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Quantum Origins of Cosmic Structure: From Vacuum Fluctuations to Primordial Inhomogeneities

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Lecture 2

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Outline

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QFT in quasi-de Sitter spacetime

- The point is to write **the action of a scalar field** $\phi = \phi_0 + \delta\phi$ **a minimally coupled to GR, using a perturbed** (flat) FLRW metric $g_{\mu\nu} = a(\eta)\eta_{\mu\nu} + \delta g_{\mu\nu}$, i.e.

$$S = \int d^4x\sqrt{-g}\left[\frac{M_P^2}{2}R - \frac{g^{\mu\nu}}{2}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right]$$

- Remember the background is expanding quasi-exponentially $a \simeq a_{\rm ini}e^{N(t)}$, witn $N(t)$ the number of e-folds

See Mukhanov, Feldman and Brandenberger, Phys. Rep. 215, 203 (1992).

QFT in quasi-de Sitter spacetime

- The QFT of the perturbations can be described by the quantization of the *Mukhanov-Sasaki variable*

$$v \equiv a\delta\phi + \phi_0' a^2/a'\Psi$$

- We can always choose a particular gauge, for example the *comoving gauge*, so the **primordial curvature perturbation** is $\mathcal{R}$. In particular

$$\mathcal{R} = v/z \qquad \text{with} \qquad z \equiv a\phi_0'/\mathcal{H}$$

.

See Mukhanov, Feldman and Brandenberger, Phys. Rep. 215, 203 (1992).

The action contains a time-dependent mass

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The second-order action for scalar perturbations is

$$S^{(2)} = \frac{1}{2}\int d\eta\, d^3x \left[(v')^2 - (\nabla v)^2 + \frac{z''}{z}v^2\right]. \qquad (1)$$

The canonical momentum is

$$\pi_v = \frac{\partial \mathcal{L}}{\partial v'} = v'. \qquad (2)$$

All background dependence enters through the effective mass term $-z''/z$.
At linear order, the cosmological problem becomes a collection of decoupled, time-dependent harmonic oscillators.

Quantizing v produces an operator

We impose the equal-time commutators

$$[\hat{v}(\eta, \mathbf{x}), \hat{\pi}_v(\eta, \mathbf{y})] = i\delta^{(3)}(\mathbf{x} - \mathbf{y}), \tag{3}$$

and

$$[\hat{v}(\eta, \mathbf{x}), \hat{v}(\eta, \mathbf{y})] = [\hat{\pi}_v(\eta, \mathbf{x}), \hat{\pi}_v(\eta, \mathbf{y})] = 0. \tag{4}$$

The background $a(\eta)$ and $\phi_0(\eta)$ are still treated classically.

Modes evolve; operators encode the algebra

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The field operator can be expanded as

$$\hat{v}(\eta, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \left[v_k(\eta) \hat{a}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} + v_k^*(\eta) \hat{a}_{\mathbf{k}}^\dagger e^{-i\mathbf{k}\cdot\mathbf{x}} \right]. \tag{5}$$

With

$$[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}^\dagger] = (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{k}'), \tag{6}$$

canonical normalization requires

$$v_k v_k^{*\prime} - v_k^* v_k' = i. \tag{7}$$

Each mode is a quantum oscillator

Schematically, the Hamiltonian is

$$\hat{H} = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\left[\hat{\pi}_{\mathbf{k}}\hat{\pi}_{-\mathbf{k}} + \omega_k^2(\eta)\hat{v}_{\mathbf{k}}\hat{v}_{-\mathbf{k}}\right], \tag{8}$$

with

$$\omega_k^2(\eta) = k^2 - \frac{z''}{z}. \tag{9}$$

The Bunch–Davies condition fixes the vacuum

At sufficiently early times, observable modes satisfy $k \gg aH$ and should behave as positive-frequency Minkowski modes:

$$v_k(\eta) \longrightarrow \frac{e^{-ik\eta}}{\sqrt{2k}} \qquad \text{as} \qquad -k\eta \longrightarrow \infty. \tag{10}$$

This defines the Bunch–Davies vacuum,

$$\hat{a}_{\mathbf{k}}|0_{\rm BD}\rangle = 0 \qquad \text{for all } \mathbf{k}. \tag{11}$$

