Quantum Cosmology

2. Boundary Proposals, Inflation and Quantum Perturbations

Prof. Gustavo Vicente

Rio de Janeiro State University

1st Arica School of Cosmology and Gravitation

13/10/2026 – 16/10/2026



Outline

1. Original Boundary Proposals
2. Inflation
3. Quantum perturbations
4. Boundary Proposals Today
5. Conclusions and open questions

Lecture 1: Canonical Quantization and Minisuperspace

You recognize this:

$$E = \frac{p^2}{2m} + V \quad \Rightarrow \quad i\hbar \frac{\partial \psi(x,t)}{\partial t} = \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right] \psi(x,t).$$

$$\boxed{[\widehat{x}, \widehat{p}] = i\hbar.}$$

Canonical quantization:

$$\mathcal{H}_{\text{grav}} = \frac{\kappa p_a^2}{12a} + U(a) \quad \Rightarrow \quad \left[-\frac{\kappa \hbar^2}{12a} \frac{d^2}{da^2} + U(a) \right] \Psi(a) = 0.$$

$$\boxed{[\widehat{a}, \widehat{p}_a] = i\hbar}.$$

Tractable WDW equation $\Rightarrow$ **no unique solution**.

Why a boundary proposal?

$$\boxed{\widehat{\mathcal{H}}\Psi \approx 0}.$$

many admissible solutions $\Rightarrow$ state-selection condition.

"Quantum creation of the Universe" $\Rightarrow$ no initial classical spacetime

Classical spacetime $\Rightarrow$ must emerge at late times.

J. J. Halliwell, *Introductory Lectures on Quantum Cosmology*, arXiv:0909.2566. S. Gielen, *Quantum Cosmology*, arXiv:2309.01790.

Original Boundary Proposals

Closed de Sitter minisuperspace

Simplest model $\Rightarrow k = +1,\ \Lambda > 0$:

$$\boxed{\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + a^2\mathrm{d}\Omega_3^2, \qquad \mathrm{Vol}(S^3) = 2\pi^2, \qquad \kappa \equiv 8\pi G = 1, \qquad H_\Lambda^2 = \frac{\Lambda}{3}.}$$

FLRW Hamiltonian constraint:

$$\boxed{p_a^2 + U(a) \approx 0, \qquad U(a) \equiv 48\pi^4 a^2\left(3k - \Lambda a^2\right).}$$

For $k = 1$,

$$U(a) = 144\pi^4 a^2\left(1 - H_\Lambda^2 a^2\right), \qquad U(a) = 0 \quad \Rightarrow \quad a = 0, \qquad \boxed{a_t = H_\Lambda^{-1} = \sqrt{\frac{3}{\Lambda}}.}$$

$$\boxed{\begin{array}{rcll} 0 < a < a_t & \Rightarrow & U > 0, \quad p_a^2 < 0 & \text{Euclidean (forbidden) region,} \\ a = a_t & \Rightarrow & U = 0, \quad p_a = 0 & \text{turning point,} \\ a > a_t & \Rightarrow & U < 0, \quad p_a^2 > 0 & \text{Lorentzian region.} \end{array}}$$

Analytical solution for Ψ: Airy equation

For $\boxed{k = 1\,,\ \Lambda > 0}$:

$$\boxed{\left[\hbar^2\left(\frac{\mathrm{d}^2}{\mathrm{d}a^2} - \frac{1}{a}\frac{\mathrm{d}}{\mathrm{d}a}\right) - 48\pi^4\left(3a^2 - \Lambda a^4\right)\right]\Psi(a) = 0.}$$

For

$$q = a^2 \quad , \quad z(a) \equiv \left(\frac{12\pi^4}{\hbar^2}\right)^{1/3}\frac{3 - \Lambda a^2}{\Lambda^{2/3}},$$

we get

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}z^2} - z\Psi = 0 \quad \Rightarrow \quad \boxed{\Psi_{\text{exact}} = c_1\,\mathrm{Ai}(z) + c_2\,\mathrm{Bi}(z).}$$

$$\boxed{\begin{aligned} 0 < a < a_t \quad &\Rightarrow \quad z > 0 \quad \text{Euclidean region,} \\ a = a_t = H_\Lambda^{-1} \quad &\Rightarrow \quad z = 0 \quad \text{turning point,} \\ a > a_t \quad &\Rightarrow \quad z < 0 \quad \text{Lorentzian region.} \end{aligned}}$$

S. Gielen, *Quantum Cosmology*, in *Encyclopedia of Mathematical Physics*, 2nd ed., Vol. 1 (Elsevier, 2025), pp. 520–530, arXiv:2309.01790 [gr-qc].

WDW $\Rightarrow$ Path integral

$$\boxed{\widehat{H}\Psi = 0}$$ Boundary conditions specify the state.

Functional-integral representation

$$\boxed{\Psi_C[h_{ij}] = \int_C \mathcal{D}g\, e^{iS[g]}}$$

C: four-geometries matching h_{ij} on the final hypersurface.

Lapse variation $\Rightarrow$ WDW

$$\int_C \mathcal{D}g \underbrace{\frac{\delta S}{\delta N}}_{H} e^{iS[g]} = 0 \quad \Longrightarrow \quad \boxed{\widehat{H}\Psi = 0.}$$

Hartle & Hawking, Phys. Rev. D **28**, 2960 (1983).

