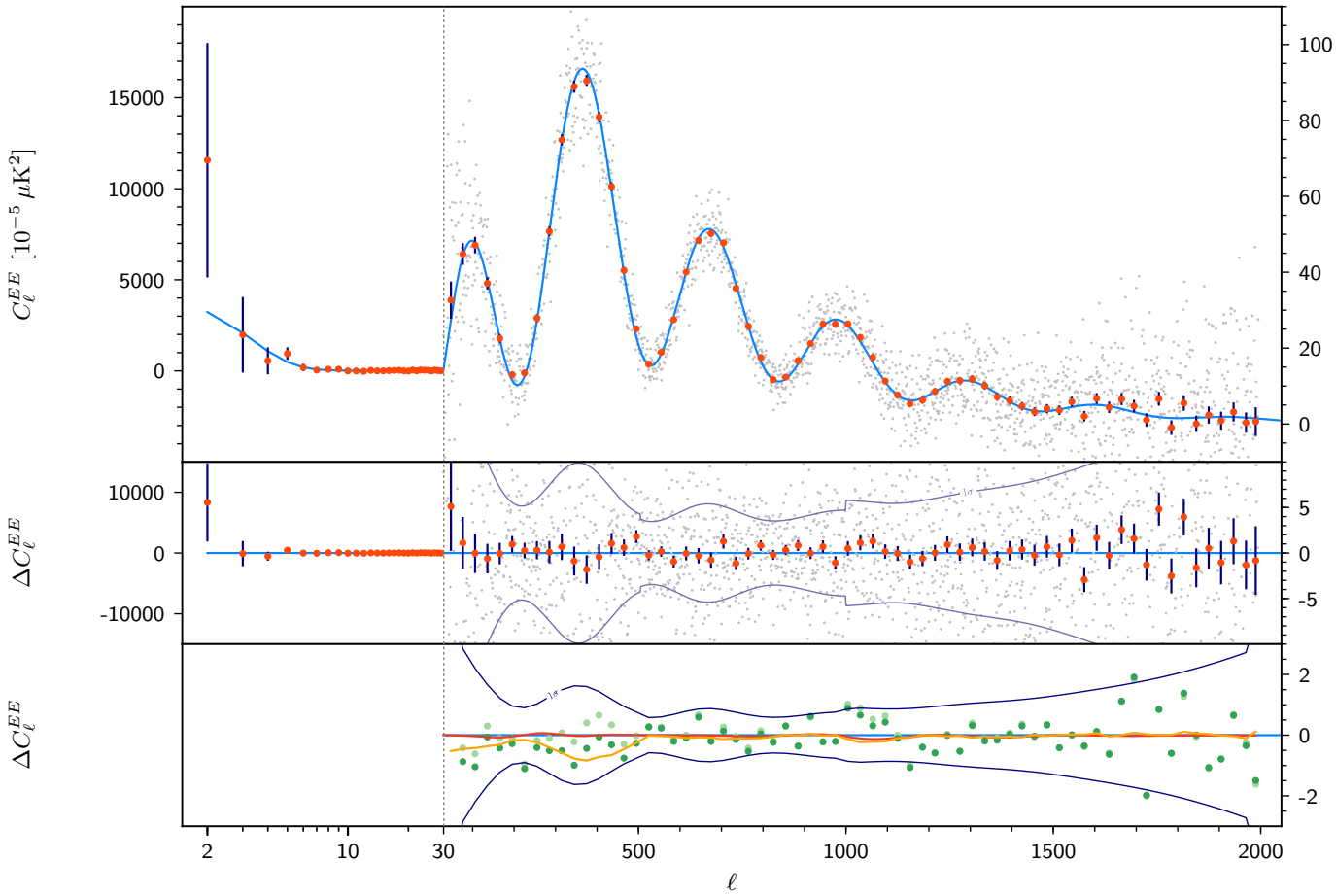
$$
\begin{aligned}
C_{TT}(\theta) &= < \frac{\Delta T}{T}(\hat{n}_1), \frac{\Delta T}{T}(\hat{n}_2) > \\
&= 2\pi \sum_l C_l^{TT} (2l+1) P_l(\cos\theta) \\
C_{EE}(\theta) &= < E(\hat{n}_1), E(\hat{n}_2) > \\
&= 2\pi \sum_l C_l^{EE} (2l+1) P_l(\cos\theta) \\
C_{BB}(\theta) &= < B(\hat{n}_1), B(\hat{n}_2) > \\
&= 2\pi \sum_l C_l^{BB} (2l+1) P_l(\cos\theta) \\
C_{TE}(\theta) &= < \frac{\Delta T}{T}(\hat{n}_1), E(\hat{n}_2) > \\
&= 2\pi \sum_l C_l^{TE} (2l+1) P_l(\cos\theta)
\end{aligned}
$$

where $\cos\theta = \hat{n}_1 \cdot \hat{n}_2$.

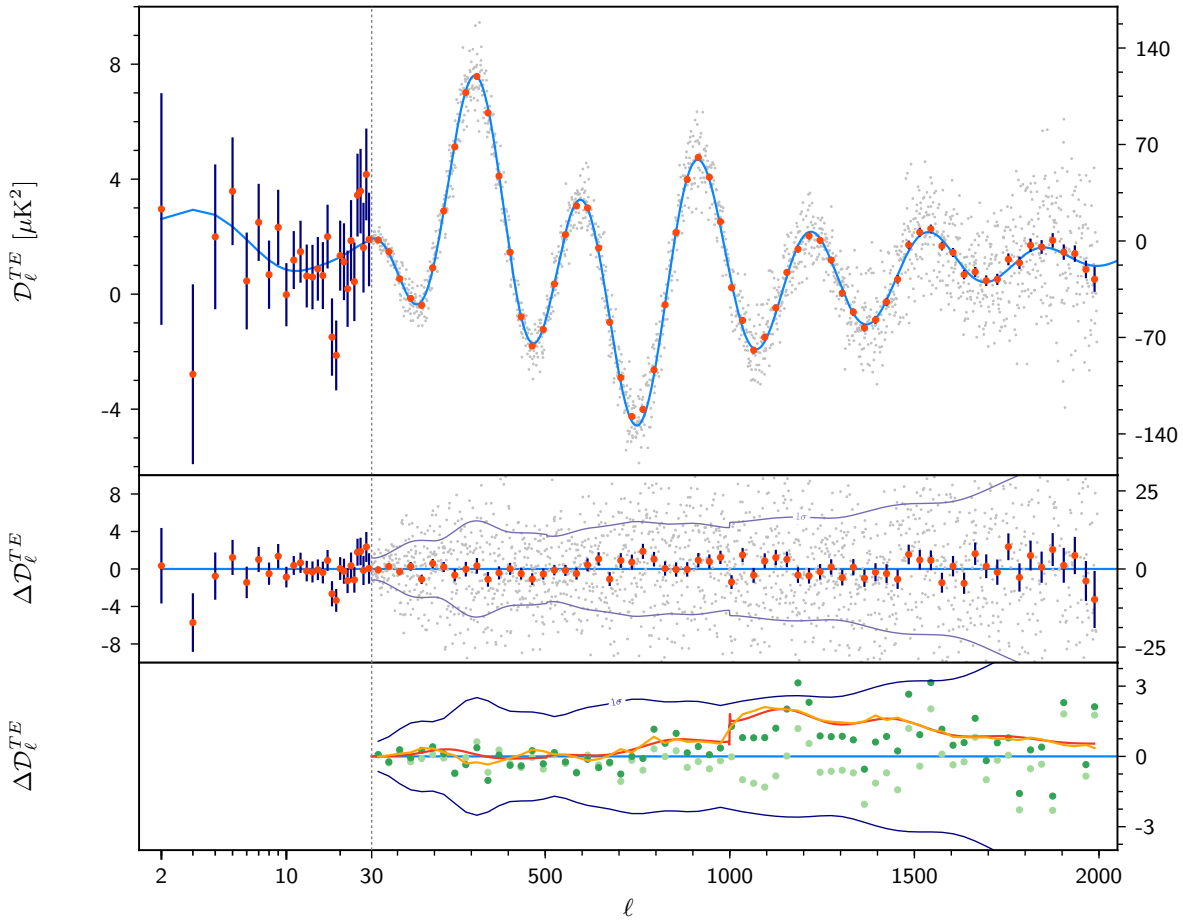
Temperature auto-correlation angular power spectra - Planck 2018