Unitary evolution changes the initial zero particle state

$$|\Psi(\eta)\rangle_{\mathbf{k},\mathbf{k}'} = \hat{\mathcal{U}}_{\mathbf{k}}(\eta)|0_{\rm BD}\rangle = \sum_{n=0}^{\infty} \mathcal{F}[r_k(\eta), \alpha_k(\eta)]|n_{\mathbf{k}}, n_{\mathbf{k}'}\rangle \tag{12}$$

However, unitary evolution does not break the symmetries of the initial BD vacuum

The de Sitter mode joins two regimes

For exact de Sitter, $\nu_s = 3/2$, the normalized Bunch–Davies solution is

$$v_k(\eta) = \frac{1}{\sqrt{2k}}\left(1 - \frac{i}{k\eta}\right)e^{-ik\eta}. \tag{13}$$

For quasi-de Sitter expansion, the corresponding solution is a Hankel function,

$$v_k(\eta) = \frac{\sqrt{-\pi\eta}}{2}e^{i(2\nu_s+1)\pi/4}H^{(1)}_{\nu_s}(-k\eta). \tag{14}$$

The slow evolution of H and ϵ_1 slightly changes the index ν_s and produces the scalar tilt.

Curvature modes freeze after Hubble crossing

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For sub-Hubble modes, $-k\eta \gg 1$,

$$v_k \simeq \frac{e^{-ik\eta}}{\sqrt{2k}}. \tag{15}$$

For super-Hubble $-k\eta \ll 1$ in de Sitter,

$$|v_k| \simeq \frac{1}{\sqrt{2k^3}|\eta|}. \tag{16}$$

Since $z \propto a \propto (-\eta)^{-1}$ at leading order,

$$\mathcal{R}_k = \frac{v_k}{z} \longrightarrow \text{constant} \tag{17}$$

for the growing mode. The decaying mode rapidly becomes negligible.

Theoretical Predictions

- The main object one is interested is:

$$\langle \mathrm{BD}|\hat{v}(\eta,\vec{x})\hat{v}(\eta,\vec{y})|\mathrm{BD}\rangle \tag{18}$$

- Afterwards, one makes the identification

$$\langle \mathrm{BD}|\hat{v}(\eta,\vec{x})\hat{v}(\eta,\vec{y})|\mathrm{BD}\rangle = \int \frac{dk}{k}\mathcal{P}(k)\frac{\sin kr}{kr} \tag{19}$$

- **The predicted object is the (scalar) power spectrum**

$$\mathcal{P}(k) = \frac{1}{2\pi^2}|\mathcal{R}_k|^2 k^3 \propto \left(\frac{H^2}{M_p^2\epsilon_1}\right)\Bigg|_{k=aH}; \tag{20}$$

$$n_s - 1 \equiv \frac{d\ln\mathcal{P}(k)}{d\ln k} = -2\epsilon_1 - \epsilon_2 \tag{21}$$

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- The parameters ϵ_1 and ϵ_2, are called the Hubble Flow functions, or simply the "slow roll" parameters.
- Defining $\epsilon_1 \equiv -\dot{H}/H^2$ and $\epsilon_2 \equiv \dot{\epsilon}_1/(H\epsilon_1)$ and using the slow roll approximation, one has

$$\epsilon_1 \simeq (M_P^2/2)(\partial_\phi V/V)^2 \ll 1$$

$$\epsilon_2 \simeq 2M_P^2[(\partial_\phi V/V)^2 - (\partial_{\phi\phi} V/V)] \ll 1$$

- The paremeters ϵ_1 and ϵ_2 are used to constrain V

Tensor perturbations obey the same logic

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Tensor perturbations are transverse and traceless,

$$g_{ij} = a^2(\eta)(\delta_{ij} + h_{ij}), \qquad \partial_i h_{ij} = 0, \qquad h_{ii} = 0. \tag{22}$$

They contain two polarizations, $\lambda = +, \times$. Their quadratic action is

$$S_t^{(2)} = \frac{M_P^2}{8} \int d\eta \, d^3x \, a^2 \left[\partial_0 h_{ij} \partial_0 h^{ij} - \partial_\ell h_{ij} \partial_\ell h^{ij} \right]. \tag{23}$$

After canonical normalization, each polarization obeys a mode equation with effective mass $-a''/a$.