Hartle–Hawking proposal

$$\Rightarrow \text{Path integral} \Rightarrow \left\{ \begin{array}{l} \text{compact (finite volume)} \\ \text{regular (nonsingular)} \end{array} \right\} \text{Euclidean geometries}$$

Euclidean region:

$$t = -i\tau \quad \Rightarrow \quad e^{iS_L/\hbar} \quad \Rightarrow \quad e^{-S_E/\hbar} \qquad \text{(Wick rotation)}.$$

$$\boxed{\Psi_{\rm HH}[h_{ij}] \sim \int_{g|_{\Sigma_t}=h} \mathcal{D}g\, e^{-S_E[g]/\hbar}.}$$

Euclidean geometry $\Rightarrow$ continues into Lorentzian one:

$$\underbrace{0 \leq a < a_t}_{\text{Euclidean}} \quad \Rightarrow \quad \underbrace{\Sigma_t : \ a = a_t}_{\textbf{final ``boundary''}} \quad \Rightarrow \quad \underbrace{a > a_t}_{\text{Lorentzian}} \ .$$

$$\Psi_{\rm HH} = \int \mathcal{D}g\, e^{-S_E[g]/\hbar} \underbrace{\simeq}_{\text{saddles}} \underbrace{\sum_s A_s e^{-S_E[g_s]/\hbar}}_{\text{sum over stationary configurations}} .$$

J. B. Hartle and S. W. Hawking, Phys. Rev. D **28** (1983) 2960–2975.

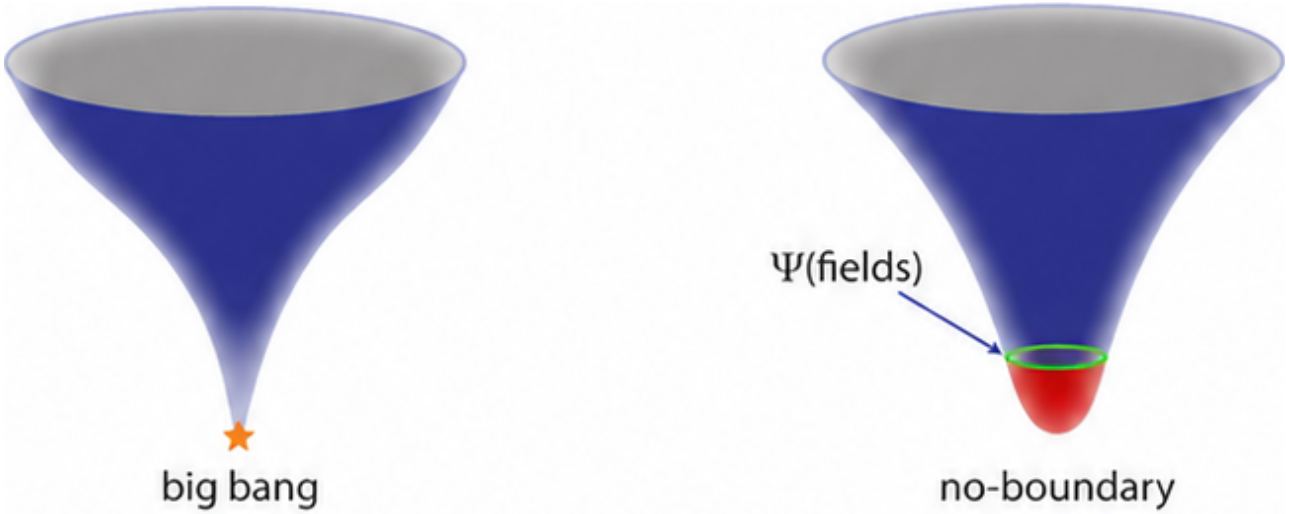
Regular no-boundary saddle (instanton)

$$\boxed{\mathrm{d}s_E^2 = N_E^2(\tau)\mathrm{d}\tau^2 + a_E^2(\tau)\mathrm{d}\Omega_3^2}.$$

$$\frac{\delta S_E}{\delta N_E} = 0 \xrightarrow{\boxed{N_E = 1}} \boxed{(a')^2 = 1 - H_\Lambda^2 a^2} \Rightarrow \boxed{a_E(\tau) = H_\Lambda^{-1}\sin(H_\Lambda \tau)}.$$

$$\text{Near } \tau = 0 \Rightarrow a_E(0) = 0\ ,\ a'_E(0) = 1.$$

$$\boxed{a_E = 0 \quad \Rightarrow \quad \textbf{smooth origin} \Rightarrow \textbf{not initial boundary}.}$$



Red region:
Euclidean geometry.

Green circle:
Matching surface.

Blue region:
Lorentzian universe.

Adapted from: Lehners, Phys. Rep. 1022 (2023), Fig. 1.

Euclidean saddle $\Rightarrow$ Lorentzian spacetime

Transition:

$$\boxed{\tau_t = \frac{\pi}{2H_\Lambda}}$$

Turning surface $\Rightarrow$ analytic continuation:

$$\boxed{\tau = \tau_t + it}.$$

$$\boxed{a_E(\tau_t + it) = H_\Lambda^{-1}\cosh(H_\Lambda t)} \Rightarrow \boxed{a_L(t) = H_\Lambda^{-1}\cosh(H_\Lambda t)}.$$

$$\boxed{\mathrm{d}s_L^2 = -\mathrm{d}t^2 + a_L^2(t)\mathrm{d}\Omega_3^2}.$$

Summary:

$$\boxed{\begin{aligned} \tau = 0 &\Rightarrow a_E = 0 \quad \text{regular pole,} \\ 0 < \tau < \tau_t &\Rightarrow \text{Euclidean hemisphere,} \\ \tau_t = \frac{\pi}{2H_\Lambda} &\Rightarrow a_E = H_\Lambda^{-1}, \quad a_E' = 0, \\ \tau = \tau_t + it &\Rightarrow a_L = H_\Lambda^{-1}\cosh(H_\Lambda t). \end{aligned}}$$

Euclidean solution $\Leftrightarrow$ Exact Airy solution

We had

$$\boxed{\Psi_{\rm exact} = c_1\,{\rm Ai}(z) + c_2\,{\rm Bi}(z).}$$

Along saddle $\Rightarrow$ Euclidean action:

$$\boxed{S_E(a) = -\hbar B + I(a)}.$$

with

$$I(a) \equiv \int_a^{a_t} |p_E(a')|{\rm d}a' \ , \ \boldsymbol{B \equiv \frac{I(0)}{\hbar} = \frac{12\pi^2}{\Lambda\hbar}}.$$

Path-integral $\Rightarrow$ for chosen saddle:

$$\boxed{\Psi_{\rm HH}(a) \sim e^{-S_E(a)/\hbar} = e^{B - I(a)/\hbar}.}$$

Matching $\Psi_{\rm exact}$ with $\Psi_{\rm HH}$:

$$\boxed{\Psi \propto N_0 \boldsymbol{e^B}\,{\rm Ai}(z(a)).}$$

Lorentzian $\Leftrightarrow$ Exact Airy and HH weight

Analogous calculation:

$$\boxed{S_L(a) = \frac{12\pi^2}{\Lambda}\left(\frac{\Lambda a^2}{3} - 1\right)^{3/2}.}$$