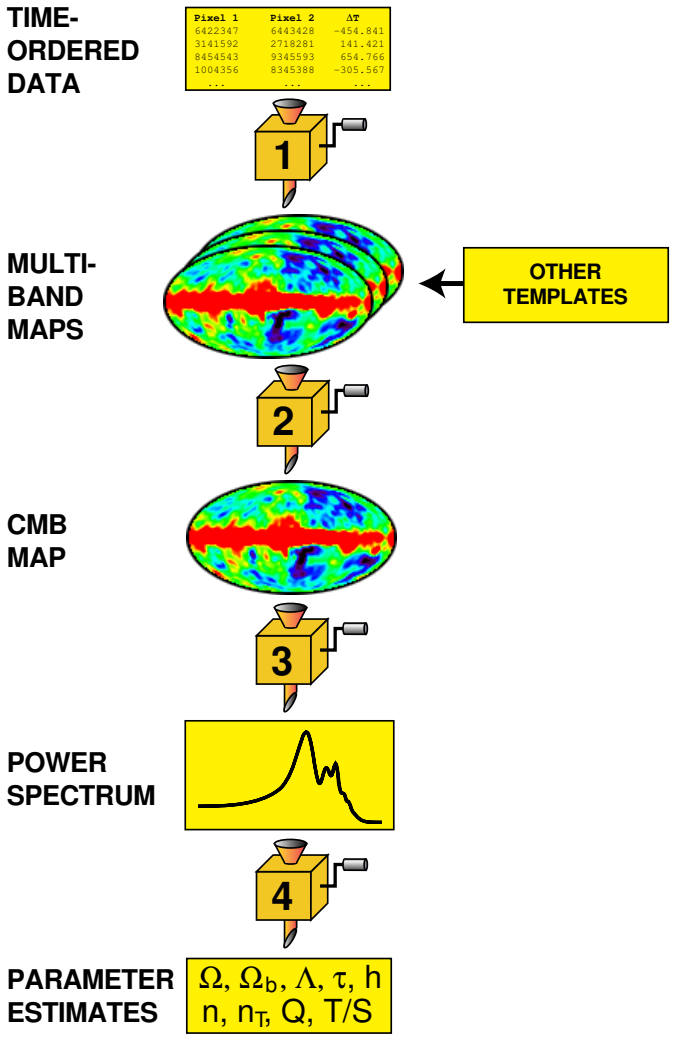
E mode auto-correlation power spectra - Planck 2018



Temperature-Polarization cross-correlation power spectra - Planck 2018



From time ordered data to cosmological parameter estimation



Conexion between observations and theory

- The conexion between the observational quantities and the theoretical predictions can be seen from the following expressions:

$$C_\ell^{TT} = \frac{2}{\pi}\int \frac{\mathrm{d}k}{k} \underbrace{P_{\mathcal{R}}(k)}_{\text{Inflation}} \underbrace{\Delta_{T\ell}(k)\Delta_{T\ell}(k)}_{\text{LateUniverse}} .$$

$$C_\ell^{EE} = \frac{2}{\pi}\int \frac{\mathrm{d}k}{k} \underbrace{P_{\mathcal{R}}(k)}_{\text{Inflation}} \underbrace{\Delta_{E\ell}(k)\Delta_{E\ell}(k)}_{\text{LateUniverse}} .$$

- By *Late Universe*, we understand the physics from the end of the inflationary era until present days.
- The physics from the *Late Universe* is obtained from solving the coupled differential equations of the dynamics of each type of particle in the universe: photons, electrons, protons, neutrinos with the equations of the metric.
- The primordial power spectrum, $P_{\mathcal{R}}(k)$, is obtained from the inflationary model.

Conexion between observations and theory

- And also

$$C_\ell^{TE} = \frac{2}{\pi}\int \frac{\mathrm{d}k}{k} \underbrace{P_{\mathcal{R}}(k)}_{\text{Inflation}} \underbrace{\Delta_{T\ell}(k)\Delta_{T\ell}(k)}_{\text{LateUniverse}} .$$

$$C_\ell^{BB} = \frac{2}{\pi}\int \frac{\mathrm{d}k}{k} \underbrace{P_t(k)}_{\text{Inflation}} \underbrace{\Delta_{B\ell}(k)\Delta_{B\ell}(k)}_{\text{LateUniverse}} .$$

where

$$P_R(k) = \frac{P_s(k)}{k^3} = \frac{A_s}{k^3}\left(\frac{k}{k*}\right)^{(n_s-1)}$$

and

$$P_t(k) = \frac{A_t}{k^3}\left(\frac{k}{k*}\right)^{n_t}$$

Usually, $\boxed{r = A_t/A_s}$ is defined as the tensor to scalar ratio and $r_{0.002}$ is its value at the pivot scale $k_* = 0.002\,\mathrm{Mpc}^{-1}$.

Equations for matter perturbations

- The baryons and CDM are described as fluids, which satisfy the continuity equation and the Euler equation. Their governing equations are given by

$$\dot{\delta}_b = -kv_b + 3\dot{\phi}_N, \quad \dot{v}_b = -\mathcal{H}v_b + k\psi_N + \frac{\dot{\tau}}{R}(v_b - \Theta_1), \tag{9}$$

$$\dot{\delta}_c = -kv_c + 3\dot{\phi}_N, \quad \dot{v}_c = -\mathcal{H}v_c + k\psi_N, \tag{10}$$

  where $\mathcal{H} := aH$.

- These equations should be modified if the matter content non-minimally couples to gravity (Bettoni 2011, 2015).

Equations for metric perturbations

- The metric perturbations obey the perturbed Einstein equations sourced by the matter density perturbations and the monopole moment of the photons (Θ_0) and massless neutrinos ($\mathcal{N}_0$). In GR:

$$\mathcal{H}\dot{\phi}_N + \frac{k^2}{3}\phi_N + \mathcal{H}^2\psi_N = -\frac{4\pi G a^2}{3}\left(\rho_b\delta_b + \rho_c\delta_c + 4\rho_\gamma\Theta_0 + 4\rho_\nu\mathcal{N}_0\right), \tag{11}$$

$$k^2(\psi_N - \phi_N) = -32\pi G a^2\left(\rho_\gamma\Theta_2 + \rho_\nu\mathcal{N}_2\right). \tag{12}$$

  where Θ_2 and $\mathcal{N}_2$ are the quadrupole moments of the photon and massless neutrino distribution functions respectively.

- In modified theories of gravity, these equations are also modified.

Equations for photon perturbations

- The photons are described by the Boltzmann equation. The Boltzmann equation has a formal solution

$$\Theta_\ell(k,\eta_0) = \int_0^{\eta_0} d\eta \left\{ g(\eta) \left[\Theta_0(k,\eta) + \Phi(k,\eta) + \frac{1}{2} P(k,\eta) \right] j_\ell(x) \right.$$
$$- g(\eta) v_b(k,\eta) j'_\ell(x) + \frac{3}{2} g(\eta) P(k,\eta) j''_\ell(x)$$
$$\left. + e^{-\tau} \left[\dot{\phi}_N(k,\eta) + \dot{\psi}_N(k,\eta) \right] j_\ell(x) \right\}, \qquad (13)$$

where $x = k(\eta_0 - \eta)$, $j_\ell(x)$ is the spherical Bessel function of the first kind, and $g(\eta) := -\dot{\tau} e^{-\tau}$ is the visibility function determining the location of the last-scattering surface.

- Note that the last line of the above equation depends on the time derivative of ϕ_N and ψ_N. This contribution is called the integrated Sachs-Wolfe (ISW) effect, which is the dominant contribution to the CMB anisotropies on large scales.

