Modified Gravity in Metric-Affine and Teleparallel Frameworks

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Lecture 1: MAG: Geometry

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Motivation

- The metric is the geometric structure defining intervals and causal cones
- The linear or affine connection is the structure defining parallel transport
- What if the relation between them is determined by field equations rather than imposed in advance? $\longrightarrow$ Metric-affine gravity
- Gauge formulation of gravity, dark sectors in cosmology
- Linear connection couples to hypermomentum, a matter source accounting for microscopic properties of matter (spin, shear, and dilation)

Stress-energy and hypermomentum

- Matter responses:

$$\Theta_{\mu\nu} := -\frac{2}{\sqrt{-g}}\frac{\delta S_{\rm m}}{\delta g^{\mu\nu}}, \qquad \Delta_\lambda{}^{\mu\nu} := -\frac{2}{\sqrt{-g}}\frac{\delta S_{\rm m}}{\delta \Gamma^\lambda{}_{\mu\nu}}. \tag{1}$$

  Writing $\Delta_{\lambda\mu\nu} := g_{\mu\alpha}g_{\nu\beta}\Delta_\lambda{}^{\alpha\beta}$, we get three independent components of the hypermomentum:

- dilation

$$\Delta_\nu := g^{\lambda\mu}\Delta_{\lambda\mu\nu}, \qquad {}^{(\rm dil)}\Delta_{\lambda\mu\nu} = \frac{1}{4}g_{\lambda\mu}\Delta_\nu. \tag{2}$$

- spin

$$\tau_{\lambda\mu\nu} := \Delta_{[\lambda\mu]\nu}. \tag{3}$$

- shear

$$\Sigma_{\lambda\mu\nu} := \Delta_{(\lambda\mu)\nu} - \frac{1}{4}g_{\lambda\mu}\Delta_\nu, \qquad g^{\lambda\mu}\Sigma_{\lambda\mu\nu} = 0. \tag{4}$$

Physical interpretation of hypermomentum

The hypermomentum describes how the internal structure of matter couples to the independent connection. Its three parts distinguish orientation, scale and shape.

- Spin, $\tau_{\lambda\mu\nu}$: intrinsic angular momentum, such as the spin of a Dirac field. In Einstein–Cartan gravity, this current sources torsion.
- Dilation, Δ_ν: an intrinsic scale current, associated with uniform rescalings of the local frame. It couples to the trace of the connection, associated with Weyl nonmetricity.
- Shear, $\Sigma_{\lambda\mu\nu}$: an intrinsic deformation current, associated with changes of shape that preserve volume. It couples to the symmetric, trace-free part of the connection, associated with traceless nonmetricity.

Conventions

- Metric signature mostly positive $(-,+,+,+)$, $\quad \mu,\nu = 0,1,2,3.$
- Derivative index: last.

$$\nabla_\nu V^\lambda = \partial_\nu V^\lambda + \Gamma^\lambda{}_{\mu\nu} V^\mu$$

- $T^\lambda{}_{\mu\nu} = 2\Gamma^\lambda{}_{[\mu\nu]}, \quad Q_{\rho\mu\nu} = +\nabla_\rho g_{\mu\nu}.$
- (Anti)symmetrization convention

$$A_{(\mu\nu)} = \tfrac{1}{2}(A_{\mu\nu} + A_{\nu\mu}), \qquad A_{[\mu\nu]} = \tfrac{1}{2}(A_{\mu\nu} - A_{\nu\mu})$$

- $\mathring{\Gamma}, \mathring{\nabla}, \mathring{R}$: computed with Levi-Civita connection.

Curvature

- The curvature tensor of the most general linear connection

$$R^{\sigma}{}_{\rho\mu\nu} = \partial_\mu \Gamma^{\sigma}{}_{\rho\nu} - \partial_\nu \Gamma^{\sigma}{}_{\rho\mu} + \Gamma^{\sigma}{}_{\lambda\mu}\Gamma^{\lambda}{}_{\rho\nu} - \Gamma^{\sigma}{}_{\lambda\nu}\Gamma^{\lambda}{}_{\rho\mu} \tag{5}$$

- The only symmetry present: $R^{\sigma}{}_{\rho\mu\nu} = -R^{\sigma}{}_{\rho\nu\mu}$.
- The three independent rank-two metric contractions are the Ricci tensor, the co-Ricci tensor, and the homothetic curvature:

$$R_{\mu\nu} := R^{\lambda}{}_{\mu\lambda\nu} = g^{\alpha\beta} R_{\alpha\mu\beta\nu}. \tag{6}$$

$$\check{R}_{\mu\nu} := g^{\alpha\beta} R_{\mu\alpha\beta\nu}. \tag{7}$$

$$\mathcal{H}_{\mu\nu} := R^{\lambda}{}_{\lambda\mu\nu} = g^{\alpha\beta} R_{\alpha\beta\mu\nu}. \tag{8}$$

Their symmetry properties are

$$\mathcal{H}_{\mu\nu} = -\mathcal{H}_{\nu\mu}, \qquad R_{[\mu\nu]} \neq 0, \qquad \check{R}_{[\mu\nu]} \neq 0 \quad \text{in general.} \tag{9}$$

Curvature

- The remaining single metric contractions are

$$g^{\alpha\nu}R_{\alpha\beta\mu\nu} = -R_{\beta\mu}, \qquad g^{\beta\nu}R_{\alpha\beta\mu\nu} = -\check{R}_{\alpha\mu}, \qquad g^{\mu\nu}R_{\alpha\beta\mu\nu} = 0. \tag{10}$$

- There is only one independent scalar obtained by contracting a single curvature tensor with the metric:

$$R := g^{\mu\nu}R_{\mu\nu} = g^{\alpha\mu}g^{\beta\nu}R_{\alpha\beta\mu\nu}. \tag{11}$$

  The other scalar traces satisfy

$$g^{\mu\nu}\check{R}_{\mu\nu} = -R, \qquad g^{\mu\nu}\mathcal{H}_{\mu\nu} = 0. \tag{12}$$

- With our nonmetricity convention $Q_{\rho\mu\nu} = +\nabla_\rho g_{\mu\nu}$, the homothetic curvature also satisfies

$$\mathcal{H}_{\mu\nu} = 2\partial_{[\mu}\Gamma^{\lambda}{}_{|\lambda|\nu]} = -\partial_{[\mu}Q_{\nu]}, \qquad Q_\mu := Q_{\mu\alpha}{}^{\alpha}. \tag{13}$$

Curvature

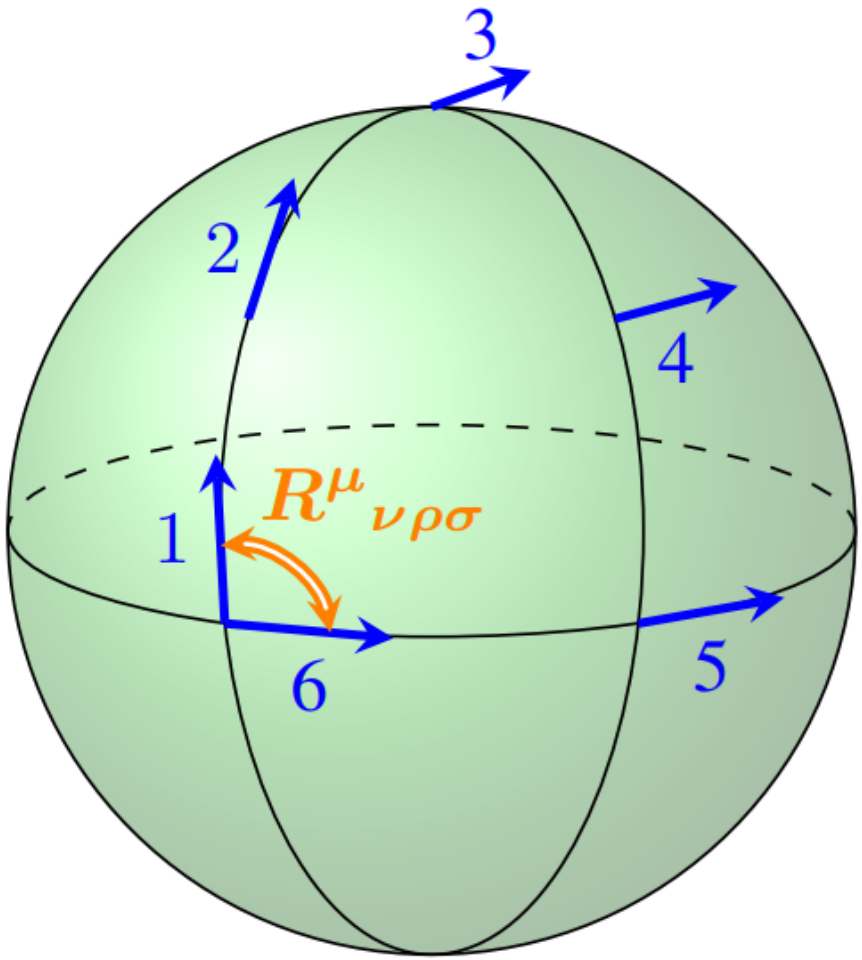


Figure: 3D representation of a 4D spacetime with only curvature.
Source: Laur Järv, Manuel Hohmann, Margus Saal, ("Laiendatud geomeetrilised gravitatsiooniteooriad," Eesti Vabariigi preemiad)

Torsion

- Our convention for the antisymmetric part of the linear connection will be

$$T^{\lambda}{}_{\mu\nu} := \Gamma^{\lambda}{}_{\mu\nu} - \Gamma^{\lambda}{}_{\nu\mu}$$

- We can work with the torsion tensor as

$$T_{\lambda\mu\nu} = g_{\lambda\alpha} T^{\alpha}{}_{\mu\nu}, \qquad T_{\lambda\mu\nu} = -T_{\lambda\nu\mu}$$

- There is only one independent trace

$$T_{\mu} := T^{\lambda}{}_{\mu\lambda}$$

- Due to its antisymmetry in last two indices, it has $4 \times 6 = 24$ independent components.