Airy solution:

$$\boxed{\Psi_{\rm HH} \propto \boldsymbol{e}^{\boldsymbol{B}}\left[\underbrace{e^{+i(S_L/\hbar+\pi/4)}}_{\text{contracting}} - \underbrace{e^{-i(S_L/\hbar+\pi/4)}}_{\text{expanding}}\right].}$$

$$\boxed{\text{HH Lorentzian region} \quad \Rightarrow \quad \text{expanding} + \text{contracting} \quad \Rightarrow \quad \text{standing WKB wave.}}$$

At turning point (our interest):

$$\Psi_{\rm HH}(a_t) = N_0 \boldsymbol{e}^{\boldsymbol{B}}\,\mathrm{Ai}(0) \propto \boldsymbol{e}^{\boldsymbol{B}}$$

Weight:

$$\boxed{\omega_{\rm HH} \equiv |\Psi_{\rm HH}(a_t)|^2 \propto \boldsymbol{e}^{\boldsymbol{2B}} = \exp\left(\frac{24\pi^2}{\Lambda\hbar}\right)} \quad \Rightarrow \quad \boxed{\text{favours smaller } \Lambda.}$$

"This is to say, less inflation".

Vilenkin proposal: tunneling condition and weight

Lorentzian solution:

$$\boxed{\Psi_{\rm V} \sim C_{\dot{a}<0} e^{+i(S_L/\hbar+\pi/4)} + C_{\dot{a}>0} e^{-i(S_L/\hbar+\pi/4)}.}$$

Tunneling from Euclidean region $\Rightarrow$ only expanding branch survives.

$$\boxed{C_{\dot{a}<0} = 0}.$$

Matching $\Psi_{\rm V}$ with $\Psi_{\rm exact}$:

$$\boxed{\Psi_{\rm T}(a) = e^{-B}\left[{\rm Ai}(z) + i\,{\rm Bi}(z)\right]}.$$

At turning point:

$$\Psi_{\rm T}(a_t) \propto e^{-B}\left[{\rm Ai}(0) + i\,{\rm Bi}(0)\right].$$

Weight:

$$\boxed{\omega_{\rm T} \equiv \left|\Psi_{\rm T}(a_t)\right|^2 \propto e^{-2B} = \exp\left(-\frac{24\pi^2}{\Lambda\hbar}\right).} \quad \Rightarrow \quad \boxed{\text{favours greater } \Lambda.}$$

Two boundary proposals

WDW equation + state-selection condition $\Rightarrow$ wave function of the Universe.

proposal	Lorentz branch	weight
Hartle–Hawking	expanding + contracting	$\omega_{\mathrm{HH}} \propto \exp\left(\frac{24\pi^2}{\Lambda\hbar}\right)$
Vilenkin	expanding only	$\omega_{\mathrm{T}} \propto \exp\left(-\frac{24\pi^2}{\Lambda\hbar}\right)$

Hartle–Hawking $\Rightarrow$ regular Euclidean origin,

Vilenkin $\Rightarrow$ Euclidean tunneling to outgoing expanding universe.

For both, no earlier Lorentzian spacetime.

Dynamical inflaton

Constant $\Lambda \Rightarrow$ exactly solvable $\Rightarrow$ no end of inflation

Introduce scalar field ϕ with $V(\phi)$:

Euclidean saddle $\Rightarrow$ if ϕ changes little:

$$\phi \simeq \phi_0, \qquad p_\phi \simeq 0, \qquad \boxed{\Lambda_{\rm eff}(\phi_0) = \kappa V(\phi_0)}.$$

de Sitter result $\Rightarrow$ to each candidate starting value ϕ_0:

boundary proposal $\Rightarrow$ weights ϕ_0 , $V(\phi) \Rightarrow$ controls later classical evolution.

S. W. Hawking, Nucl. Phys. B **239** (1984) 257–276. A. D. Linde, Lett. Nuovo Cim. **39** (1984) 401–405. A. Vilenkin, Phys. Rev. D **37** (1988) 888–897.

Relative weights $\Rightarrow$ initial field value

$$\boxed{B(\phi_0) = \frac{3}{16G^2V(\phi_0)\hbar}.}$$

Relative weights

$$\boxed{w_{\rm HH}(\phi_0) \propto e^{+3/[8G^2V(\phi_0)\hbar]} \quad , \quad w_{\rm T}(\phi_0) \propto e^{-3/[8G^2V(\phi_0)\hbar]}.}$$

Therefore,

$$\boxed{\begin{array}{rcl} \text{Hartle–Hawking} & \Rightarrow & \text{weights smaller initial } V(\phi_0), \\ \text{Vilenkin} & \Rightarrow & \text{weight larger initial } V(\phi_0). \end{array}}$$

Compares initial energy scales $\Rightarrow$ not amount of inflation (e-folds).

Weights $\Rightarrow$ not probabilities $\Rightarrow$ "preference".

J. J. Halliwell, arXiv:0909.2566. J.-L. Lehners, Phys. Rept. **1022** (2023) 1–82, arXiv:2303.08802.

From boundary proposal to inflation

Oscillatory region:

$$\boxed{\Psi_0(a,\phi) = A(a,\phi)e^{iS_0(a,\phi)/\hbar}.}$$

Leading WKB order$\Rightarrow$ WDW gives Hamilton–Jacobi:

$$\boxed{p_a = \partial_a S_0 \ , \ p_\phi = \partial_\phi S_0 \quad \Rightarrow \quad \text{classical Lorentzian solution } a(t), \phi(t).}$$

Sequence:

$$\boxed{\text{boundary proposal}} \Rightarrow \boxed{\text{relative weights}} \Rightarrow \boxed{\text{classical history selected by } S_0}$$

$$\Downarrow$$

$$\boxed{\text{test inflation using } V(\phi) \qquad \text{define and quantize perturbations on that background}}$$

Inflation $\Rightarrow$ Lorentzian histories emerging from it.

Later we come back to perturbations.