CMB Temperature Spectrum in Galileon Gravity

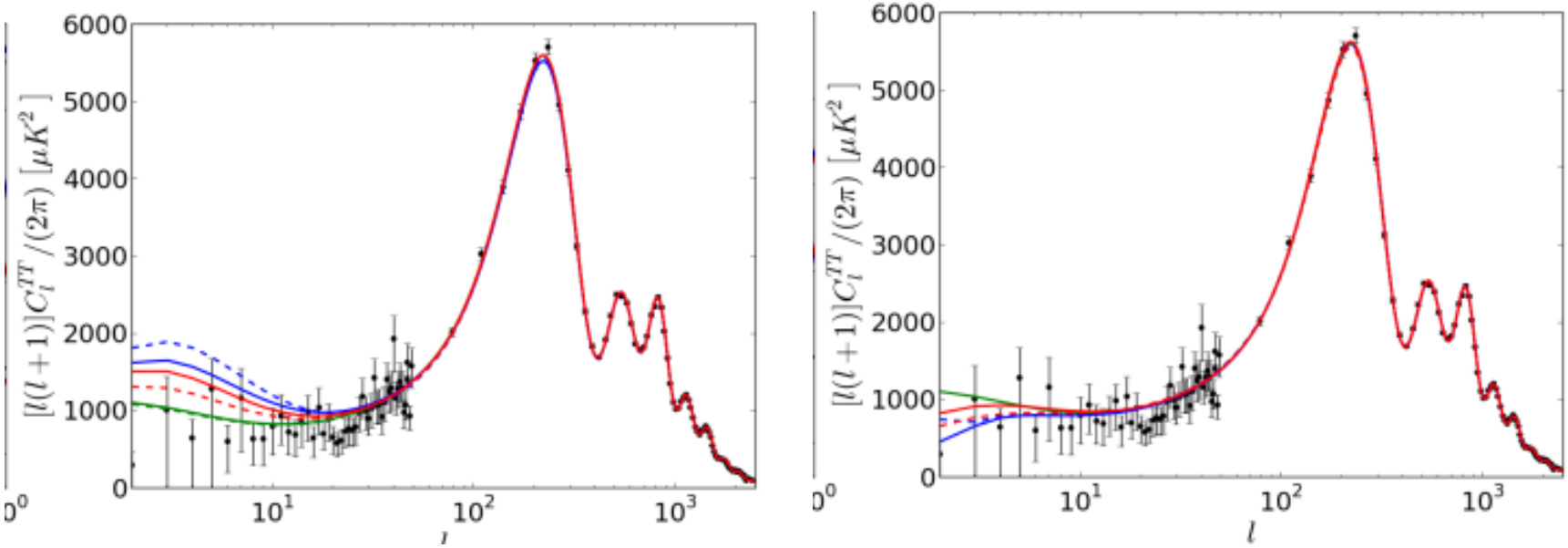


FIGURE: Best fits models to CMB + CMB lensing + BAO data (solid line). Only CMB + CMB lensing (dashed line)

- Left figure: Green Line: ΛCDM model with massive neutrinos, Blue line: Qubic Galileon, Red lines: Qubic Galileon + Massive neutrinos.
- Right figure: Green Line: ΛCDM model, Blue line: Quintic Galileon, Red line: Cuartic Galileon.

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Constaints on inflationary parameters from CMB observations

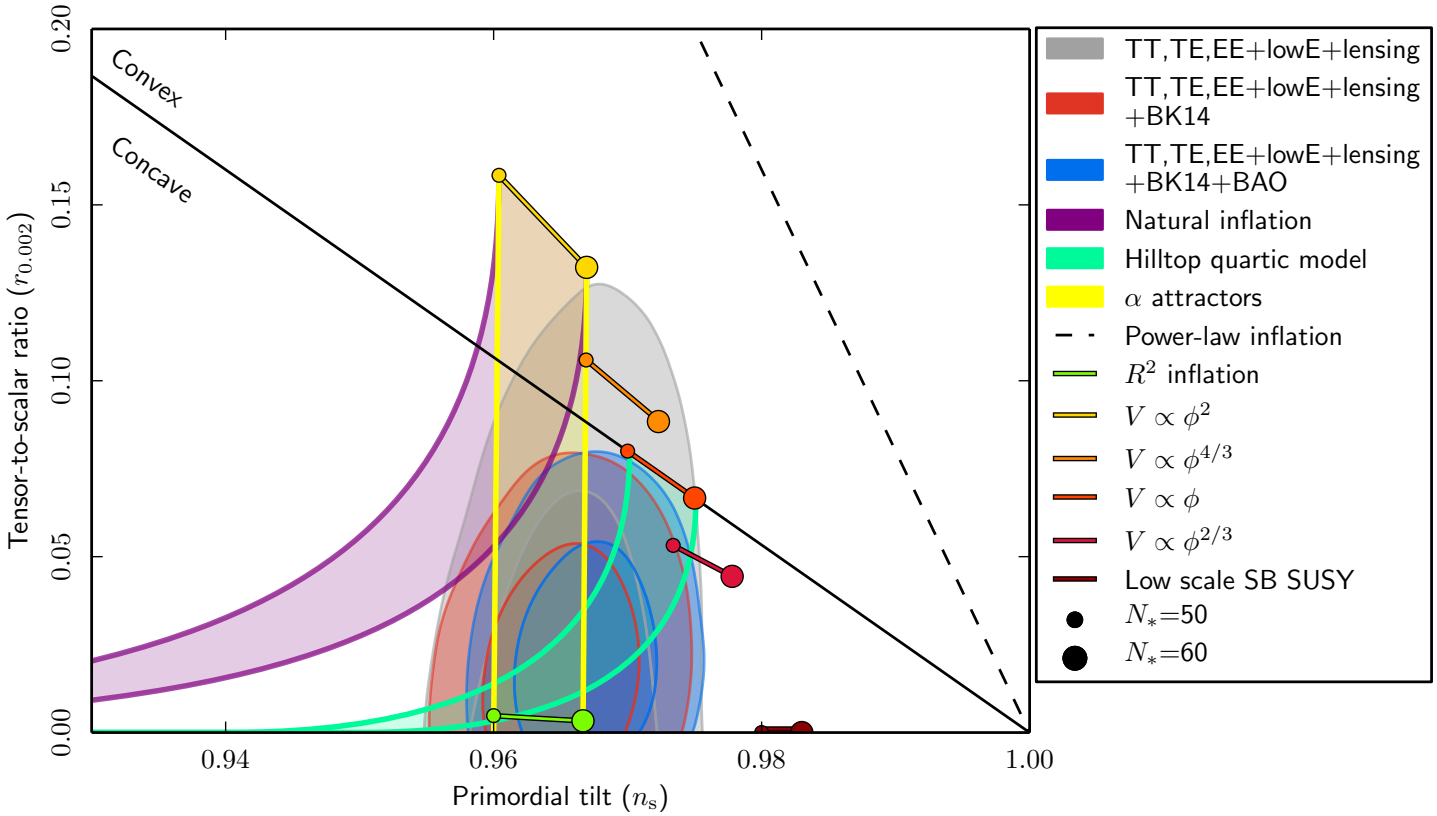

FIGURE: Constraints on the inflationary parameters n_s and r from Planck 2018 data.

Constraints on MG models

- The Induced Gravity (IG) model lagrangian is given by:

$$\mathcal{L} = \frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi - \lambda\phi^2 + \frac{1}{2}\xi R\phi^2 + \mathcal{L}_m \tag{14}$$

- P18+DESI+SDSS is constructed by replacing the first two redshift bins of DESI with SDSS low-z results.

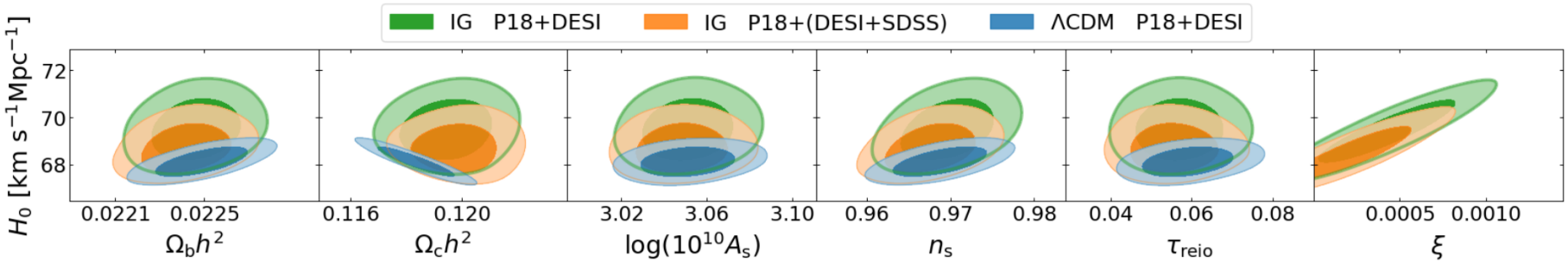

Figure: Constraints on the IG model from CMB + DESI DR1 BAO data. Ferrari et al. PRD111, 083523, (2025)

$$\boxed{H_0 = 69.52^{+0.79}_{-0.90} \quad \text{km/s/Mpc} \quad (68\%\,\text{C.L.})} \tag{15}$$

Constraints on MG models

- The Brans-Dicke Galileon (BDG) model lagrangian is given by:

$$\mathcal{L} = -\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi(1-\frac{8\alpha}{\phi^2}) - \Lambda - \frac{\alpha}{\phi}\Box\phi\,\nabla_\mu\phi\nabla^\mu\phi + \frac{1}{2}\xi R\phi^2 + \mathcal{L}_m \quad (16)$$

- $\xi = 5\times10^{-5}$ $\quad \frac{1}{\tilde{\alpha}_8} = 10^{-8}\tilde{\alpha}$ where $\tilde{\alpha} = \alpha \times [\mathrm{MpcGev}^{-1}]^{-2}$.