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- The theoretical prediction is the tensor power spectrum

$$\mathcal{P}_t(k) \propto \left(\frac{H^2}{M_p^2}\right)\Bigg|_{k=aH}; \qquad n_t \equiv \frac{d\ln P_t(k)}{d\ln k} = -2\epsilon_1 \tag{24}$$

- The tensor-to-scalar ratio

$$r \equiv P_t(k)/P_s(k) = 16\epsilon_1 \tag{25}$$

Summary of slow roll Inflation

- The simplest inflationary model (a.k.a *slow-roll inflation*) postulates the existence of a scalar field ϕ called the inflaton. One needs to specify its potential V (which has some parameters).
- The inflaton is a matter field that requires negative pressure: $p \simeq -\rho$, imposed by GR (because $V \gg \dot{\phi}^2/2$).
- Once the inflaton has done its work, it needs to be *turned-off* ($V \simeq \dot{\phi}^2/2$), so it cannot be detected directly today. That process is called *reheating* and the precise details are unknown.
- Other sophisticated models can be constructed (more fields, more parameters), but they must yield essentially the *same predictions*.

The observational data

- The data is obtained from the 2-p correlation function

$$C_{TT}(\gamma) = \langle \Theta(\hat{n_1})\Theta(\hat{n_2}) \rangle$$

where $\langle \cdot \rangle$ avg. over orientations, and

$$\Theta(\hat{n}) \equiv \frac{\delta T}{T_0}(\hat{n}) = \sum_{l,m} a_{lm} Y_{lm}(\theta, \varphi)$$

- The data is presented as

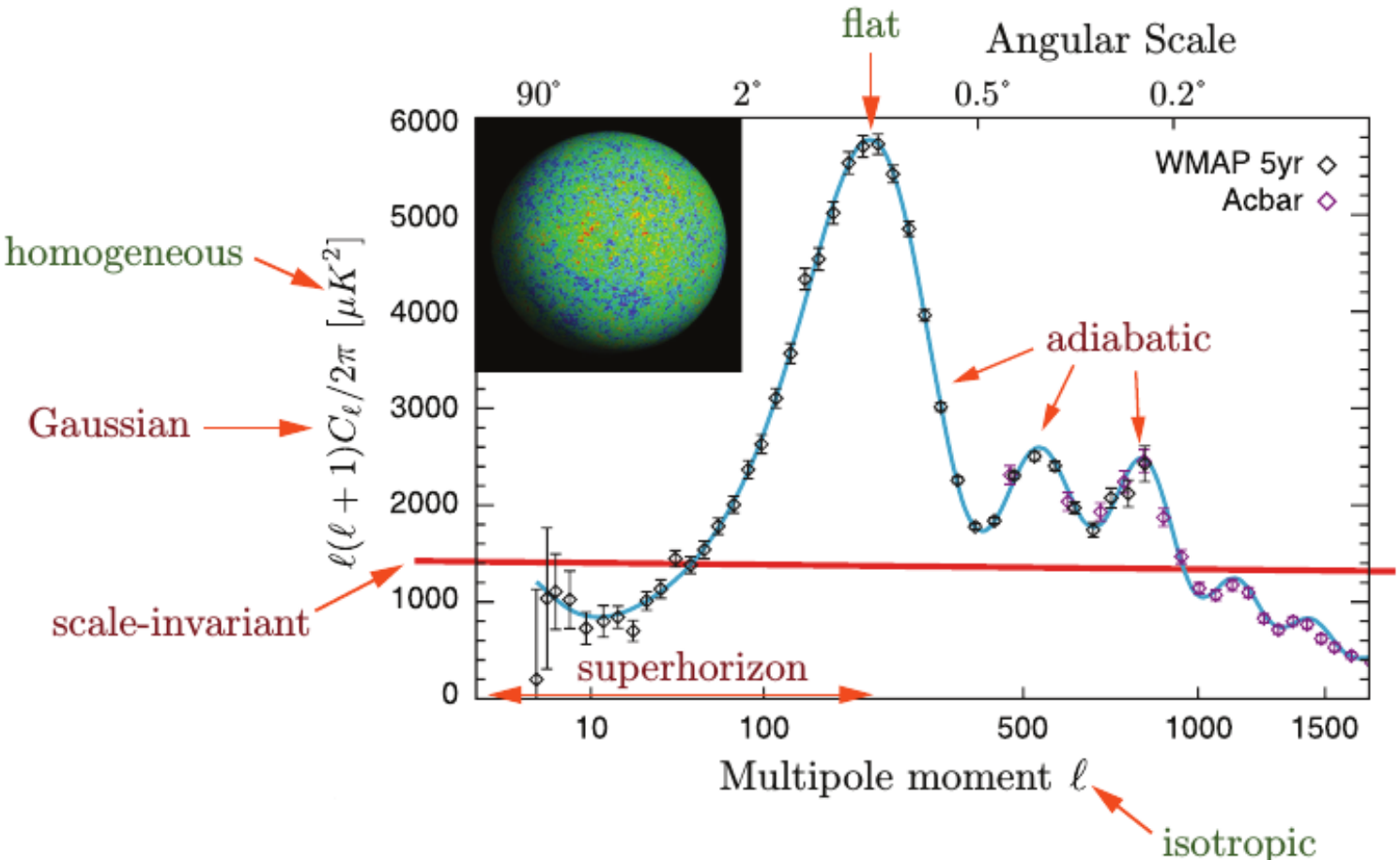
$$C_l^{(\text{obs})} \equiv \frac{1}{2l+1} \sum_m a_{lm}^* a_{lm}$$