Torsion

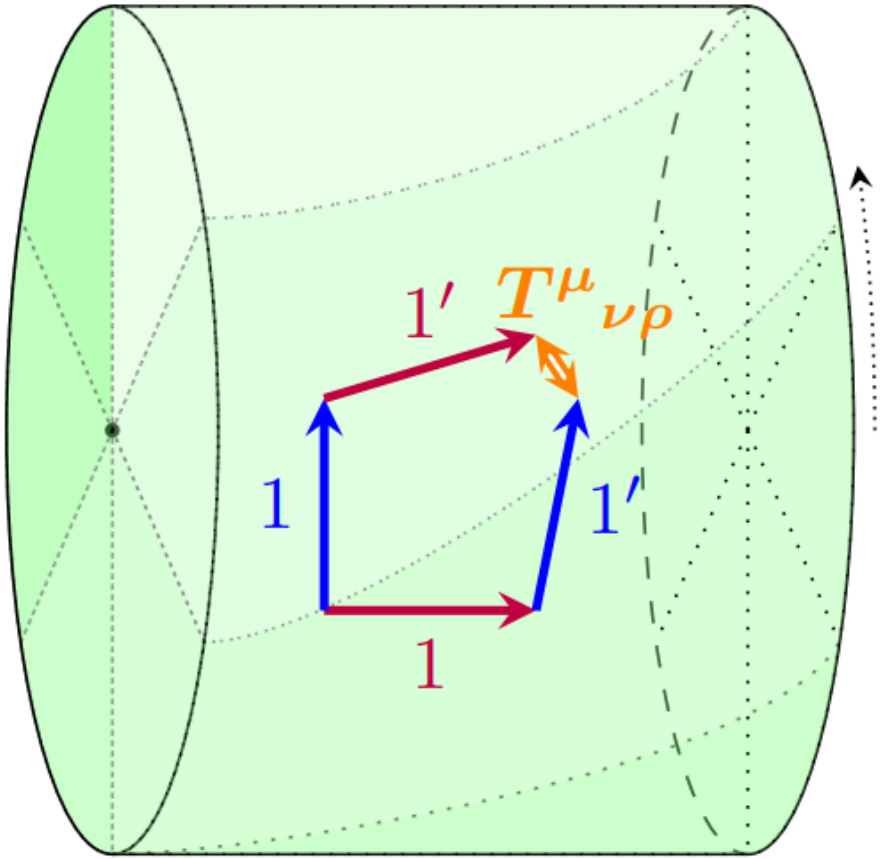


Figure: 3D representation of a 4D spacetime with only torsion.
Source: Laur Järv, Manuel Hohmann, Margus Saal, ("Laiendatud geomeetrilised gravitatsiooniteooriad," Eesti Vabariigi preemiad)

Nonmetricity

- The nonmetricity tensor is defined as

$$Q_{\rho\mu\nu} := \nabla_\rho g_{\mu\nu} = \partial_\rho g_{\mu\nu} - \Gamma^\beta{}_{\mu\rho} g_{\beta\nu} - \Gamma^\beta{}_{\nu\rho} g_{\mu\beta} \tag{14}$$

- Due to the symmetry $Q_{\rho\mu\nu} = Q_{\rho\nu\mu}$, it has $4 \times 10 = 40$ independent components.
- It has two independent traces

$$Q_\mu := Q_{\mu\nu}{}^\nu, \qquad \tilde{Q}_\mu := Q_{\nu\mu}{}^\nu \tag{15}$$

- $Q_{\rho\mu\nu} = 0$: metric compatibility.

Nonmetricity

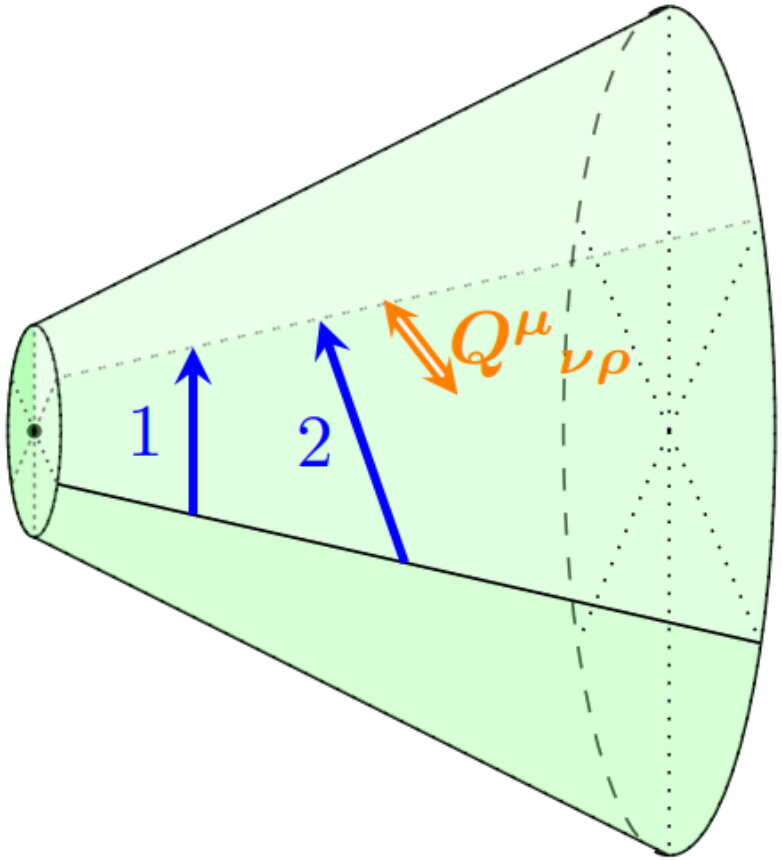


Figure: 3D representation of a 4D spacetime with only nonmetricity
Source: Laur Järv, Manuel Hohmann, Margus Saal, (“Laiendatud geomeetrilised gravitatsiooniteooriad,” Eesti Vabariigi preemiad)

Connection decomposition

- The most general linear connection can be decomposed as

$$\Gamma^{\lambda}{}_{\mu\nu} = \mathring{\Gamma}^{\lambda}{}_{\mu\nu} + K^{\lambda}{}_{\mu\nu} + L^{\lambda}{}_{\mu\nu}$$

- Levi-Civita part:

$$\mathring{\Gamma}^{\lambda}{}_{\mu\nu} = \frac{1}{2} g^{\lambda\beta} \left(\partial_{\mu} g_{\beta\nu} + \partial_{\nu} g_{\beta\mu} - \partial_{\beta} g_{\mu\nu} \right)$$

- Distortion:

$$N^{\lambda}{}_{\mu\nu} := \Gamma^{\lambda}{}_{\mu\nu} - \mathring{\Gamma}^{\lambda}{}_{\mu\nu} = K^{\lambda}{}_{\mu\nu} + L^{\lambda}{}_{\mu\nu}$$

- K and L are so-called contortion and disformation tensors, respectively
- N or K, L tensors span $24 + 40 = 64$ independent components, as expected.

Contortion and disformation tensors

- We define the contortion tensor as

$$K^{\lambda}{}_{\mu\nu} = \frac{1}{2} g^{\lambda\beta} \left(-T_{\mu\beta\nu} - T_{\nu\beta\mu} + T_{\beta\mu\nu} \right)$$

- It satisfies the properties $K_{\lambda\mu\nu} = -K_{\mu\lambda\nu}$ and $T^{\lambda}{}_{\mu\nu} = 2K^{\lambda}{}_{[\mu\nu]}$, also $K^{\lambda}{}_{(\mu\nu)} \neq 0$ in general.
- The disformation tensor is defined as

$$L^{\lambda}{}_{\mu\nu} = \frac{1}{2} g^{\lambda\beta} \left(-Q_{\mu\beta\nu} - Q_{\nu\beta\mu} + Q_{\beta\mu\nu} \right).$$

- It satisfies $Q_{\rho\mu\nu} = -2L_{(\mu\nu)\rho} = -2N_{(\mu\nu)\rho}$ and $L^{\lambda}{}_{\mu\nu} = L^{\lambda}{}_{\nu\mu}$,, then $L^{\lambda}{}_{[\mu\nu]} = 0$.