Inflation

Background setup

Equations of motion:

$$\boxed{H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\phi}^2 + V\right), \qquad \ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0.}$$

Chosen initial field value ϕ_0 and potential $V(\phi)$:

$$\boxed{\phi_0,\ V(\phi) \Rightarrow \text{evolution} \Rightarrow \text{inflate or not?}}$$

Not only inflate, but long enough:

$$\boxed{N_e(\phi_0) \equiv \ln\frac{a_{\rm end}}{a_i} = \int_{t_i}^{t_{\rm end}} H\,{\rm d}t \gtrsim \mathcal{O}(60)} \Rightarrow \boxed{\begin{array}{c}\text{primordial causal patch}\\ \Downarrow \\ \text{expands beyond our observable Universe}\\ \Downarrow \\ \text{horizon and flatness problems solved.}\end{array}}$$

A. H. Guth, Phys. Rev. D **23** (1981) 347–356. A. D. Linde, Phys. Lett. B **108** (1982) 389–393. A. Albrecht and P. J. Steinhardt, Phys. Rev. Lett. **48** (1982) 1220–1223. A. D. Linde, Phys. Lett. B **129** (1983) 177–181.

Slow roll inflation and exit

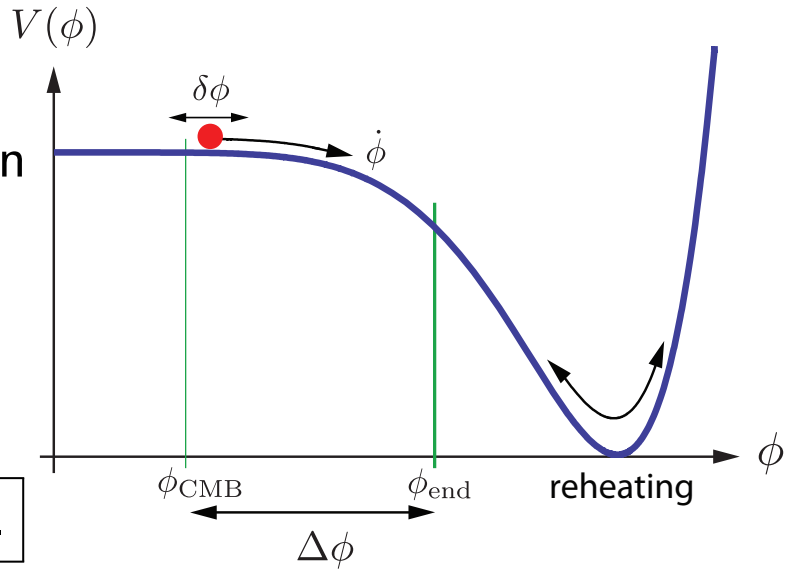
Slow-roll inflation:

$\boxed{V(\phi) \gg \dot{\phi}^2}$ $\Rightarrow$ potential-dominated inflation

Nearly flat $V \Rightarrow$ **slow-roll approximation**.

End of inflation:

$\boxed{\text{inflaton energy} \Rightarrow \text{transferred to other d.o.f.}}$



Source: Baumann, TASI Lectures (2009), Fig. 4.

How does inflaton field release its energy?

$$\begin{cases} \text{Reheating} & \Rightarrow \phi \text{ oscillates around } V_{\rm min}, \\ \text{Warm inflation} & \Rightarrow \text{dissipation during inflation}, \\ \text{Preheating} & \Rightarrow \text{resonant nonperturbative particle production}, \\ \text{False vacuum} & \Rightarrow \text{bubble nucleation and collisions}. \end{cases}$$

A. H. Guth, Phys. Rev. D **23** (1981) 347–356. A. D. Linde, Phys. Lett. B **108** (1982) 389–393. A. Albrecht and P. J. Steinhardt, Phys. Rev. Lett. **48** (1982) 1220–1223. A. D. Linde, Phys. Lett. B **129** (1983) 177–181.

Quantum perturbations

Quantum perturbations: semiclassical split

FLRW universe $\Rightarrow$ no seeds $\Rightarrow$ large-scale structure or CMB anisotropies:

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + \delta g_{\mu\nu}, \qquad \phi = \phi_0(t) + \delta\phi(t, \boldsymbol{x}).$$

homogeneous background $+$ small inhomogeneities $\Rightarrow$ linear perturbation theory.

Inflaton $\Rightarrow$ quantum field $\Rightarrow$ **practical** inflationary calculations $\Rightarrow$ split:

$$\begin{aligned} a(t),\ \phi_0(t) \quad &\Rightarrow \quad \text{classical homogeneous background,} \\ \delta g_{\mu\nu},\ \delta\phi \quad &\Rightarrow \quad \textbf{linear perturbations to be quantized.} \end{aligned}$$

J. M. Bardeen, Phys. Rev. D **22** (1980) 1882–1905. V. F. Mukhanov, JETP Lett. **41** (1985) 493–496. M. Sasaki, Prog. Theor. Phys. **76** (1986) 1036–1046.

Scalar perturbations and gauge freedom

At linear order $\Rightarrow$ $\boxed{\delta g_{\mu\nu} = \text{scalar} + \text{vector} + \text{tensor}}$ $\Rightarrow$ each sector evolves independently.

Density perturbations coupled to a single inflaton $\Rightarrow$ **considering only scalar sector**:

$$\boxed{\mathrm{d}s^2 = a^2(\eta)\Big\{ -(1+2A)\mathrm{d}\eta^2 + 2\partial_i B\,\mathrm{d}\eta\,\mathrm{d}x^i + \big[(1+2\psi)\delta_{ij} + 2\partial_i\partial_j E\big]\mathrm{d}x^i\mathrm{d}x^j \Big\}.}$$

$$\boxed{\phi(\eta, \boldsymbol{x}) = \phi_0(\eta) + \delta\phi(\eta, \boldsymbol{x}).}$$

$$\boxed{\begin{array}{rcl} A,\ B & \Rightarrow & \text{lapse and shift perturbations,} \\ \psi,\ E & \Rightarrow & \text{scalar perturbations of the spatial metric,} \\ \delta\phi & \Rightarrow & \text{inflaton perturbation.} \end{array}}$$

$$\boxed{\delta\phi \text{ and } \psi \Rightarrow \text{not separately gauge invariant.}}$$

Gauge-invariant combination: $\boxed{\mathcal{R} \equiv \psi - \dfrac{\mathcal{H}}{\phi_0'}\delta\phi, \ \text{(comoving curvature perturbation).}}$