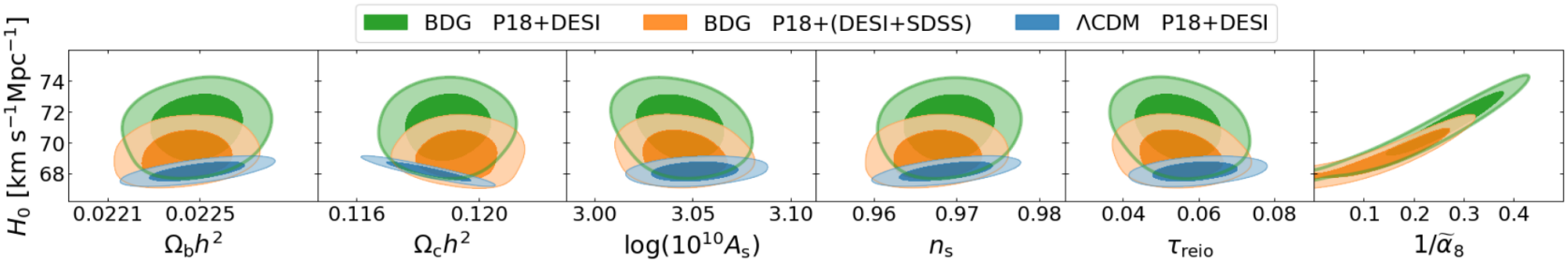

Figure: Constraints on the IG model from CMB + DESI DR1 BAO data. Ferrari et al. PRD111, 083523, (2025)

$$\boxed{H_0 = 71^{+1.5}_{-1.3} \quad \mathrm{km/s/Mpc} \quad (68\%\,\mathrm{C.L.})} \quad (17)$$

CMB shift parameters

- Evaluating the full CMB likelihood requires solving the Boltzmann hierarchy and computing the whole C_ℓ at every point of the parameter space: expensive, especially for MG models with extra parameters.
- For models that only modify the late time expansion, most of the constraining power of the CMB is geometrical: it comes from the way the acoustic scale imprinted at z_* is projected onto the sky today. That information can be compressed into the shift parameters or distance priors:

$$\ell_A = \pi \frac{D_M(z_*)}{r_s(z_*)} \qquad R = \frac{\sqrt{\Omega_m H_0^2}}{c} D_M(z_*)$$

where $D_M(z_*) = \int_0^{z_*} c\,\mathrm{d}z/H(z) = c\,(\eta_0 - \eta_*)$ is the comoving angular diameter distance to last scattering (flat universe).

- ℓ_A is the acoustic scale: it fixes the angular position of the peaks, $\ell_n \simeq (n - \varphi_n)\,\ell_A$. R (Bond, Efstathiou & Tegmark 1997) is the same distance in units set by $\Omega_m H_0^2$, and controls the shift of the peaks. Both are used together with $\omega_b = \Omega_b h^2$, which fixes their relative height.

CMB shift parameters: use and limitations

- The compressed data vector $v = (R, \ell_A, \omega_b)$ enters the likelihood as

$$\chi^2_{\rm CMB} = \left(v - v^{\rm obs}\right)^T \mathbf{C}^{-1} \left(v - v^{\rm obs}\right)$$

with $\mathbf{C}$ the covariance matrix of the three (strongly correlated) parameters. From Planck 2018 TT,TE,EE+lowE (Chen, Huang & Wang 2019):

$$R = 1.7502 \pm 0.0046, \qquad \ell_A = 301.471 \pm 0.090, \qquad \omega_b = 0.02236 \pm 0.00015$$

- Advantage: very fast, and useful to combine the CMB with SNIa, BAO and cosmic chronometers for models that modify only the background.
- Limitations, severe for MG, and the reason why the constraints shown before use the full C_ℓ likelihood:
  - z_* and $r_s(z_*)$ must still be obtained from the numerical integration of the recombination equation (Lecture 2);
  - they assume standard pre recombination physics: if the model changes $H(z)$ before recombination (early dark energy, early MG, extra $N_{\rm eff}$), r_s changes and the compression is not valid;
  - they discard the amplitude and shape of the spectra, which is what constrains the perturbative sector of MG.