Identities

- For a metric-compatible connection,

$$Q_{\rho\mu\nu} = 0 \quad \Longrightarrow \quad \mathcal{H}_{\mu\nu} = 0, \qquad \check{R}_{\mu\nu} = -R_{\mu\nu}. \tag{16}$$

- For a torsion-free connection, the first Bianchi identity gives

$$T^{\lambda}{}_{\mu\nu} = 0 \quad \Longrightarrow \quad \mathcal{H}_{\mu\nu} = 2R_{[\mu\nu]}. \tag{17}$$

- For the Levi-Civita connection,

$$R_{\mu\nu} = \mathring{R}_{\mu\nu} = \mathring{R}_{\nu\mu}, \qquad \check{R}_{\mu\nu} = -\mathring{R}_{\mu\nu}, \qquad \mathcal{H}_{\mu\nu} = 0. \tag{18}$$

Geometric sectors

- The Levi-Civita connection is recovered when $T^{\lambda}{}_{\mu\nu} = 0$, $Q_{\rho\mu\nu} = 0$. This describes standard Riemannian geometry.
- Riemann–Cartan geometry has curvature and torsion, therefore there $Q_{\rho\mu\nu} = 0$.
- Metric teleparallel gravity satisfies

$$R^{\alpha}{}_{\beta\mu\nu} = 0, \qquad Q_{\rho\mu\nu} = 0$$

- Symmetric teleparallel satisfies

$$R^{\alpha}{}_{\beta\mu\nu} = 0, \qquad T^{\lambda}{}_{\mu\nu} = 0$$

- The most general curvature tensor $R^{\alpha}{}_{\beta\mu\nu} = 0$ allows $\mathring{R}^{\alpha}{}_{\beta\mu\nu} \neq 0$.

Geometric sectors

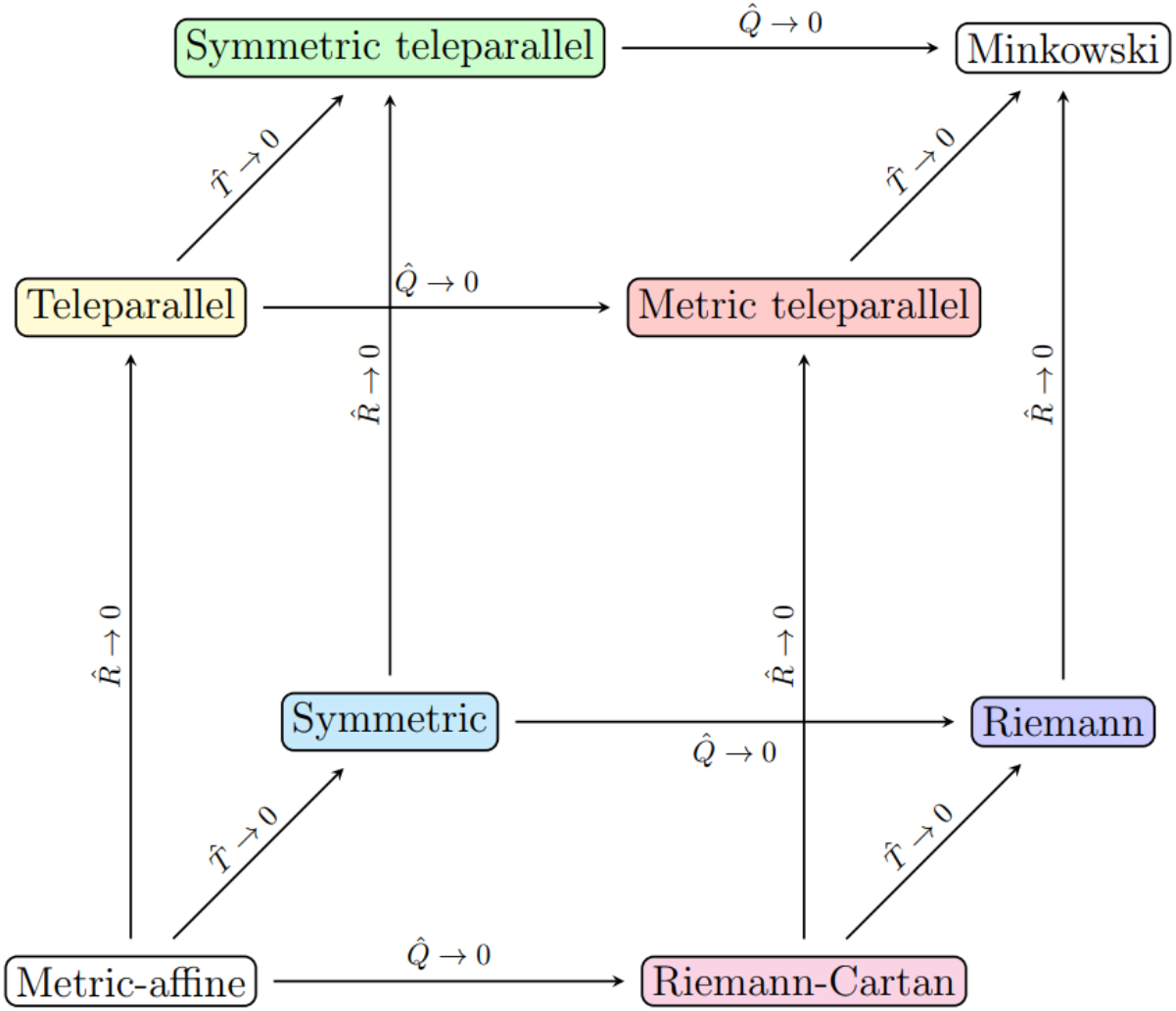


Figure: Possible geometric sectors based on values of R, T and Q. Taken from 2106.13793

Curvature tensor and distortion

- If the linear connection is split in terms of distortion N as

$$\Gamma^{\rho}{}_{\sigma\nu} = \mathring{\Gamma}^{\rho}{}_{\sigma\nu} + N^{\rho}{}_{\sigma\nu}$$

  then its associated curvature tensor is

$$R^{\rho}{}_{\sigma\mu\nu} = \mathring{R}^{\rho}{}_{\sigma\mu\nu} + \mathring{\nabla}_{\mu} N^{\rho}{}_{\sigma\nu} - \mathring{\nabla}_{\nu} N^{\rho}{}_{\sigma\mu} + N^{\rho}{}_{\lambda\mu} N^{\lambda}{}_{\sigma\nu} - N^{\rho}{}_{\lambda\nu} N^{\lambda}{}_{\sigma\mu}$$

- This can also be written as

$$R^{\rho}{}_{\sigma\mu\nu} = \mathring{R}^{\rho}{}_{\sigma\mu\nu} + 2\mathring{\nabla}_{[\mu} N^{\rho}{}_{|\sigma|\nu]} + 2N^{\rho}{}_{\lambda[\mu} N^{\lambda}{}_{|\sigma|\nu]}$$

Ricci and scalar curvature contractions

- The first and third contraction of the curvature tensor gives the Ricci curvature tensor

$$R_{\mu\nu} := R^{\lambda}{}_{\mu\lambda\nu}$$

- It therefore corresponds to

$$R_{\mu\nu} = \mathring{R}_{\mu\nu} + \mathring{\nabla}_{\lambda} N^{\lambda}{}_{\mu\nu} - \mathring{\nabla}_{\nu} N^{\lambda}{}_{\mu\lambda} + N^{\lambda}{}_{\rho\lambda} N^{\rho}{}_{\mu\nu} - N^{\lambda}{}_{\rho\nu} N^{\rho}{}_{\mu\lambda}$$

- We know that $\mathring{R}_{[\mu\nu]} = 0$, but $R_{[\mu\nu]} \neq 0$ in general.
- The scalar curvature is $R = g^{\mu\nu} R_{\mu\nu}$, which is explicitly

$$R = \mathring{R} + \mathring{\nabla}_{\lambda}\left(g^{\mu\nu} N^{\lambda}{}_{\mu\nu} - g^{\lambda\mu} N^{\nu}{}_{\mu\nu}\right) + g^{\mu\nu} N^{\lambda}{}_{\rho\lambda} N^{\rho}{}_{\mu\nu} - g^{\mu\nu} N^{\lambda}{}_{\rho\nu} N^{\rho}{}_{\mu\lambda}$$

Identities for the distortion

- Two selected traces for the distortion are

$$A^\mu := g^{\rho\nu} N^\mu{}_{\rho\nu}, \qquad B^\mu := g^{\mu\rho} N^\lambda{}_{\rho\lambda}$$