One physical scalar perturbation

Comoving gauge (tensor perturbations omitted):

$$\boxed{\delta\phi = 0, \quad , \quad E = 0} \quad \Rightarrow \quad \boxed{\mathcal{R} = \psi}, \qquad h_{ij} = a^2(1+2\mathcal{R})\delta_{ij} = a^2 e^{2\mathcal{R}}\delta_{ij}.$$

Lapse and shift:

$$\frac{\delta S}{\delta A} = 0, \quad , \quad \frac{\delta S}{\delta B} = 0 \quad \Rightarrow \quad \boxed{A,\ B \text{ fixed by constraint equations.}}$$

Therefore,

$$\boxed{(A, B, \psi, E, \delta\phi) \xrightarrow{\text{gauge fixing + constraints}} \mathcal{R} \Rightarrow \text{one physical scalar degree of freedom.}}$$

Canonical Mukhanov–Sasaki variable:

$$\boxed{v \equiv z\mathcal{R}, \quad , \quad z \equiv a\frac{\phi_0'}{\mathcal{H}} = a\frac{\dot{\phi}_0}{H}.}$$

$$\boxed{\mathcal{R} \Rightarrow \text{geometrical perturbation}}$$

$$\boxed{v \Rightarrow \text{canonically normalized variable to be quantized.}}$$

Mukhanov–Sasaki action and modes

Expanding action to $2^{\rm nd}$ order in scalar perturbations $\Rightarrow$ gauge fixing + constraints $\Rightarrow$ quadratic action for v:

$$\boxed{S_2[v] = \frac{1}{2}\int \mathrm{d}\eta\, \mathrm{d}^3x \left[(v')^2 - (\nabla v)^2 + \frac{z''}{z}v^2\right].}$$

After fourier expansion:

$$\boxed{S_2 = \frac{1}{2}\int \mathrm{d}\eta\, \mathrm{d}^3k \left(|v_k'|^2 - \omega_k^2|v_k|^2\right), \qquad \omega_k^2 = k^2 - \frac{z''}{z}.}$$

Euler–Lagrange equations:

$$\boxed{v_k'' + \left(k^2 - \frac{z''}{z}\right)v_k = 0.}$$

$$\boxed{\text{each } k\text{-mode} = \text{time-dependent harmonic oscillator} \quad \Rightarrow \quad \text{quantize } v_k.}$$

Hubble crossing

Comoving Hubble radius:

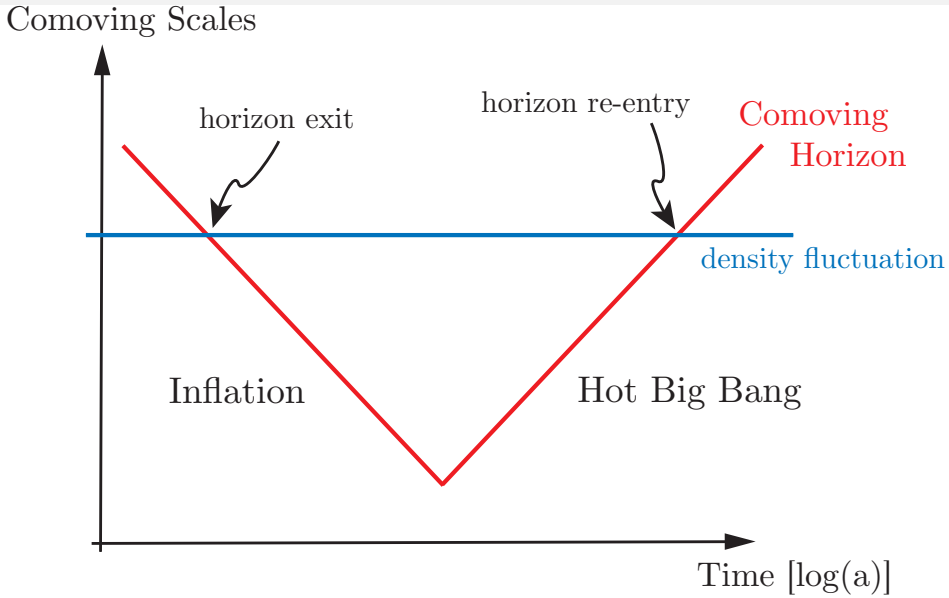
$$\boxed{r_H \equiv \frac{R_H}{a} = (aH)^{-1} \quad \Rightarrow \quad \text{decresces during inflation.}}$$

Slow-roll inflation:

$$\boxed{\omega_k^2 = k^2 - \frac{z''}{z}} \quad \Rightarrow \quad \boxed{\frac{z''}{z} \sim (aH)^2} \quad \Rightarrow \quad \text{compare } k^{-1} \text{ with } (aH)^{-1}.$$

$$\boxed{k^{-1} \ll (a_i H_i)^{-1}: \qquad \underbrace{k^{-1} \ll (aH)^{-1}}_{\text{inside}} \Rightarrow \underbrace{k^{-1} = (aH)^{-1}}_{\text{crossing}} \Rightarrow \underbrace{k^{-1} \gg (aH)^{-1}}_{\text{outside}}.}$$

Hubble exit, freezing and reentry



Source: Baumann, TASI Lectures (2009), Fig. 8.

Outside Hubble scale:

$$k \ll aH \quad \Rightarrow \quad k^2 \text{ term negligible} \quad \Rightarrow \quad \left(z^2 \mathcal{R}_k'\right)' \simeq 0.$$

$$\boxed{\mathcal{R}_k = C_1 + C_2 \int^{\eta} \frac{\mathrm{d}\tilde{\eta}}{z^2} \xrightarrow{z \text{ grows}} C_1} \Rightarrow \boxed{\mathcal{R}_k \simeq \text{constant.}}$$

Sub-Hubble modes	$k \gg aH$	$\Rightarrow$	$k^{-1} \ll (aH)^{-1} \Rightarrow$ oscillatory,
Modes at Hubble crossing	$k \simeq aH$	$\Rightarrow$	$k^{-1} \simeq (aH)^{-1} \Rightarrow \mathcal{R}_k$ freezes.
Super-Hubble modes	$k \ll aH$	$\Rightarrow$	$k^{-1} \gg (aH)^{-1} \Rightarrow \mathcal{R}_k$remains constant.

quantum fluctuation $\Rightarrow$ Hubble exit $\Rightarrow$ $\mathcal{R}_k$ preserved $\Rightarrow$ Hubble reentry $\Rightarrow$ CMB anisotropies and large-scale structure.