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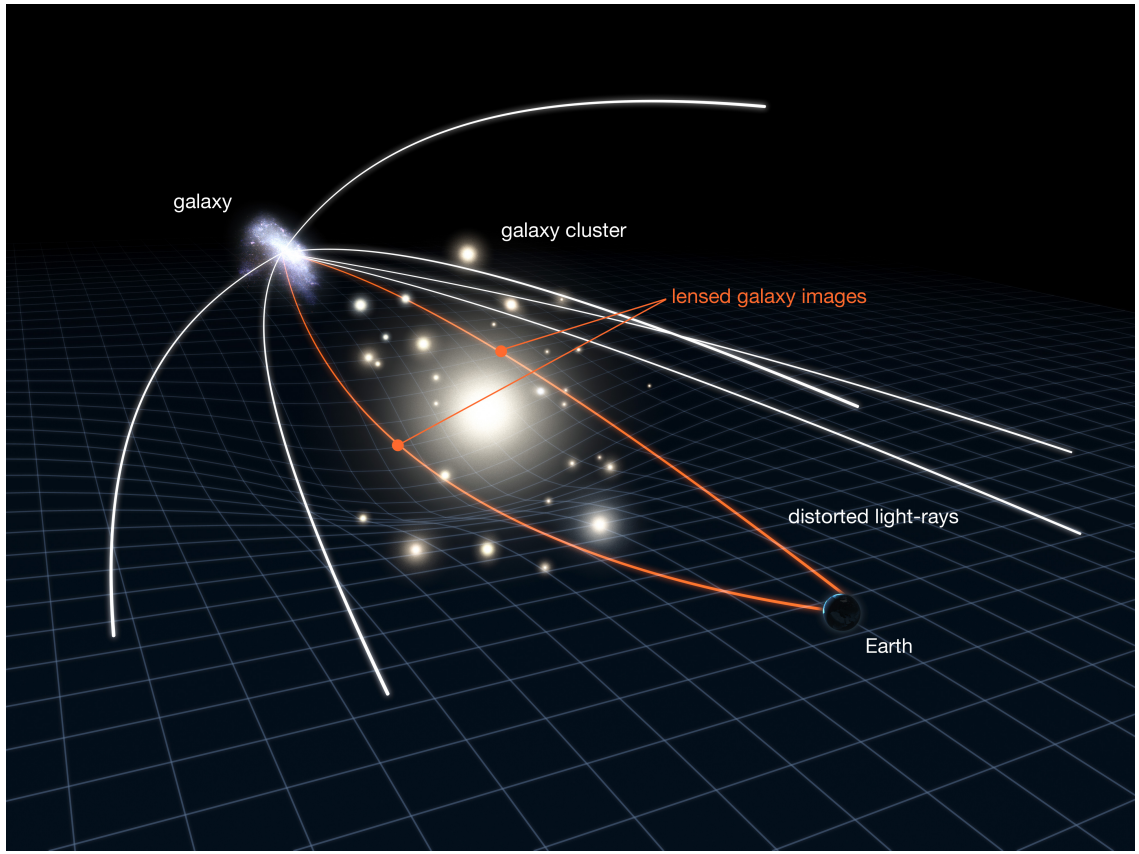
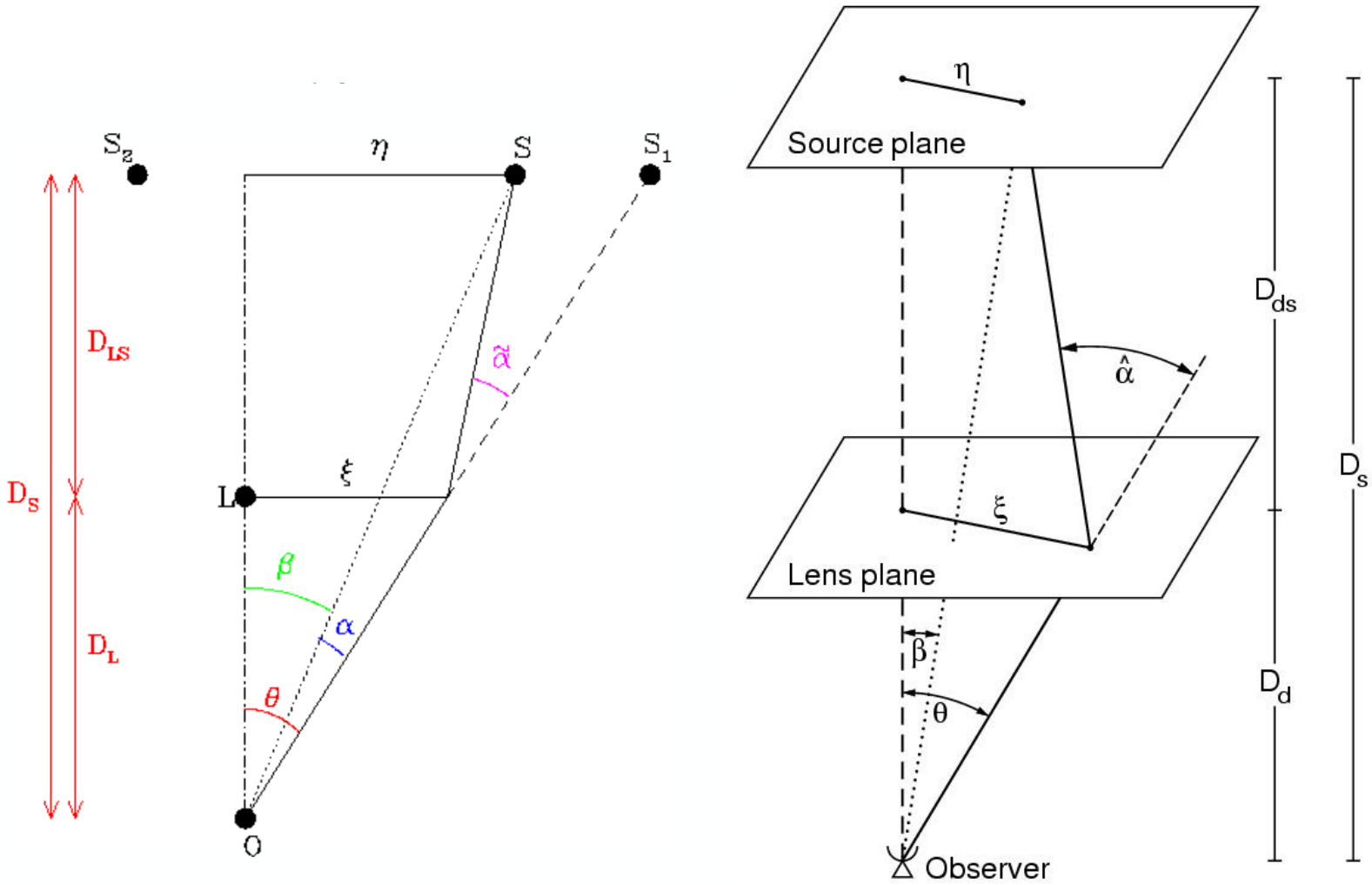
Gravitational lensing

- Gravitational lensing refers to the deflectiong of light rays due to mass sources along the line of sight.




S_2
η
S
S_1
D_{LS}
D_S
D_L
L
ξ
$\tilde{\alpha}$
β
α
θ
O
η
Source plane
D_{ds}
$\hat{\alpha}$
D_s
ξ
Lens plane
β
D_d
θ
Observer

The lensing potential

- The net deflection angle α at a distance χ between two rays is given by

$$\alpha = -\frac{2}{c^2}\int_0^{\chi} d\chi' \frac{f_K(\chi-\chi')}{f_K(\chi)} \nabla_\perp \Phi_\gamma(f_K(\chi')\theta, \chi') \tag{18}$$

where $f_K(\chi)$ is the comoving angular diameter distance, and $\nabla_\perp$ is the gradient with respect to the comoving transverse distance and $\Phi_\gamma = \frac{\Phi_N+\Psi_N}{2}$ is the Weyl potential.

- The lensing potential, ψ, is defined such that the deflection angle is given by the angular gradient of the lensing potential:

$$\alpha = \nabla_\theta \psi \tag{19}$$

where ∇_θ the gradient with respecto to the angular position is related to the gradient of the comoving transverse distance by $\nabla_\theta = f_K(\chi)\nabla_\perp$.

- The lensing potential is given by:

$$\psi(\theta,\chi) = \frac{2}{c^2}\int_0^{\chi} d\chi' \frac{f_K(\chi-\chi')}{f_K(\chi)f_K(\chi')} \Phi_\gamma(f_K(\chi')\theta, \chi') \tag{20}$$

Weak lensing formalism

- The convergence κ is related to the isotropic magnification of an image:

$$\kappa(\vec{\theta}) = \frac{1}{2}\left(\frac{\partial^2\psi}{\partial\theta_1^2} + \frac{\partial^2\psi}{\partial\theta_2^2}\right) = \frac{1}{2}\nabla^2\psi(\vec{\theta}) \tag{21}$$

  where $\vec{\theta} = (\theta_1, \theta_2)$ is the angular position in the sky.

- The shear $\gamma = (\gamma_1, \gamma_2)$ is related to the anisotropic distortion of the image:

$$\gamma_1(\vec{\theta}) = \frac{1}{2}\left(\frac{\partial^2\psi}{\partial\theta_1^2} - \frac{\partial^2\psi}{\partial\theta_2^2}\right) \quad , \quad \gamma_2(\vec{\theta}) = \frac{\partial^2\psi}{\partial\theta_1\partial\theta_2} \tag{22}$$

Convergence for weak lensing

- Therefore, the convergence can be expressed as:

$$\kappa(\theta,\chi) \quad == \quad -\frac{1}{c^2}\int_0^{\chi} d\chi' \, \frac{f_K(\chi-\chi')}{f_K(\chi')f_K(\chi)} \, \nabla^2_\theta \Phi_\gamma(f_K(\chi')\theta,\chi') \qquad (23)$$

where ∇^2_θ is the Laplacian with respect to the angular position.

Convergence for weak lensing

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  where ∇^2_θ is the Laplacian with respect to the angular position.

- Using the Poisson equation, we can rewrite the convergence as:

$$\kappa(\theta,\chi) = \frac{3H_0^2\Omega_m}{2c^2}\int_0^{\chi} d\chi' \, \frac{f_K(\chi-\chi')f_K(\chi')}{f_K(\chi)} \, \frac{\delta(f_K(\chi')\theta,\chi')}{a(\chi')} \tag{24}$$

  where $\delta = \frac{\delta\rho_m}{\rho_m}$ is the matter density contrast.

Convergence for weak lensing

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$$\kappa(\theta,\chi) = \frac{3H_0^2\Omega_m}{2c^2}\int_0^{\chi} d\chi' \, \frac{f_K(\chi-\chi')f_K(\chi')}{f_K(\chi)} \, \frac{\delta(f_K(\chi')\theta,\chi')}{a(\chi')} \tag{24}$$

  where $\delta = \frac{\delta\rho_m}{\rho_m}$ is the matter density contrast.