- They are related to torsion and nonmetricity traces as

$$A_\mu = -\hat{Q}_\mu + \frac{1}{2} Q_\mu - T_\mu, \qquad B_\mu = -\frac{1}{2} Q_\mu + T_\mu$$

- Therefore,

$$A^\mu - B^\mu = Q^\mu - \tilde{Q}^\mu - 2T^\mu$$

- The curvature scalar can then be alternatively written as

$$R = \mathring{R} + \mathring{\nabla}_\mu (A^\mu - B^\mu) + B_\mu A^\mu - N_{\alpha\mu\nu} N^{\mu\nu\alpha}$$

Quadratic distortion

- If we define a function of the distortion as

$$\mathcal{U}(N) := B_\mu A^\mu - N_{\alpha\mu\nu} N^{\mu\nu\alpha}$$

and remember that distortion is contortion plus disformation ($N = K + L$), then we recover the well-known torsion and nonmetricity scalars $\mathcal{T}$ and $\mathcal{Q}$ as

$$\mathcal{U}(K) = \mathcal{T}, \qquad \mathcal{U}(L) = -Q \tag{19}$$

- Then the distortion scalar corresponds to

$$\mathcal{U}(K + L) = \mathcal{T} - Q + \mathcal{C}$$

where

$$\mathcal{C} = -Q_{\alpha\mu\nu} T^{\nu\alpha\mu} + T_\mu (Q^\mu - \tilde{Q}^\mu)$$

is a term explicitly mixing torsion and nonmetricity. It is clear that if $T^\lambda{}_{\mu\nu} = 0$ or $Q_{\rho\mu\nu} = 0$, then $\mathcal{C} = 0$.

Torsion, nonmetricity and general scalars

- Meanwhile, the torsion scalar is

$$\mathcal{T} := \frac{1}{4}T_{\lambda\mu\nu}T^{\lambda\mu\nu} + \frac{1}{2}T_{\lambda\mu\nu}T^{\nu\mu\lambda} - T_\mu T^\mu$$

- The nonmetricity scalar is

$$Q := -\frac{1}{4}Q_{\lambda\mu\nu}Q^{\lambda\mu\nu} + \frac{1}{2}Q_{\lambda\mu\nu}Q^{\mu\nu\lambda} + \frac{1}{4}Q_\mu Q^\mu - \frac{1}{2}Q_\mu \tilde{Q}^\mu$$

- General curvature scalar

$$R = \mathring{R} + \mathcal{T} - Q + \mathcal{C} + \mathring{\nabla}_\mu(Q^\mu - \tilde{Q}^\mu - 2T^\mu)$$

or

$$R = \mathring{R} + \mathcal{T} - Q + \mathcal{C} - B_T - B_Q$$

after defining the boundary terms $B_T := 2\mathring{\nabla}_\mu T^\mu$, $B_Q := \mathring{\nabla}_\mu(\tilde{Q}^\mu - Q^\mu)$.

Boundary terms

- The vectorial parts of torsion and nonmetricty form part of a boundary term

$$V^\mu := 2T^\mu + \tilde{Q}^\mu - Q^\mu, \qquad B_T + B_Q = \mathring{\nabla}_\mu V^\mu$$

- They enter the action as a boundary term due to the identity

$$\sqrt{-g}(B_T + B_Q) = \partial_\mu(\sqrt{-g}V^\mu)$$

- Therefore, once integrating out

$$\int_{\mathcal{M}} d^4x\sqrt{-g}(B_T + B_Q) = \int_{\partial\mathcal{M}} d\Sigma_\mu V^\mu$$

- For compactly supported variations then there is no bulk contribution.

Lecture 2: MAG: Dynamics



MAG: Dynamics

- Specify the gravitational action and matter coupling:

$$S[g,\tilde{\Gamma},\Psi]=\frac{1}{2\kappa}\int \mathrm{d}^4x\,\sqrt{-g}\,\mathcal{L}_G+S_{\mathrm{m}}[g,\tilde{\Gamma},\Psi],\qquad \kappa=8\pi G.$$

- Vary the independent fields:

$$\frac{\delta S}{\delta g^{\mu\nu}}=0,\qquad \frac{\delta S}{\delta \tilde{\Gamma}^{\lambda}{}_{\mu\nu}}=0,\qquad \frac{\delta S}{\delta \Psi}=0.$$

- The connection equation determines its physical role:

gauge freedom, auxiliary field, propagating field.

Matter couplings

- Standard scalar:

$$\mathcal{L}_\phi = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi - V(\phi), \qquad \Delta_\lambda{}^{\mu\nu} = 0.$$

- Standard electromagnetism:

$$F_{\mu\nu} = 2\partial_{[\mu} A_{\nu]}, \qquad \mathcal{L}_{\mathrm{EM}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \qquad \Delta_\lambda{}^{\mu\nu} = 0.$$

- Replacing the exterior derivative changes the coupling:

$$2\nabla_{[\mu} A_{\nu]} = F_{\mu\nu} + T^\lambda{}_{\mu\nu} A_\lambda.$$

- Standard minimally coupled Dirac matter supplies a spin current.
- The matter action determines the hypermomentum.

Matter equations and physical motion

- For diffeomorphism-invariant $S_{\mathrm{m}}[g, \Psi]$ (no dependence on connection), the matter equations imply

$$\mathring{\nabla}_{\mu} \Theta^{\mu}{}_{\nu} = 0.$$

- A structureless massive test particle:

$$S_{\mathrm{p}} = -m \int \mathrm{d}s_g.$$

- Its equation of motion, with proper time τ:

$$\frac{\mathrm{d}^2 x^{\mu}}{\mathrm{d}\tau^2} + \mathring{\Gamma}^{\mu}{}_{\alpha\beta} \frac{\mathrm{d}x^{\alpha}}{\mathrm{d}\tau} \frac{\mathrm{d}x^{\beta}}{\mathrm{d}\tau} = 0.$$

- Connection-dependent matter introduces hypermomentum terms in the balance laws and equations of motion.

Independent-connection Einstein–Hilbert gravity

- Gravitational action:

$$S = \frac{1}{2\kappa}\int \mathrm{d}^4x\,\sqrt{-g}\left(\tilde{R} - 2\Lambda\right) + S_{\mathrm{m}}[g, \tilde{\Gamma}, \Psi].$$

- Metric variation at fixed connection:

$$\delta_g \tilde{R} = \tilde{R}_{\mu\nu}\,\delta g^{\mu\nu}, \qquad \delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\delta g^{\mu\nu}.$$

- Metric field equation:

$$\boxed{\tilde{R}_{(\mu\nu)} - \frac{1}{2}g_{\mu\nu}\tilde{R} + \Lambda g_{\mu\nu} = \kappa\Theta_{\mu\nu}}$$

- The connection requires a separate field equation.

The connection equation

- Recall that

$$\Delta_\lambda{}^{\mu\nu} := -\frac{2}{\sqrt{-g}}\frac{\delta S_{\rm m}}{\delta \Gamma^\lambda{}_{\mu\nu}} \tag{20}$$

- In general, Palatini tensor is

$$P^\lambda{}_{\mu\nu} := -\frac{1}{\sqrt{-g}}\frac{\delta}{\delta \Gamma^\lambda{}_{\mu\nu}}\int d^4x\sqrt{-g}\mathcal{L}_{grav} \tag{21}$$

- Variation with respect to $\tilde{\Gamma}$:

$$\boxed{P_\lambda{}^{\mu\nu} = \kappa\Delta_\lambda{}^{\mu\nu}}$$

- The Palatini tensor, expressed through distortion, is

$$P_\lambda{}^{\mu\nu} = \delta^\nu_\lambda g^{\alpha\beta}N^\mu{}_{\alpha\beta} + g^{\mu\nu}N^\rho{}_{\lambda\rho} - g^{\rho\nu}N^\mu{}_{\rho\lambda} - g^{\mu\rho}N^\nu{}_{\lambda\rho}. \tag{22}$$

- The gravitational operator is linear and algebraic in N.
- Einstein–Hilbert gravity supplies no connection kinetic term.

Projective symmetry

- An arbitrary one-form ξ_ν generates

$$\tilde{\Gamma}^{\lambda}{}_{\mu\nu} \longmapsto \tilde{\Gamma}^{\lambda}{}_{\mu\nu} + \delta^{\lambda}_{\mu}\xi_\nu .$$

- Curvature transforms as

$$\delta_\xi \tilde{R}^{\rho}{}_{\sigma\mu\nu} = 2\delta^{\rho}_{\sigma}\partial_{[\mu}\xi_{\nu]},$$
$$\delta_\xi \tilde{R}_{\mu\nu} = 2\partial_{[\mu}\xi_{\nu]}.$$

- The symmetric Ricci tensor and scalar are invariant:

$$\delta_\xi \tilde{R}_{(\mu\nu)} = 0, \qquad \delta_\xi \tilde{R} = 0.$$

- The Einstein–Hilbert action has projective gauge freedom.