Quantization of scalar perturbations

Canonical quantization gives ($\hbar = 1$)

$$\boxed{[\hat{v}(\eta, \boldsymbol{x}), \hat{\pi}_v(\eta, \boldsymbol{y})] = i\delta^3(\boldsymbol{x} - \boldsymbol{y}), \qquad \pi_v = v'.}$$

After Fourier expansion:

$$[\hat{a}_{\boldsymbol{k}}, \hat{a}^\dagger_{\boldsymbol{q}}] = (2\pi)^3\delta^3(\boldsymbol{k} - \boldsymbol{q}), \qquad v_k v_k^{*'} - v_k^* v_k' = i.$$

each k is an independent quantum oscillator.

Bunch–Davies vacuum $\Rightarrow$ positive-frequency modes:

$$\boxed{\hat{a}_{\boldsymbol{k}}|0_{\rm BD}\rangle = 0, \qquad v_k(\eta) \longrightarrow \frac{e^{-ik\eta}}{\sqrt{2k}}.}$$

sub-Hubble quantum vacuum $\Rightarrow$ evolve through Hubble crossing $\Rightarrow$ frozen primordial mode.

T. S. Bunch and P. C. W. Davies, Proc. Roy. Soc. Lond. A **360** (1978) 117–134.

Quantum evolution through Hubble crossing

Quasi-de Sitter inflation:

$$\boxed{v_k'' + \left(k^2 - \frac{z''}{z}\right) v_k = 0} \Rightarrow a(\eta) \simeq -\frac{1}{H\eta}, \qquad \frac{z''}{z} \simeq \frac{2}{\eta^2}, \quad \Rightarrow \quad \boxed{|aH| \simeq \frac{1}{|\eta|}.}$$

Positive-frequency solution:

$$\boxed{v_k(\eta) = \frac{e^{-ik\eta}}{\sqrt{2k}} \left(1 - \frac{i}{k\eta}\right).}$$

$$\begin{aligned} |k\eta| \gg 1 \quad &\Rightarrow \quad k \gg aH \Rightarrow \text{quantum vacuum oscillation,} \\ |k\eta| \sim 1 \quad &\Rightarrow \quad k \sim aH \Rightarrow \text{amplification and squeezing,} \\ |k\eta| \ll 1 \quad &\Rightarrow \quad k \ll aH \Rightarrow v_k \propto z \Rightarrow \mathcal{R}_k = \frac{v_k}{z} \simeq \text{constant.} \end{aligned}$$

$$\boxed{* \Rightarrow \text{evaluation at Hubble crossing} \quad \Rightarrow \quad k = a_* H_*.}$$

Quantum perturbations

$$\Psi(q,\{v_k\}) \simeq A(q)e^{iS_0(q)/\hbar}\prod_k \psi_k(q,v_k), \qquad S_0 \Rightarrow \text{classical background } q^A(\eta).$$

Which perturbation vacuum is selected by the boundary proposal?

Each mode $\Rightarrow$ quantum oscillator:

$$i\hbar\frac{\partial\psi_k}{\partial\eta} = \frac{1}{2}\left[-\hbar^2\frac{\partial^2}{\partial v_k^2} + \omega_k^2 v_k^2\right]\psi_k \quad , \quad \omega_k^2(\eta) = k^2 - \frac{z''}{z}.$$

Vauum state

$$\psi_k(v_k,\eta) = \mathcal{A}_k(\eta)\exp\left[-\frac{\Omega_k(\eta)}{2\hbar}v_k^2\right]. \quad \Rightarrow \quad \boxed{\Omega_k = -i\frac{(f_k^*)'}{f_k^*}} \quad \Rightarrow \quad \boxed{f_k'' + \omega_k^2 f_k = 0.}$$

Solution:

$$\boxed{f_k = \alpha_k f_k^{\rm BD} + \beta_k (f_k^{\rm BD})^*} \quad , \quad f_k^{\rm BD} = \frac{1}{\sqrt{2k}}\left(1 - \frac{i}{k\eta}\right)e^{-ik\eta}.$$

Select $\boxed{\alpha_k = 1\ ,\ \beta_k = 0}$ and sub-Hubble modes $(k|\eta| \gg 1)$:

$$f_k^{\rm BD} \quad \Rightarrow \quad \frac{e^{-ik\eta}}{\sqrt{2k}} \quad \Rightarrow \quad \Omega_k \Rightarrow k \quad \Rightarrow \quad \boxed{\psi_k^{\rm BD}(v_k) \propto \exp\left(-\frac{k}{2\hbar}v_k^2\right).}$$

Hartle–Hawking: regular Euclidean perturbations

Analytic continuation:

$$\sigma = -i\eta \quad \Rightarrow \quad \boxed{\frac{d^2 v_k}{d\sigma^2} - k^2 v_k = 0.} \quad \Rightarrow \quad \boxed{v_k^E(\sigma) = A_k e^{k\sigma} + B_k e^{-k\sigma}.}$$

Regular Euclidean origin ($\sigma \to -\infty$) $\Rightarrow \boxed{B_k = 0.}$:

Euclidean action on this solution for IC $\boxed{v_k^E(\sigma) = v_{k0} e^{k\sigma}}$:

$$S_{E,k}^{\text{on-shell}} = \frac{k}{2} v_{k0}^2.$$

State induced on matching surface:

$$\boxed{\Psi_{k,\text{HH}}^E(v_{k0}) \propto \exp\left(-\frac{k}{2\hbar} v_{k0}^2\right)} \quad \Rightarrow \quad \text{ground-state Gaussian.}$$

R. Laflamme, Phys. Lett. B **198** (1987) 156–160.

Hartle–Hawking: Lorentzian Bunch–Davies state

Previous solution ($B_k = 0$):

$$\boxed{v_k^E(\sigma) = A_k e^{k\sigma}.}$$

Short mode,

$$\frac{e^{k(\sigma-\sigma_t)}}{\sqrt{2k}} \xrightarrow{\text{analytic continuation}} \frac{e^{-ik(\eta-\eta_t)}}{\sqrt{2k}}.$$

Absorb $e^{ik\eta_t}$ into normalization:

$$\boxed{v_k^L \xrightarrow{-k\eta\gg 1} \frac{e^{-ik\eta}}{\sqrt{2k}}} \quad \Rightarrow \quad \boxed{\alpha_k = 1, \qquad \beta_k = 0.}$$

Evolving with the complete Lorentzian Mukhanov–Sasaki equation:

$$\boxed{v_{k,\mathrm{HH}}^{\mathrm{BD}}(\eta) = \frac{1}{\sqrt{2k}}\left(1 - \frac{i}{k\eta}\right) e^{-ik\eta}.}$$

no-boundary regularity $\Rightarrow$ Euclidean Gaussian $\Rightarrow$ Lorentzian Bunch–Davies state.