- In a flat universe, $K = 0$, the comoving angular diameter distance is simply the comoving distance, $f_K(\chi) = \chi$, and the lensing potential can be written as:

$$\psi(\theta,\chi) = \frac{2}{c^2}\int_0^{\chi} d\chi' \, \frac{\chi-\chi'}{\chi\chi'} \, \Phi_\gamma(\chi'\theta,\chi') \tag{25}$$

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CMB lensing

- The CMB photons are deflected by the gravitational potential of the large scale structure as they travel from the last scattering surface to us.
- This leads to a distortion in the spatial pattern of observed CMB temperature and polarisation maps
- CMB lensing is a direct probe of intervening gravitational fields along the line of sight and is one of the most powerful means of testing theories of gravity from CMB observations.
- The lensed CMB anisotropies, $X(\hat{n})(X = T, Q, U)$, are given as a remapping of the primary unlensed CMB anisotropies at the last scattering: $X(\hat{n}) = X[\hat{n} + d(\hat{n})]$ where $\hat{n}$ is the observer line-of-sight direction and $\alpha(\hat{n})$ is the deflection angle.
- We can approximate $\alpha(\hat{n})$ as the angular gradient of a lensing potential, $\psi(\hat{n})$ (which we recall is related to the Weyl potencial $\Phi_\gamma = \frac{\phi_N + \psi_N}{2}$), i.e. $\alpha(\hat{n}) = \nabla_\theta \psi(\hat{n})$.

CMB lensing

- The convergence can be expressed as:

$$\begin{aligned} \kappa(\hat{n}) &= \frac{2}{c^2}\nabla_\theta^2 \int_0^{\chi_{CMB}} d\chi \, \frac{\chi_{CMB}-\chi}{\chi_{CMB}\chi}\Phi_\gamma \\ &= \frac{1}{c^2}\nabla_\theta^2 \int_0^{\chi_{CMB}} d\chi \frac{\chi_{CMB}-\chi}{\chi_{CMB}\chi}\left(\phi_N+\psi_N\right) \end{aligned} \tag{26}$$

  where χ is the comoving distance, χ_{CMB} is the comoving distance to the last scattering surface, and Φ_γ is the Weyl potential.

- Therefore the lensing convergence directly probes the gravitational potential of the LSS along the line-of-sight
- We can reconstruct a map of the lensing convergence using the fact that, for a given lensing mass distribution in our Universe, the lensing effect violates the statistical isotropy of the lensed CMB fluctuations.
- For a complete review on CMB lensing see: Lewis & Challinor, Phys. Rep. 429, 1 (2006)

CMB lensing power spectrum

- The CMB lensing power spectrum is given by:

$$C_L^{\psi\psi} \sim \left\langle a_{l_1 m_1} a_{l_2 m_2} a_{l_3 m_3} a_{l_4 m_4} \right\rangle_c \tag{27}$$

where $\langle \cdots \rangle_c$ denotes the connected part of the four-point correlation function.

CMB Lensing Spectrum in Galileon Gravity

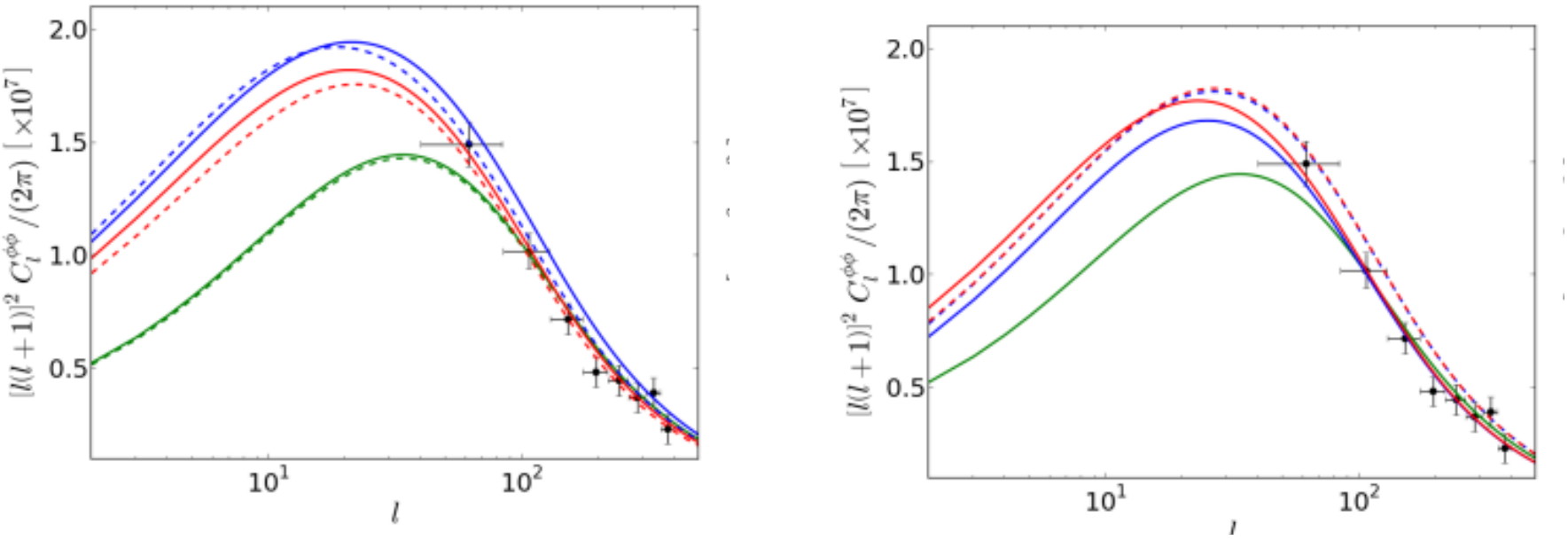

FIGURE: Best fits models to CMB + CMB lensing + BAO data (solid line). Only CMB + CMB lensing (dashed line)

- Left figure: Green Line: ΛCDM model with massive neutrinos, Blue line: Qubic Galileon, Red lines: Qubic Galileon + Massive neutrinos.
- Right figure: Green Line: ΛCDM model, Blue line: Quintic Galileon, Red line: Cuartic Galileon.

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The perturbative sector in MG

- In GR and in the absence of anisotropic stress, the two scalar potentials are equal: $\Phi_N = \Psi_N$. However, in modified gravity theories, this equality may not hold, leading to a gravitational slip between the two potentials : $\eta = \frac{\Phi_N}{\Psi_N}$.
- The newtonian potentials Φ_N and Ψ_N obey the modified Poisson equations:

$$\begin{aligned} k^2\Psi_N &= -4\pi G a^2(1+\mu(a,k))\rho\delta \\ k^2(\Phi_N+\Psi_N) &= -8\pi G a^2(1+\Sigma(a,k))\rho\delta \end{aligned} \tag{28}$$

- The functions $\mu(a,k)$ and $\Sigma(a,k)$ parametrize deviations from GR at the perturbative level.

Constraints from CMB and CMB lensing

- Asuming a ΛCDM background, and a parametrized (MG) model for the perturbations, amd considering only time dependence for μ and Σ:

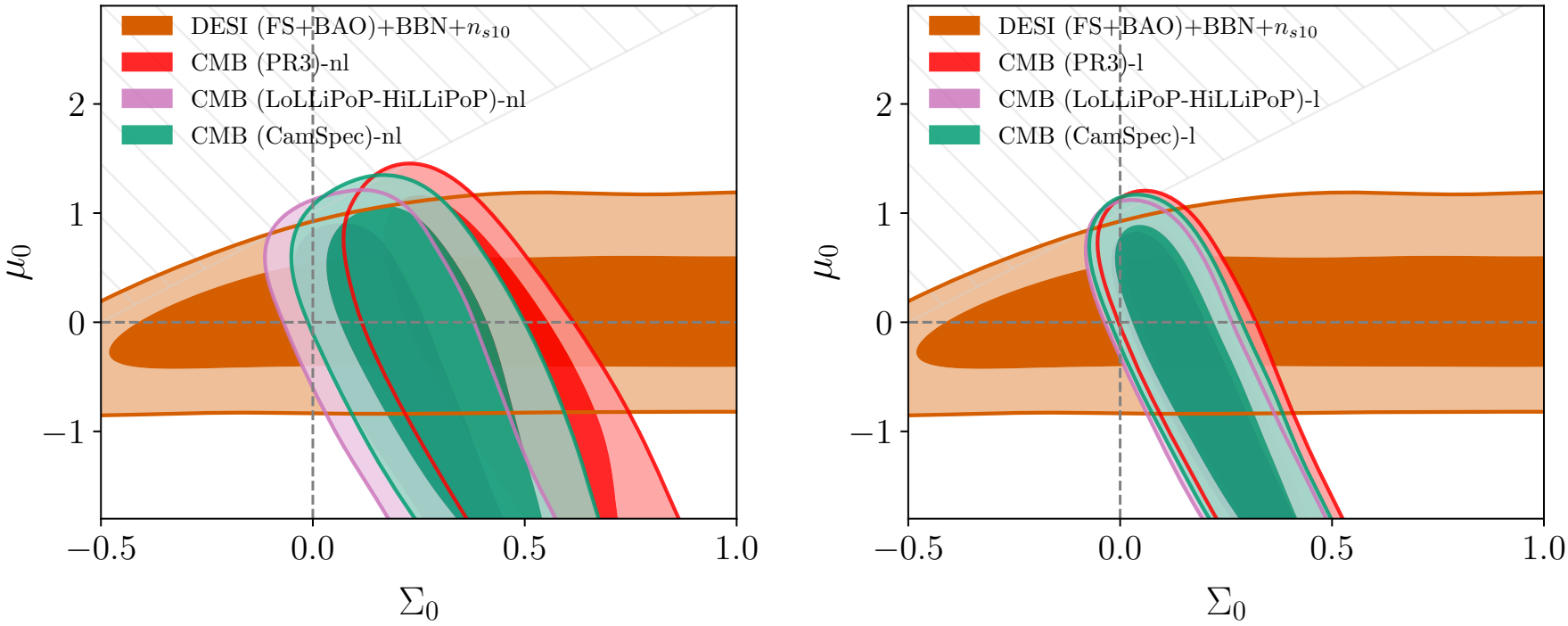

Figure: Constraints on MG models from DESI DR1 M. Ishak et al JCAP09(2025)053: Left: without CMB lensing data, Right: with CMB lensing data

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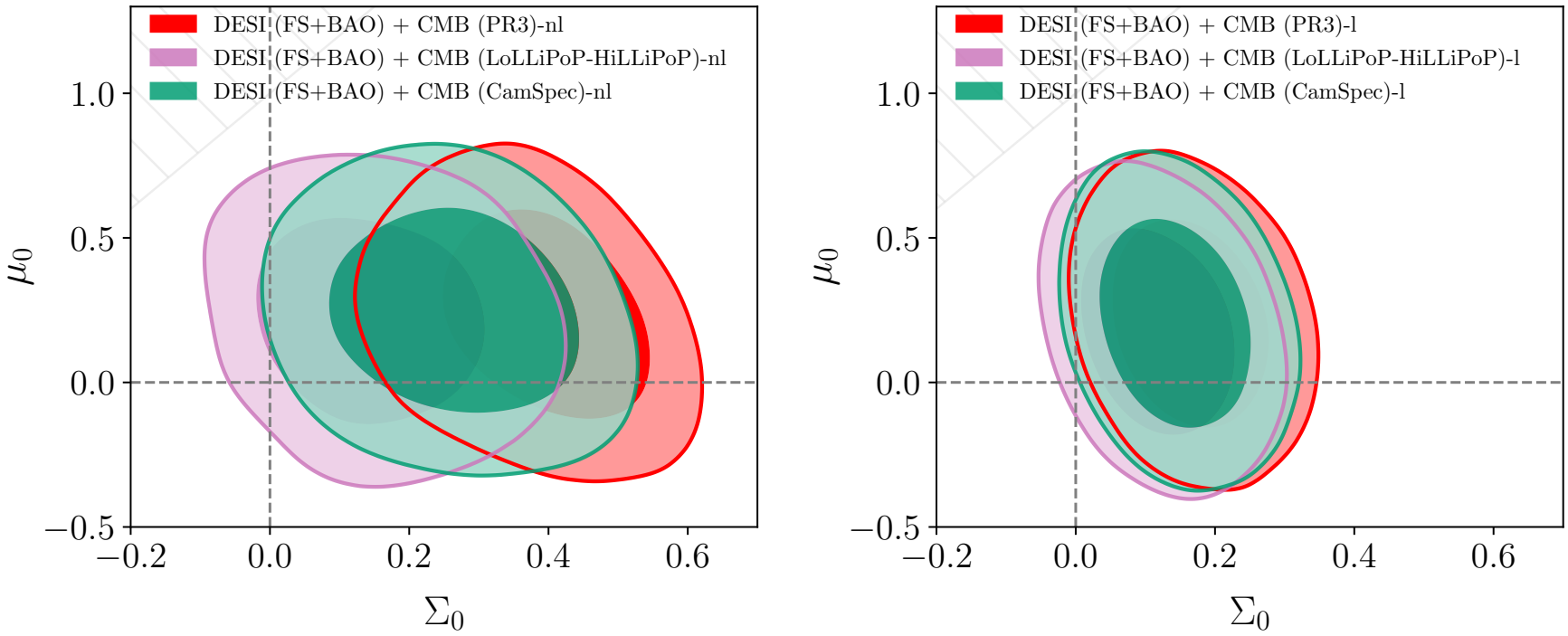

Figure: Constraints on MG models from DESI DR1 M. Ishak et al JCAP09(2025)053: Left: without CMB lensing data, Right: with CMB lensing data

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Relevant papers for this lecture

- Weak Lensing: Bartelmann & Schneider, Phys. Rep. 340, 291 (2001), R. Mandelbaum Annu. Rev. Astron. Astrophys. 56, 393 (2018)
- CMB lensing: Lewis & Challinor, Phys. Rep. 429, 1 (2006)
- CMB: Planck 2018 results. VIII. Gravitational lensing, A & A 641, A8 (2020)
- CMB: Planck 2018 results. VI. Cosmological parameters, A & A 641, A6 (2020)
- CMB: Planck 2018 results. V. CMB power spectra and likelihoods, A & A 641, A5 (2020)
- Link between theory and observations: S.Arai et al. arXiv: 2212.09094, Hou et al Universe 9, 302 (2023), arXiv: 2306.13726
- Galileon gravity and CMB: Barreira et al. JCAP 08, 059 (2014)
- Constraints on MG models from CMB and BAO observations: Ferrari et al. PRD111, 083523, (2025)
- Constraints on MG models from CMB and CMB lensing observations: M. Ishak et al JCAP09(2025)053

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Separation of light rays

- The transverse physical separation between two nearby light rays in an thin light bundle which is inmersed in an **homogeneous and isotropic universe** is described by the geodesic deviation equation:

$$\frac{D^2\xi^\mu}{D\lambda^2} = R^\mu_{\nu\rho\sigma} k^\nu k^\rho \xi^\sigma \tag{29}$$

where ξ^μ is the separation vector between two nearby rays, k^μ is the photon four-momentum, λ is an affine parameter along the light ray which has to be chosen such that it locally reproduces the proper distance and increases with decreasing time, $d\lambda = -cadt$, and $R^\mu_{\nu\rho\sigma}$ is the Riemann tensor of an homogeneous and isotropic universe.

- Now, we replace the covariant derivative by a partial derivative and introduce the optical tidal matrix, $\mathcal{T}$, defined as:

$$\mathcal{T}^a_b = -R_{\mu\nu\rho\sigma} k^\nu k^\rho e^{\mu a} e^\sigma_b \tag{30}$$

where $e^{\mu a}$ are the two spacelike vectors orthogonal to the photon four-momentum and the observer four-velocity.

Separation of light rays

- The geodesic deviation equation can be written as:

$$\frac{d^2\xi^a}{d\lambda^2} = \mathcal{T}^a_b \xi^b \tag{31}$$

where $a, b = 1, 2$ denote the two directions orthogonal to the photon four-momentum.

- The optical tidal matrix ($\mathcal{T}^{ab}$) can be decomposed into a Ricci and a Weyl part:

$$\mathcal{T} = \underbrace{\begin{pmatrix} \mathcal{R} & 0 \\ 0 & \mathcal{R} \end{pmatrix}}_{\text{Ricci focusing}} + \underbrace{\begin{pmatrix} -\Re(\mathcal{F}) & \Im(\mathcal{F}) \\ Im(\mathcal{F}) & \Re(\mathcal{F}) \end{pmatrix}}_{\text{Weyl focusing}} \tag{32}$$

where $\mathcal{R} = -\frac{1}{2} R_{\mu\nu} k^\mu k^\nu$ is the Ricci focusing term and $\mathcal{F} = -\frac{1}{2} C_{\alpha\beta\mu\nu} (\epsilon^*)^\alpha k^\beta (\epsilon^*)^\mu k^\nu$ is the complex Weyl focusing term, with $\epsilon = e_1 + i e_2$.