Recovering general relativity

- For connection-independent matter:

$$\Delta_\lambda{}^{\mu\nu} = 0 \quad \Longrightarrow \quad \tilde{\Gamma}^\lambda{}_{\mu\nu} = \mathring{\Gamma}^\lambda{}_{\mu\nu} + \delta^\lambda_\mu \mathcal{A}_\nu .$$

- The connection equation leaves $\mathcal{A}_\nu$ arbitrary:

$$\tilde{R}_{(\mu\nu)} = \mathring{R}_{\mu\nu}, \qquad \tilde{R} = \mathring{R}.$$

- The metric equation becomes

$$\boxed{\mathring{G}_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa \Theta_{\mu\nu}}$$

- Choosing $\mathcal{A}_\nu = 0$ selects the Levi-Civita representative.
- No additional vector particle appears in this theory.

Projective symmetry constrains matter

- The Palatini tensor obeys an off-shell identity:

$$P_\mu{}^{\mu\nu} \equiv 0.$$

- Its field equation requires vanishing dilation trace:

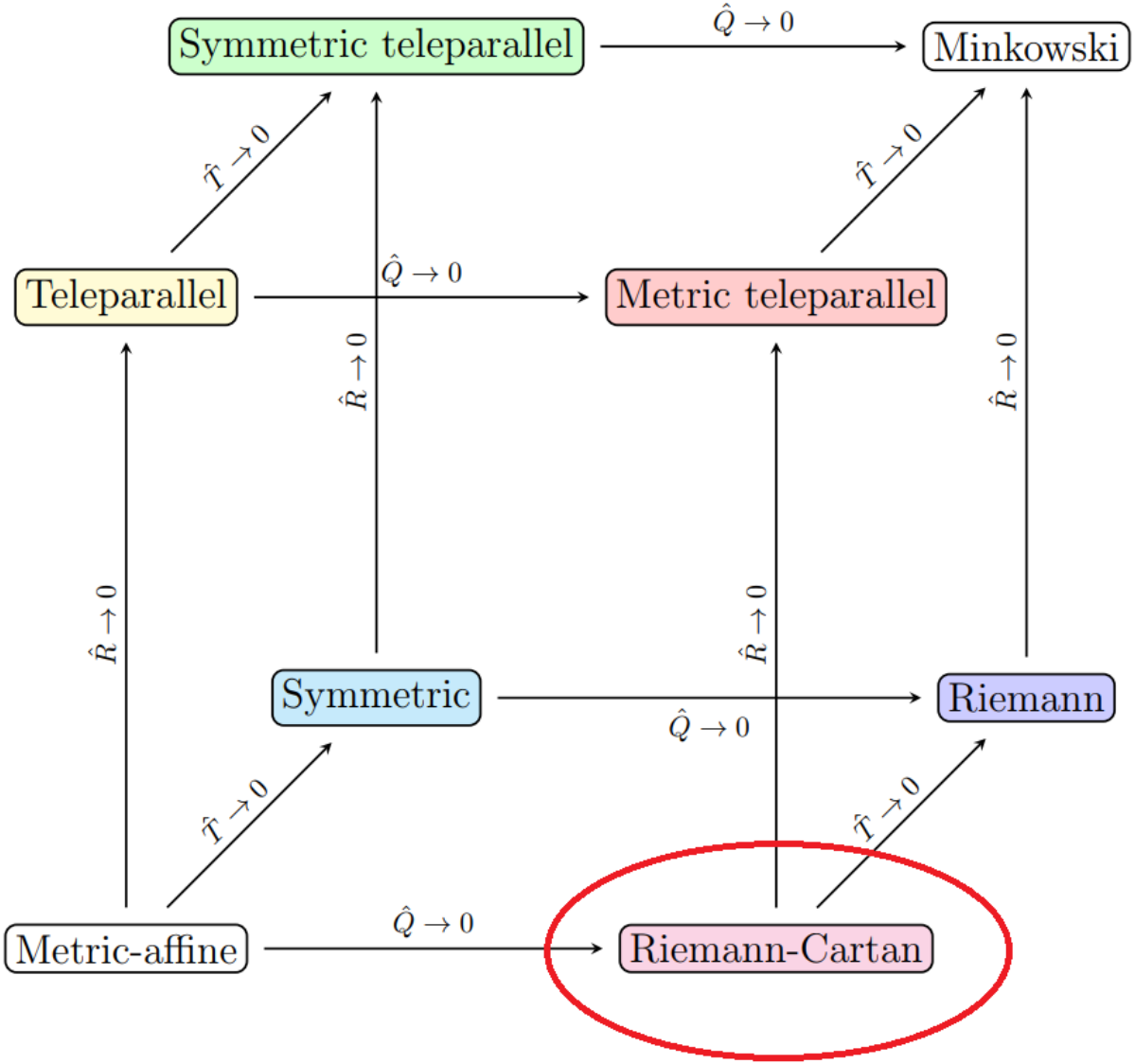
$$\boxed{\Delta_\mu{}^{\mu\nu} = 0}$$

- At fixed matter fields:

$$\delta_\xi S_{\rm m} = -\frac{1}{2}\int \mathrm{d}^4x\,\sqrt{-g}\,\Delta_\mu{}^{\mu\nu}\xi_\nu.$$

- Projectively invariant matter satisfies this identity.
- Supporting nonzero dilation sources requires changing the gravitational action or the constrained variational problem.

Riemann-Cartan geometry



Einstein–Cartan gravity

- Impose metric compatibility in the variational principle:

$$Q_{\rho\mu\nu} = 0, \qquad \tilde{\Gamma}^{\lambda}{}_{\mu\nu} = \mathring{\Gamma}^{\lambda}{}_{\mu\nu} + K^{\lambda}{}_{\mu\nu}.$$

- At fixed metric, compatible variations satisfy

$$\delta K_{\lambda\mu\nu} = -\delta K_{\mu\lambda\nu}.$$

- The Einstein–Hilbert connection equation becomes

$$\boxed{T^{\nu}{}_{\lambda\mu} + \delta^{\nu}_{\lambda} T_{\mu} - \delta^{\nu}_{\mu} T_{\lambda} = -\kappa \tau_{\lambda\mu}{}^{\nu}}$$

- Spin determines torsion locally.

Dirac matter and algebraic torsion

- The standard minimally coupled Dirac spin current satisfies

$$\tau_{\lambda\mu\nu} = \tau_{[\lambda\mu\nu]}.$$

- The Cartan equation gives

$$T_\mu = 0, \qquad T_{\lambda\mu\nu} = -\kappa\tau_{\lambda\mu\nu}, \qquad K_{\lambda\mu\nu} = -\frac{\kappa}{2}\tau_{\lambda\mu\nu}.$$

- In this antisymmetric sector:

$$\mathcal{L}_{\text{tor}} = -\frac{1}{8\kappa}T_{\lambda\mu\nu}T^{\lambda\mu\nu} - \frac{1}{4}\tau_{\lambda\mu\nu}T^{\lambda\mu\nu}.$$

- Torsion has no independent wave equation.

The effective fermion interaction

- Substitute the torsion solution into the action:

$$S_{\text{eff}} = \int \mathrm{d}^4x \sqrt{-g} \left[\frac{\mathring{R}}{2\kappa} + \mathring{\mathcal{L}}_{\text{D}} + \frac{\kappa}{8} \tau_{\lambda\mu\nu} \tau^{\lambda\mu\nu} \right].$$

- $\mathring{\mathcal{L}}_{\text{D}}$ uses the Levi-Civita spin connection.
- The spin current is bilinear in fermion fields:

$$\tau \sim \bar{\psi}\psi \quad \Longrightarrow \quad \mathcal{L}_{\text{contact}} \sim \kappa(\bar{\psi}\psi)^2.$$

- A local four-fermion interaction of gravitational strength.
- An auxiliary connection can change matter physics.