Vilenkin: outgoing background and perturbation vacuum

Tunneling condition $\Rightarrow$ selects outgoing expanding Lorentzian background branch.

Outgoing condition $\Rightarrow$ does not determine perturbation state.

Additional condition $\Rightarrow$ Euclidean continuation of Lorentzian **positive-frequency condition**:

$$(\partial_\eta + ik)v_k^L = 0 \quad \xleftrightarrow{\partial_\eta = -i\partial_\sigma} \quad \boxed{(\partial_\sigma - k)v_k^E = 0.}$$

Applying it to $\boxed{v_k^E = A_k e^{k\sigma} + B_k e^{-k\sigma}}$:

$$(\partial_\sigma - k)v_k^E = -2kB_k e^{-k\sigma} = 0 \quad \Rightarrow \quad \boxed{B_k = 0}.$$

Vilenkin $\Rightarrow$ selects same Euclidean mode as Hartle-Hawking:

$$B_k = 0 \quad \Rightarrow \quad v_k^E \propto e^{k\sigma}.$$

Therefore,

$$\boxed{v_{k,\mathrm{T}}^{\mathrm{BD}}(\eta) = \frac{1}{\sqrt{2k}}\left(1 - \frac{i}{k\eta}\right)e^{-ik\eta}.}$$

T. Vachaspati and A. Vilenkin, Phys. Rev. D **37** (1988) 898–903.

Boundary Proposals Today

Euclidean $\Rightarrow$ complex saddle

de Sitter $\Rightarrow$ saddle represented by real Euclidean solution:

$$a_E(\tau) = H^{-1}\sin(H\tau).$$

With a dynamical scalar field $\Rightarrow$ proposals impose different conditions:

HH: saddle closes smoothly $\Rightarrow$ $a_E(0) = 0,\ a'_E(0) = 1,\ \phi(0) = \phi_0,\ \phi'(0) = 0.$

Vilenkin: outgoing expanding background

Proposal-specific conditions + late-time classicality $\Rightarrow$ generally require complex saddle:

$$a(\tau) \in \mathbb{C}, \qquad \phi(\tau) \in \mathbb{C}.$$

Late times $\Rightarrow$ negligible imaginary parts $\Rightarrow$ effectively real Lorentzian universe.

Euclidean origin $\Rightarrow$ complex evolution $\Rightarrow$ Lorentzian spacetime.

G. W. Lyons, Phys. Rev. D **46** (1992) 1546–1550. J. B. Hartle, S. W. Hawking and T. Hertog, Phys. Rev. D **77** (2008) 123537, arXiv:0803.1663. J. Feldbrugge, J.-L. Lehners and N. Turok, Phys. Rev. D **95** (2017) 103508, arXiv:1703.02076.

Complex saddle and classical spacetime

Complex saddle $\Rightarrow$ wave function has amplitude and phase:

$$\boxed{\Psi(q) \simeq \exp\left[-\frac{I(q)}{\hbar}\right] \exp\left[\frac{i}{\hbar} S(q)\right].}$$

Phase S becomes rapidly varying as the universe expands:

$$\boxed{|\nabla I| \ll |\nabla S|,}$$

Hamilton–Jacobi equation $\Rightarrow$ classical history:

$$p_A = \nabla_A S \quad \Rightarrow \quad a(t), \phi(t) \text{ approximately classical.}$$

J. B. Hartle, S. W. Hawking and T. Hertog, Phys. Rev. D **77** (2008) 123537, arXiv:0803.1663. O. Janssen, Class. Quant. Grav. **38** (2021) 095003, arXiv:2009.06282.

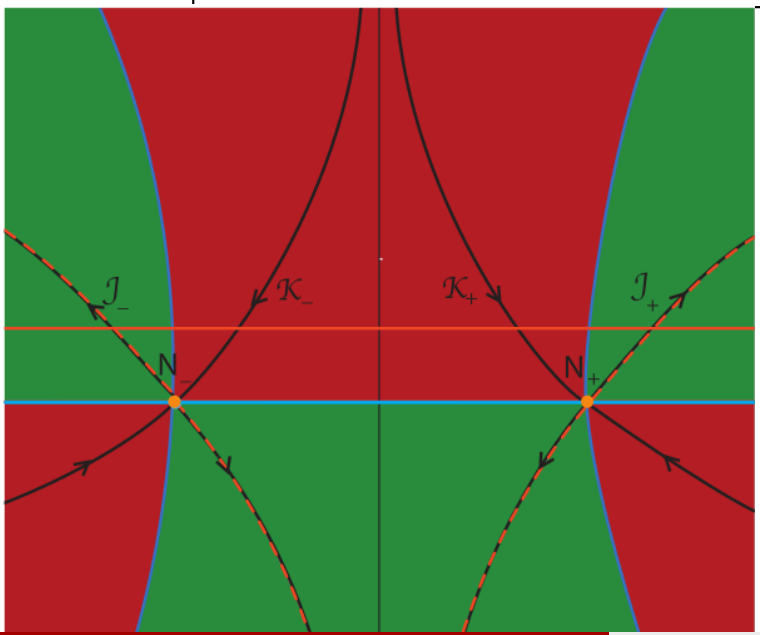
Contour Problem

Wave function contains an integral over the lapse:

$$\boxed{\Psi(a_\Sigma) = \int_{\mathcal{C}} dN\, e^{iS(N,a_\Sigma)/\hbar}.} \quad \Rightarrow \quad a = a_\Sigma \text{ at three-geometry } \Sigma.$$

Contour $\mathcal{C} \Rightarrow$ integration of lapse N in complex plane.

$$\boxed{\begin{array}{c} N_\sigma = \text{location of complex saddle,} \\ \mathcal{J}_\sigma = \text{steepest-descent contour through } N_\sigma, \\ \text{steepest descent} = \text{integrand magnitude decreases away from saddle.} \end{array}}$$



Adapted from Di Tucci–Lehners–Sberna, PRD 100(2019), Fig. 3. - HH example.