Separation of light rays

- Now we make two change of variables: i) we consider the comoving separation vector $\mathbf{x} = \frac{\xi}{a}$. ii) The affine parameter λ has to be chosen such that it locally reproduces the proper distance and increases with decreasing time, hence $d\lambda = -cadt$. We replace λ by the comoving distance χ, defined as $d\lambda = a^2 d\chi$. We obtain

$$\frac{d^2\mathbf{x}}{d\chi^2} = -K\mathbf{x} \tag{33}$$

where K is the curvature.

- The solution of this equation can be written as:

$$\mathbf{x}(\theta, \chi) = f_K(\chi)\theta \tag{34}$$

where $f_K(\chi)$ is the comoving angular diameter distance and θ is the angle between the two rays at the observer.

Separation of light rays in a perturbed Universe

- Let us now consider a universe with small density perturbations. Assuming that the newtonian potential of the perturbations is small $\Phi_\gamma < c^2$, the separation of two nearby light rays is described by:

$$\frac{d^2\mathbf{x}}{d\chi^2} = -K\mathbf{x} - \frac{2}{c^2}\Delta\left[\nabla_\perp \Phi_\gamma(\mathbf{x}(\theta,\chi))\right] \tag{35}$$

  where $\nabla_\perp$ is the gradient operator in the plane orthogonal to the photon trajectory and $\Phi_\gamma = \frac{\phi_N + \Psi_N}{2}$ is the Weyl potential.

- The notation on the right-hand side indicates that the difference of the perpendicular potential gradients has to be evaluated between the two light rays which have comoving separation $\mathbf{x}(\theta,\chi)$ at comoving distance χ from the observer.

- The boundary conditions are $\mathbf{x}(\theta,0) = 0$ and $\frac{d\mathbf{x}}{d\chi}(\theta,0) = \theta$, which means that the two rays start at the observer with an angular separation $\mathbf{x} = 0$ and that they set out into directions which differ by θ.

- To solve Equation 35 we construct a Green function which is a combination of either trigonometric or hyperbolic functions, depending on the curvature

$$G(\chi,\chi') = f_K(\chi-\chi') = \begin{cases} \frac{1}{\sqrt{K}}\sin(\sqrt{K}(\chi-\chi')) & K>0 \\ \chi-\chi' & K=0 \\ \frac{1}{\sqrt{|K|}}\sinh(\sqrt{|K|}(\chi-\chi')) & K<0 \end{cases} \quad (36)$$

- In such way, the solution of Equation 35 is:

$$\begin{aligned} \mathbf{x}(\theta,\chi) &= f_K(\chi)\theta - \frac{2}{c^2}\int_0^\chi d\chi' f_K(\chi-\chi') \\ &\quad \times\left[\nabla_\perp\Phi_\gamma(\mathbf{x}(\theta,\chi')) - \nabla_\perp\Phi_\gamma(0,\chi')\right] \\ &= f_K(\chi)\theta - \frac{2}{c^2}\int_0^\chi d\chi' f_K(\chi-\chi')\Delta\left[\nabla_\perp\Phi_\gamma(\mathbf{x}(\theta,\chi'))\right] \quad (37) \end{aligned}$$

- Assuming that the change in the comoving separation χ between two actual rays due light deflections in a gravitational potential is small comprared to the comoving separation of unperturbed rays:

$$\frac{|\mathbf{x}(\theta,\chi) - f_K(\chi)\theta|}{|\mathbf{x}(\theta,\chi)|} \ll 1 \tag{38}$$

- We can replace $\mathbf{x}(\theta,\chi')$ by $f_K(\chi')\theta$ in the argument of the potential in the integral. This is the Born approximation.
- The Born approximation allows us to replace the difference in the perpendicular potencial gradients by the perpendicular gradient of the potential difference:
- We rename the difference in the potential from $\Delta\Phi_\gamma$ to Φ_γ.
- The net deflection angle α at a distance χ between two rays is the difference between $\mathbf{x}(\theta,\chi)$ and $\mathbf{x}(\theta,\chi')$, divided by the comoving angular diameter distance to the source:

$$\alpha = \frac{\mathbf{x}(\theta,\chi) - f_K(\chi)\theta}{f_K(\chi)} = -\frac{2}{c^2}\int_0^\chi d\chi' \frac{f_K(\chi-\chi')}{f_K(\chi)} \nabla_\perp \Phi_\gamma(f_K(\chi')\theta,\chi') \tag{39}$$

The lensing four-point function

- For a fixed lens, $\tilde{T}(\hat{n}) = T(\hat{n} + \nabla\psi)$ couples different modes (flat sky, $\mathbf{L} = \boldsymbol{\ell}_1 + \boldsymbol{\ell}_2$):
$$\langle \tilde{T}(\boldsymbol{\ell}_1)\tilde{T}(\boldsymbol{\ell}_2)\rangle_{\rm CMB} = (2\pi)^2\delta_D(\boldsymbol{\ell}_1+\boldsymbol{\ell}_2)\, C^{TT}_{\ell_1} + f(\boldsymbol{\ell}_1,\boldsymbol{\ell}_2)\,\psi(\mathbf{L})$$
with $f(\boldsymbol{\ell}_1,\boldsymbol{\ell}_2) = (\mathbf{L}\cdot\boldsymbol{\ell}_1)\, C^{TT}_{\ell_1} + (\mathbf{L}\cdot\boldsymbol{\ell}_2)\, C^{TT}_{\ell_2}$.
- Averaging also over the lenses, the four-point function has a Gaussian (disconnected) part and a connected part, the trispectrum:
$$\langle \tilde{T}(\boldsymbol{\ell}_1)\tilde{T}(\boldsymbol{\ell}_2)\tilde{T}(\boldsymbol{\ell}_3)\tilde{T}(\boldsymbol{\ell}_4)\rangle_c = (2\pi)^2\delta_D(\boldsymbol{\ell}_1+\boldsymbol{\ell}_2+\boldsymbol{\ell}_3+\boldsymbol{\ell}_4)\,\mathcal{T}(\boldsymbol{\ell}_1,\boldsymbol{\ell}_2,\boldsymbol{\ell}_3,\boldsymbol{\ell}_4)$$
$$\boxed{\begin{aligned}\mathcal{T} = {} & C^{\psi\psi}_{|\boldsymbol{\ell}_1+\boldsymbol{\ell}_2|}\, f(\boldsymbol{\ell}_1,\boldsymbol{\ell}_2)\, f(\boldsymbol{\ell}_3,\boldsymbol{\ell}_4) + C^{\psi\psi}_{|\boldsymbol{\ell}_1+\boldsymbol{\ell}_3|}\, f(\boldsymbol{\ell}_1,\boldsymbol{\ell}_3)\, f(\boldsymbol{\ell}_2,\boldsymbol{\ell}_4) \\ & + C^{\psi\psi}_{|\boldsymbol{\ell}_1+\boldsymbol{\ell}_4|}\, f(\boldsymbol{\ell}_1,\boldsymbol{\ell}_4)\, f(\boldsymbol{\ell}_2,\boldsymbol{\ell}_3)\end{aligned}}$$
one term for each way of pairing the four modes through a lens.
- A quadratic estimator $\hat{\psi}(\mathbf{L}) \propto \int d^2\ell\, F\, \tilde{T}(\boldsymbol{\ell})\tilde{T}(\mathbf{L}-\boldsymbol{\ell})$ reconstructs the lensing map; its power spectrum is a four-point function of $\tilde{T}$:
$$\langle \hat{\psi}(\mathbf{L})\hat{\psi}^*(\mathbf{L}')\rangle = (2\pi)^2\delta_D(\mathbf{L}-\mathbf{L}')\left[C^{\psi\psi}_L + N^{(0)}_L + N^{(1)}_L\right]$$
where $N^{(0)}_L$ (the Gaussian part, present even without lensing) must be subtracted, and $N^{(1)}_L = \mathcal{O}(C^{\psi\psi})$ comes from the other pairings (Hu & Okamoto, ApJ 574, 566 (2002)).