Spin ensembles and macroscopic matter

- An unpolarized ensemble can satisfy

$$\langle \tau_{\lambda\mu\nu} \rangle = 0.$$

- Spin correlations can nevertheless survive:

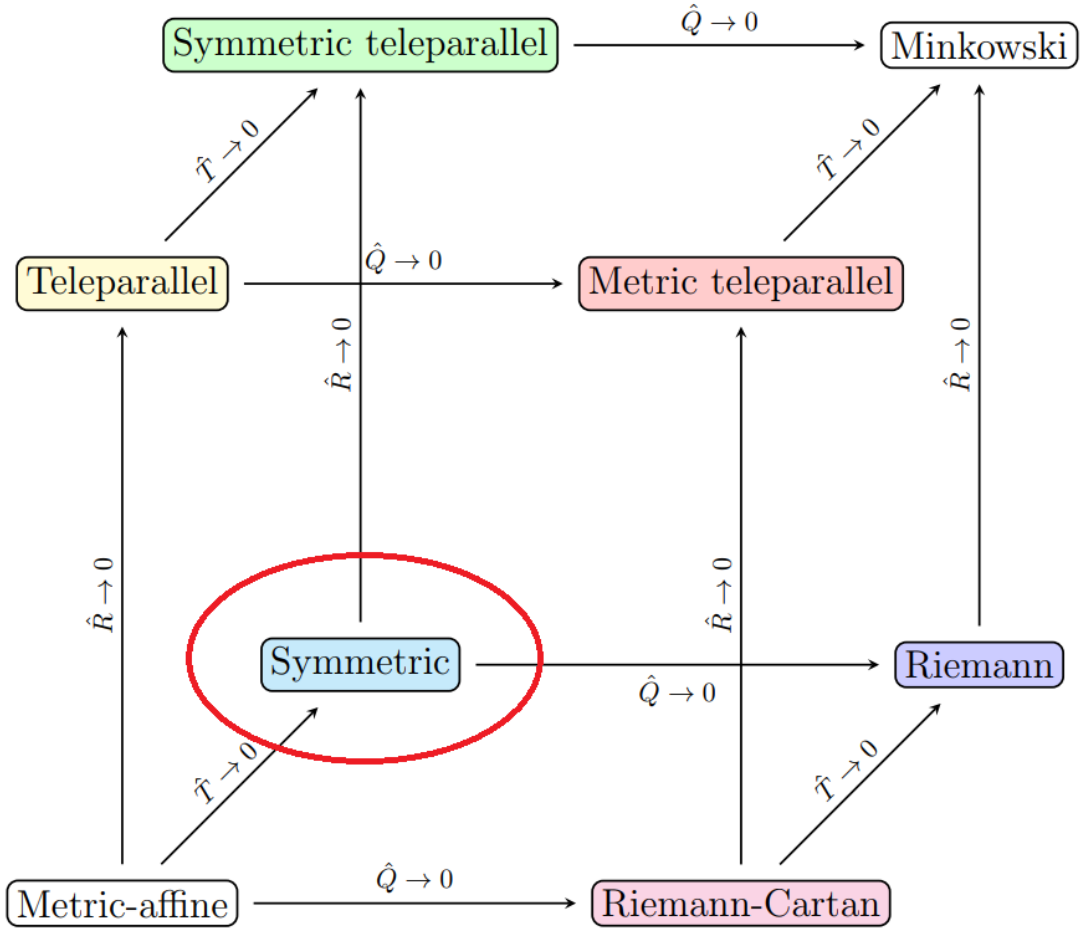
$$\left\langle \tau_{\lambda\mu\nu} \tau^{\lambda\mu\nu} \right\rangle \neq 0.$$

- Microscopic interactions can survive macroscopic averaging.
- The averaged interaction depends on these correlations:

$$\langle \mathcal{L}_{\text{contact}} \rangle = \frac{\kappa}{8} \left\langle \tau_{\lambda\mu\nu} \tau^{\lambda\mu\nu} \right\rangle .$$

- Cosmological predictions require a matter state, spin correlations and an equation of state.

Symmetric geometry, Palatini f(R)



Palatini $f(\tilde{R})$ gravity

- Consider connection-independent matter:

$$S = \frac{1}{2\kappa}\int \mathrm{d}^4x\,\sqrt{-g}\,f(\tilde{R}) + S_{\mathrm{m}}[g,\Psi], \qquad \Delta_\lambda{}^{\mu\nu} = 0.$$

- Define $\varphi := f'(\tilde{R})$. The metric equation is

$$\varphi\tilde{R}_{(\mu\nu)} - \frac{1}{2}f(\tilde{R})g_{\mu\nu} = \kappa\Theta_{\mu\nu}.$$

- Its trace:

$$\boxed{\varphi\tilde{R} - 2f(\tilde{R}) = \kappa\Theta}, \qquad \Theta := g^{\mu\nu}\Theta_{\mu\nu}.$$

- On a regular branch, $\tilde{R}$ is fixed algebraically by the matter trace.

The auxiliary metric

- The connection equation:

$$\varphi P_\lambda{}^{\mu\nu} + \delta^\nu_\lambda g^{\mu\rho}\partial_\rho\varphi - g^{\mu\nu}\partial_\lambda\varphi = 0.$$

- For $\varphi > 0$, define $h_{\mu\nu} := \varphi g_{\mu\nu}$:

$$\tilde{\Gamma}^\lambda{}_{\mu\nu} = \mathring{\Gamma}^\lambda{}_{\mu\nu}[h] + \delta^\lambda_\mu \mathcal{A}_\nu.$$

- Torsion is pure gauge (standard Palatini f(R) formulation is therefore torsionless)

$$T^\lambda{}_{\mu\nu} = \delta^\lambda_\mu \mathcal{A}_\nu - \delta^\lambda_\nu \mathcal{A}_\mu, \qquad T_\mu = -3\mathcal{A}_\mu. \tag{23}$$

- In the projective gauge $\mathcal{A}_\nu = 0$:

$$Q_{\rho\mu\nu} = -\partial_\rho \ln\varphi \, g_{\mu\nu}.$$

- Matter couples to $g_{\mu\nu}$; $h_{\mu\nu}$ determines the eliminated connection.
- Nonmetricity can arise with vanishing hypermomentum.

The same function, different dynamics

- Example:

$$f(X) = X + \alpha X^2, \qquad \alpha > 0.$$

- Metric variation, with $X = \mathring{R}$, produces wave equation

$$6\alpha \mathring{\Box}\mathring{R} - \mathring{R} = \kappa\Theta, \qquad \mathring{\Box} := g^{\mu\nu}\mathring{\nabla}_\mu\mathring{\nabla}_\nu.$$

  Curvature carries an additional scalar mode with mass $m^2 = 1/(6\alpha)$.

- Independent connection variation, with $X = \tilde{R}$:

$$-\tilde{R} = \kappa\Theta, \qquad \varphi = 1 - 2\alpha\kappa\Theta.$$

  The Palatini scalar is determined algebraically, no independent initial data.

- The variational principle changes the physical content. The eliminated connection contains derivatives of φ.

Vacuum branches

- In vacuum, the trace equation becomes

$$f'(\tilde{R})\tilde{R} - 2f(\tilde{R}) = 0.$$

- Choose an isolated root $\tilde{R} = c$ with $f'(c) > 0$:

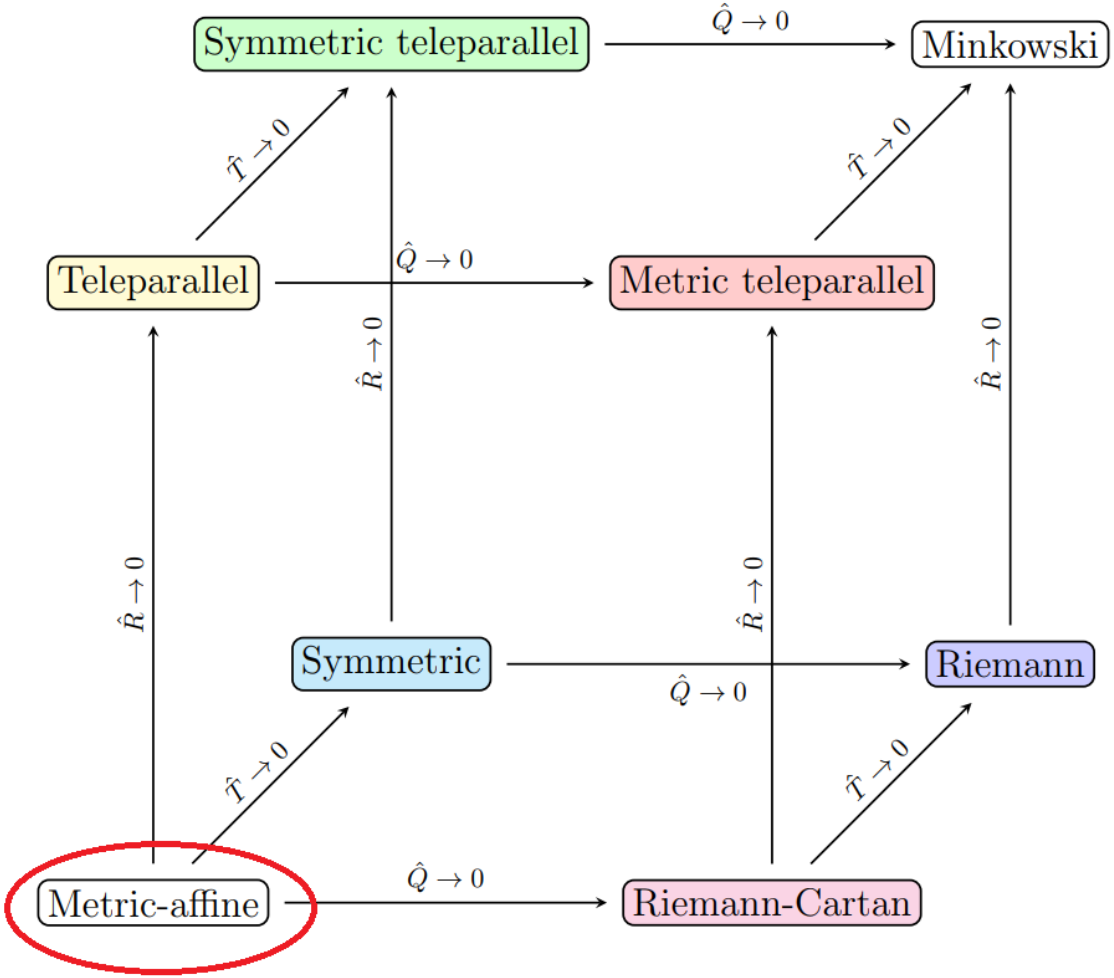
$$\tilde{\Gamma}^{\lambda}{}_{\mu\nu} = \mathring{\Gamma}^{\lambda}{}_{\mu\nu} + \delta^{\lambda}_{\mu}\mathcal{A}_{\nu}, \qquad \mathring{R}_{\mu\nu} = \frac{c}{4}g_{\mu\nu}.$$

- Each regular root defines an effective cosmological constant:

$$\Lambda_{\text{eff}} = \frac{c}{4}.$$

- Degenerate branches require separate treatment. For $f(\tilde{R}) \propto \tilde{R}^2$, the vacuum trace equation is an identity.