Orange horizontal line $\Rightarrow$ real Lorentzian contour $\mathcal{C}$;
Orange points $N_\pm \Rightarrow$ complex saddles.
Dashed curves $\mathcal{J}_\pm \Rightarrow$ steepest-descent thimbles;
$\mathcal{C}$ can be deformed into $\mathcal{J}_- + \mathcal{J}_+ \Rightarrow$ both HH saddles contribute.

Picard–Lefschetz theory

Before: on shell $(\delta S = 0) \Rightarrow$ Now: full path integral $\Rightarrow$ also off-shell geometries.

Saddle points are complex $\Rightarrow$ contour $\mathcal{C}$ determines which saddles contribute.

$$\text{different contours} \quad \Rightarrow \quad \begin{cases} \text{different combinations of saddles} \\ \text{different wave functions.} \end{cases}$$

The contour is part of the prescription because it selects which saddles contribute.

$$\frac{\delta S[q,N]}{\delta q} = 0 \quad \Rightarrow \quad \bar{q}(t;N) \quad \Rightarrow \quad S_{\text{eff}}(N) = S[\bar{q}(t;N),N] \quad \Rightarrow \quad \frac{\partial S_{\text{eff}}}{\partial N} = 0 \quad \Rightarrow \quad \{N_\sigma\}$$

$$\{N_\sigma\} + \mathcal{C} \quad \xrightarrow{\text{Picard–Lefschetz}} \quad \{N_\sigma \text{ that contribute}\}.$$

J. J. Halliwell and J. Louko, Phys. Rev. D **39** (1989) 2206. J. Feldbrugge, J.-L. Lehners and N. Turok, Phys. Rev. D **95** (2017) 103508, arXiv:1703.02076; Phys. Rev. Lett. **119** (2017) 171301, arXiv:1705.00192.

Hartle–Hawking: boundary conditions in full path integral

Before $\Rightarrow$ Saddle level on q, not N $\Rightarrow$ no-boundary geometry satisfies

$$\boxed{\begin{array}{c} a_E(0) = 0 \quad \text{(compactness)}, \\ a'_E(0) = 1 \quad \text{(regularity)}. \end{array}}$$

Full path integral $\Rightarrow$ both cannot be fixed $\Uparrow$.

$$\text{One implementation} \Rightarrow \boxed{\text{regularity: } a'_E(0) = 1.}$$

At saddles of full path integral: Unstable saddles removed $\Rightarrow$ Gaussian perturbations.

$$\text{From } \boxed{a'_E(0) = 1} \Rightarrow \frac{\delta S_E}{\delta N_E} = 0 \Rightarrow [a'_E(0)]^2 = 1 - H_\Lambda^2 a_E^2(0) \Rightarrow \boxed{a_E(0) = 0}.$$

$$\boxed{\text{regularity imposed} \quad \Rightarrow \quad \text{compactness emerges} \left(\text{from } \frac{\partial S_{\text{eff}}}{\partial N} = 0 \right).}$$

A. Di Tucci and J.-L. Lehners, Phys. Rev. Lett. **122** (2019) 201302, arXiv:1903.06757. A. Di Tucci, J.-L. Lehners and L. Sberna, Phys. Rev. D **100** (2019) 123543, arXiv:1911.06701.

Vilenkin: boundary conditions in full path integral

Tunneling wave function satisfies:

$$\boxed{\text{outgoing expanding background } + \text{ positive-frequency perturbations.}}$$

Full path integral $\Rightarrow$ outgoing condition fixes background $\Rightarrow$ does not fix perturbation state.

One implementation $\Rightarrow$ $\boxed{\text{add perturbation boundary term } \Rightarrow \text{ stable perturbations}}$

Euclidean form:

$$\boxed{S^{\rm bdy}_{E,k} = \frac{k}{2}\left[v^E_k(\sigma_i)\right]^2} \xrightarrow{\delta(S_{E,k}+S^{\rm bdy}_{E,k})=0} (\partial_\sigma - k)v^E_k = 0 \Rightarrow B_k = 0 \Rightarrow \text{Bunch–Davies.}$$

At saddles of full path integral:

Outgoing complex saddle + positive frequency + Gaussian perturbations.

$$\boxed{\text{outgoing background } + \text{ perturbation boundary term} \Rightarrow \text{stable perturbations.}}$$

A. Vilenkin and M. Yamada, Phys. Rev. D **98** (2018) 066003, arXiv:1808.02032; Phys. Rev. D **99** (2019) 066010, arXiv:1812.08084.

Conclusions and open questions

Conclusions and open questions

- No general principle $\Rightarrow$ uniquely selects contour $\mathcal{C}$.
- In studied de Sitter minisuperspace models, the *regular-momentum condition* for HH and the *perturbation boundary term* for Vilenkin give stable perturbations $\Rightarrow$ uniqueness and naturalness remain open.
- Results mainly in de Sitter minisuperspace + linear perturbations
- How should the probabilities for inflation be normalized?

A. Di Tucci and J.-L. Lehners, Phys. Rev. Lett. **122** (2019) 201302, arXiv:1903.06757. A. Di Tucci, J.-L. Lehners and L. Sberna, Phys. Rev. D **100** (2019) 123543, arXiv:1911.06701. A. Vilenkin and M. Yamada, Phys. Rev. D **98** (2018) 066003, arXiv:1808.02032; Phys. Rev. D **99** (2019) 066010, arXiv:1812.08084. J. J. Halliwell, J. B. Hartle and T. Hertog, Phys. Rev. D **99** (2019) 043526, arXiv:1812.01760. C. Jonas, J.-L. Lehners and V. Meyer, Phys. Rev. D **105** (2022) 043529, arXiv:2112.07986.

Thank you!

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