Full MAG


Symmetric teleparallel
$\hat{Q} \to 0$
Minkowski
$\hat{T} \to 0$
$\hat{T} \to 0$
Teleparallel
$\hat{Q} \to 0$
Metric teleparallel
$\hat{R} \to 0$
$\hat{R} \to 0$
$\hat{R} \to 0$
$\hat{R} \to 0$
Symmetric
Riemann
$\hat{Q} \to 0$
$\hat{T} \to 0$
$\hat{T} \to 0$
Metric-affine
$\hat{Q} \to 0$
Riemann-Cartan

Quadratic torsion and nonmetricity

- Extend the gravitational Lagrangian:

$$\mathcal{L}_G = \tilde{R} - 2\Lambda + \mathcal{L}_{T^2} + \mathcal{L}_{Q^2} + \mathcal{L}_{TQ}.$$

- Constant-coefficient quadratic contractions, without derivatives of T or Q.
- Apply the curvature identity from Lecture 1:

$$\mathcal{L}_G \simeq \mathring{R} - 2\Lambda + \mathcal{U}_2(N),$$

  where $\simeq$ denotes equality up to a total divergence.
- If matter contains no derivatives of N, the connection equation remains algebraic.

Solving for the connection

- The part of the Lagrangian depending on the connection can be written as

$$\mathcal{L}_N = \frac{1}{4\kappa} N^I M_{IJ} N^J - \frac{1}{2} \mathcal{J}_I N^I, \qquad M_{IJ} = M_{JI} \tag{24}$$

- Quadratic dependence on N and a linear matter source give

$$\mathrm{M}_{IJ} N^J = \kappa \mathcal{J}_I.$$

  I, J collect connection components $(1, \ldots, 64)$; $\mathcal{J}_I$ is independent of N.
- On an invertible sector, after fixing gauge freedom:

$$N^I = \kappa (\mathrm{M}^{-1})^{IJ} \mathcal{J}_J.$$

- Vanishing source: $N = 0$ and recovery of GR.
- Nonzero source: effective local matter interactions.
- Degeneracy requires a constraint or gauge analysis.

A kinetic term for the connection

- Curvature-squared terms can supply connection derivatives.
- A concrete model, with constant $Z > 0$:

$$S = \int \mathrm{d}^4x \sqrt{-g} \left[\frac{\tilde{R} - 2\Lambda}{2\kappa} - \frac{Z}{64} \mathcal{H}_{\mu\nu} \mathcal{H}^{\mu\nu} \right] + S_{\mathrm{m}}[g, \Psi].$$

- Separate the connection trace:

$$\mathcal{A}_\nu := \frac{1}{4} N^\lambda{}_{\lambda\nu},$$
$$N^\lambda{}_{\mu\nu} = \bar{N}^\lambda{}_{\mu\nu} + \delta^\lambda_\mu \mathcal{A}_\nu, \qquad \bar{N}^\lambda{}_{\lambda\nu} = 0.$$

- The homothetic curvature becomes

$$\mathcal{H}_{\mu\nu} = 4\mathcal{F}_{\mu\nu}, \qquad \mathcal{F}_{\mu\nu} := 2\partial_{[\mu}\mathcal{A}_{\nu]}.$$

A propagating connection vector

- The remaining connection equations give $\bar{N} = 0$:

$$\tilde{\Gamma}^{\lambda}{}_{\mu\nu} = \mathring{\Gamma}^{\lambda}{}_{\mu\nu} + \delta^{\lambda}_{\mu}\mathcal{A}_{\nu}.$$

- The effective action has Einstein–Maxwell form:

$$S_{\text{eff}} = \int \mathrm{d}^4x\,\sqrt{-g}\left[\frac{\mathring{R} - 2\Lambda}{2\kappa} - \frac{Z}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\right] + S_{\text{m}}.$$

- Vector equation and residual gauge symmetry:

$$\mathring{\nabla}_{\mu}\mathcal{F}^{\mu\nu} = 0, \qquad \mathcal{A}_{\mu} \longmapsto \mathcal{A}_{\mu} + \partial_{\mu}\chi.$$

- Two additional vector polarizations. One vector is encoded jointly in torsion and nonmetricity.

Physical degrees of freedom

- Hamiltonian counting on a regular constraint sector:

$$N_{\rm dof} = N_{\rm conf} - N_{\rm first} - \frac{1}{2} N_{\rm second}.$$

  $N_{\rm first}$ and $N_{\rm second}$ count first- and second-class constraints.

- Examples, excluding matter degrees of freedom:

$$\begin{aligned} \text{Einstein–Hilbert, } \Delta = 0: \quad & N_{\rm dof} = 2, \\ \text{The } \mathcal{H}^2 \text{ model above}: \quad & N_{\rm dof} = 2 + 2. \end{aligned}$$

- Identify constraints and gauge freedom before counting modes.
- Check physical kinetic signs and well-posed evolution for the chosen model and background.

Quadratic MAG (1912.01023)

$$
\begin{aligned}
S(g,A) \;=\; & -\frac{1}{2}\int d^dx\,\sqrt{|g|}\,\Big[-a_0F+F^{\mu\nu\rho\sigma}\big(c_1F_{\mu\nu\rho\sigma}+c_2F_{\mu\nu\sigma\rho}+c_3F_{\rho\sigma\mu\nu}+c_4F_{\mu\rho\nu\sigma}\\
& +c_5F_{\mu\sigma\nu\rho}+c_6F_{\mu\sigma\rho\nu}\big)+F^{(13)\mu\nu}\big(c_7F^{(13)}_{\mu\nu}+c_8F^{(13)}_{\nu\mu}\big)+F^{(14)\mu\nu}\big(c_9F^{(14)}_{\mu\nu}+c_{10}F^{(14)}_{\nu\mu}\big)\\
& +F^{(14)\mu\nu}\big(c_{11}F^{(13)}_{\mu\nu}+c_{12}F^{(13)}_{\nu\mu}\big)+F^{\mu\nu}\big(c_{13}F_{\mu\nu}+c_{14}F^{(13)}_{\mu\nu}+c_{15}F^{(14)}_{\mu\nu}\big)+c_{16}F^2\\
& +T^{\mu\rho\nu}\big(a_1T_{\mu\rho\nu}+a_2T_{\mu\nu\rho}\big)+a_3T^\mu T_\mu+Q^{\rho\mu\nu}\big(a_4Q_{\rho\mu\nu}+a_5Q_{\nu\mu\rho}\big)\\
& +a_6Q^\mu Q_\mu+a_7\widetilde{Q}^\mu\widetilde{Q}_\mu+a_8Q^\mu\widetilde{Q}_\mu+a_9T^{\mu\rho\nu}Q_{\mu\rho\nu}+T^\mu\Big(a_{10}Q_\mu+a_{11}\widetilde{Q}_\mu\Big)\Big]\,, \qquad (2.4)
\end{aligned}
$$

where

$$
\begin{aligned}
T_\mu \;&:=\; T_\lambda{}^\lambda{}_\mu\,, \qquad Q_\mu := Q_{\mu\lambda}{}^\lambda\,, \qquad \widetilde{Q}_\mu := Q_\lambda{}^\lambda{}_\mu\,,\\
F_{\mu\nu} \;&:=\; F_{\mu\nu\lambda}{}^\lambda\,, \quad F^{(14)}_{\mu\nu} := F_{\lambda\mu\nu}{}^\lambda\,, \quad F^{(13)}_{\mu\nu} := F_{\lambda\mu}{}^\lambda{}_\nu\,, \quad F := F_{\mu\nu}{}^{\mu\nu}\,. \qquad (2.5)
\end{aligned}
$$

Cubic MAG (2411.12954)

$$\mathcal{L}^{(3)}_{\text{curv}-\text{tors}} = \mathcal{L}^{(3)}_{\tilde{R}TT} + \mathcal{L}^{(3)}_{\tilde{R}SS} + \mathcal{L}^{(3)}_{\tilde{R}tt} + \mathcal{L}^{(3)}_{\tilde{R}TS} + \mathcal{L}^{(3)}_{\tilde{R}Tt} + \mathcal{L}^{(3)}_{\tilde{R}St}\,, \tag{58}$$

where

$$\mathcal{L}^{(3)}_{\tilde{R}TT} = (h_1 - h_{27})\tilde{R}_{\mu\nu}T^\mu T^\nu + h_2\tilde{R}T_\mu T^\mu + h_{27}\hat{R}^{\alpha\rho}T_\alpha T_\rho\,, \tag{59}$$

$$\mathcal{L}^{(3)}_{\tilde{R}SS} = (h_3 - h_{28})\tilde{R}_{\mu\nu}S^\mu S^\nu + h_4\tilde{R}S_\mu S^\mu + h_{28}\hat{R}^{\alpha\rho}S_\alpha S_\rho\,, \tag{60}$$

$$\begin{aligned}\mathcal{L}^{(3)}_{\tilde{R}tt} = {}& \frac{1}{2}(2h_5 + h_{30})\,\tilde{R}_{\lambda\rho\mu\nu}t_\sigma{}^{\lambda\rho}t^{\sigma\mu\nu} + h_6\tilde{R}_{\lambda\rho\mu\nu}t_\sigma{}^{\lambda\mu}t^{\sigma\rho\nu} + h_7\tilde{R}_{\lambda\rho\mu\nu}t^{\lambda\rho}{}_\sigma t^{\sigma\mu\nu} + (h_8 + h_{29})\tilde{R}_{\lambda\rho\mu\nu}t^{\lambda\mu}{}_\sigma t^{\sigma\rho\nu} \\ & + (h_9 + h_{29})\tilde{R}_{\lambda\rho\mu\nu}t^{\lambda\mu}{}_\sigma t^{\rho\nu\sigma} + (h_{10} + h_{31} - h_{32})\tilde{R}_{\lambda\rho}t_{\mu\nu}{}^\lambda t^{\rho\mu\nu} + (h_{11} - h_{32})\tilde{R}_{\lambda\rho}t_{\mu\nu}{}^\lambda t^{\mu\nu\rho} + h_{12}\tilde{R}t_{\lambda\rho\mu}t^{\lambda\rho\mu} \\ & + h_{29}\tilde{R}^\alpha{}_{\rho\tau\gamma}t^{\rho\tau}{}_\mu t^\gamma{}_\alpha{}^\mu + h_{30}\tilde{R}^\alpha{}_{\rho\tau\gamma}t^{\tau\gamma}{}_\mu t^\mu{}_\alpha{}^\rho + h_{31}\hat{R}^\alpha{}_\tau t^\tau{}_{\gamma\mu}t^\gamma{}_\alpha{}^\mu + h_{32}\hat{R}^\alpha{}_\tau t^\gamma{}_\alpha{}^\mu t_\mu{}^\tau{}_\gamma\,,\end{aligned} \tag{61}$$

$$\begin{aligned}\mathcal{L}^{(3)}_{\tilde{R}TS} = {}& \frac{1}{2}(2h_{13} - h_{33})\,\varepsilon^{\lambda\rho\mu\nu}\tilde{R}_{\lambda\rho\mu\nu}T_\sigma S^\sigma + (h_{14} + h_{33})\varepsilon_\nu{}^{\lambda\rho\sigma}\tilde{R}_{\lambda\rho\mu\sigma}T^\mu S^\nu + (h_{15} + h_{34})\varepsilon^{\lambda\rho\mu\nu}\tilde{R}_{\lambda\rho}T_\mu S_\nu \\ & + h_{33}\varepsilon^{\alpha\tau\gamma\mu}\tilde{R}^\rho{}_{\tau\gamma\mu}T_\rho S_\alpha + h_{34}\varepsilon^{\alpha\rho}{}_\tau{}^\mu\hat{R}^\tau{}_\mu T_\rho S_\alpha + h_{35}\varepsilon^{\alpha\rho\gamma\mu}\tilde{R}^\omega{}_{\omega\gamma\mu}T_\rho S_\alpha\,,\end{aligned} \tag{62}$$

$$\begin{aligned}\mathcal{L}^{(3)}_{\tilde{R}Tt} = {}& (h_{16} - h_{36})\tilde{R}_{\lambda\rho\mu\nu}T^\nu t^{\lambda\rho\mu} + \frac{1}{2}(2h_{17} + h_{37})\,\tilde{R}_{\lambda\rho\mu\nu}T^\rho t^{\lambda\mu\nu} + (h_{18} - h_{38} + h_{39})\tilde{R}_{\lambda\rho}T_\mu t^{\mu\lambda\rho} + (h_{19} + h_{39})\tilde{R}_{\lambda\rho}T_\mu t^{\lambda\rho\mu} \\ & + h_{36}\tilde{R}^\rho{}_\tau{}^\alpha{}_\gamma t^\tau{}_\rho{}^\gamma T_\alpha + h_{37}\tilde{R}^\alpha{}_{\rho\tau\gamma}t^{\tau\rho\gamma}T_\alpha + h_{38}\hat{R}^\rho{}_\gamma t^\alpha{}_\rho{}^\gamma T_\alpha + h_{39}\hat{R}^\rho{}_\gamma t^{\gamma\alpha}{}_\rho T_\alpha + h_{40}\tilde{R}^\omega{}_{\omega\tau\gamma}t^{\tau\alpha\gamma}T_\alpha\,,\end{aligned} \tag{63}$$

$$\begin{aligned}\mathcal{L}^{(3)}_{\tilde{R}St} = {}& \frac{1}{2}(2h_{20} + h_{41})\,\varepsilon_{\alpha\rho\mu\nu}\tilde{R}_\tau{}^{\rho\mu\nu}S^\gamma t^{\alpha\tau}{}_\gamma + h_{21}\varepsilon_{\alpha\rho\mu\nu}\tilde{R}_\tau{}^{\rho\mu\nu}S^\gamma t_\gamma{}^{\alpha\tau} + (h_{22} + h_{43})\varepsilon_{\alpha\rho}{}^{\mu\nu}\tilde{R}^\rho{}_{\mu\tau\nu}S^\gamma t_\gamma{}^{\alpha\tau} \\ & + (h_{23} - h_{42} + 2h_{43})\varepsilon_{\alpha\rho}{}^{\mu\nu}\tilde{R}_{\gamma\mu\tau\nu}S^\alpha t^{\gamma\rho\tau} + (h_{24} + h_{41} - h_{42})\varepsilon_{\alpha\rho}{}^{\mu\nu}\tilde{R}_{\gamma\mu\tau\nu}S^\alpha t^{\rho\tau\gamma} + h_{41}\varepsilon^{\alpha\gamma\mu\nu}\tilde{R}^\rho{}_{\tau\gamma\mu}S_\alpha t^\tau{}_{\rho\nu} \\ & + (h_{25} + h_{44} + h_{45})\varepsilon_{\alpha\rho\tau\mu}\tilde{R}^\mu{}_\gamma S^\alpha t^{\rho\tau\gamma} + \frac{1}{2}(2h_{26} - h_{44})\,\varepsilon_{\lambda\rho\mu\nu}\tilde{R}^{\lambda\rho}S_\sigma t^{\sigma\mu\nu} + h_{42}\varepsilon^\alpha{}_\rho{}^{\mu\nu}\tilde{R}^\rho{}_{\tau\gamma\mu}S_\alpha t^{\gamma\tau}{}_\nu \\ & + h_{43}\varepsilon^\alpha{}_\rho{}^{\tau\nu}\tilde{R}^\rho{}_{\tau\gamma\mu}S_\alpha t^{\gamma\mu}{}_\nu + h_{44}\varepsilon^{\alpha\gamma\mu\nu}\hat{R}^\rho{}_\gamma S_\alpha t_{\mu\rho\nu} + h_{45}\varepsilon^\alpha{}_\rho{}^{\mu\nu}\hat{R}^\rho{}_\gamma S_\alpha t_\mu{}^\gamma{}_\nu + h_{46}\varepsilon^{\alpha\gamma\mu\nu}\tilde{R}^\omega{}_{\omega\tau\gamma}S_\alpha t_\mu{}^\tau{}_\nu\,.\end{aligned} \tag{64}$$

Further reading

- Einstein–Hilbert theory and projective freedom: Bernal et al., arXiv:1606.08756.
- Matter couplings and metric-affine $f(\tilde{R})$: Sotiriou and Liberati, arXiv:gr-qc/0604006.
- Auxiliary and propagating connections: Vitagliano, Sotiriou and Liberati, arXiv:1008.0171.
- Quadratic torsion and nonmetricity: Vitagliano, arXiv:1308.1642.
- Projective symmetry and a dynamical vector: Janssen and Jiménez-Cano, arXiv:1807.10168.