this is called the temperature *angular power spectrum* of the CMB.

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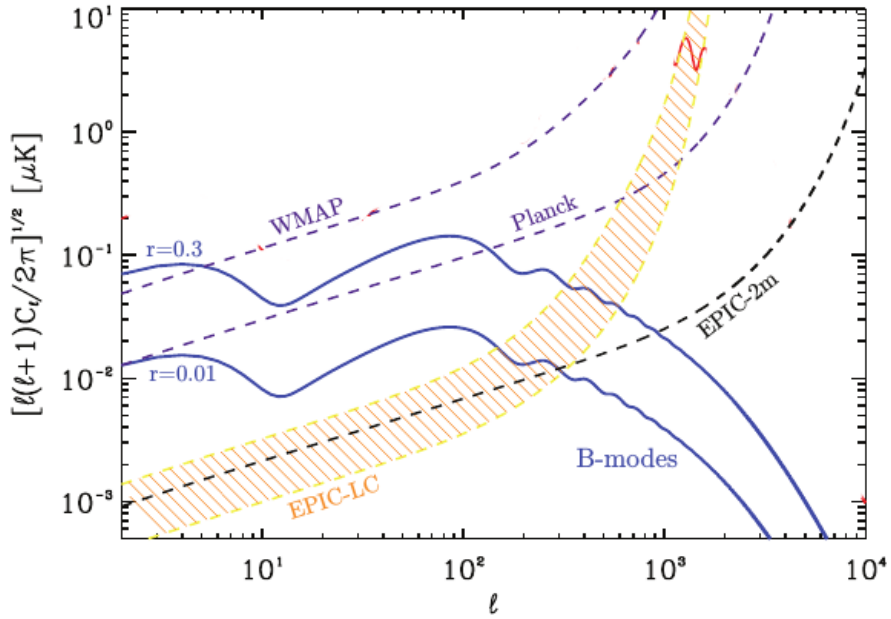
The *theo. angular* PS is related to the *primordial* PS

$$C_l^{(\mathrm{th})} = 4\pi \int \frac{dk}{k} \underbrace{\Delta_l^2(k)}_{\text{plasma physics}} \underbrace{\mathcal{P}(k)}_{\text{Inflation}}$$



B-mode polarization

- The primordial gravitational waves induce a particular polarization pattern that can be distinguished from the scalar (and vector) sources, this is called the B-mode polarization.
- The observational data is presented in the BB angular spectrum C_l^{BB} (analogous to the scalar C_l's)



Observational constraints for inflation

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- The scalar PPS is parameterized as

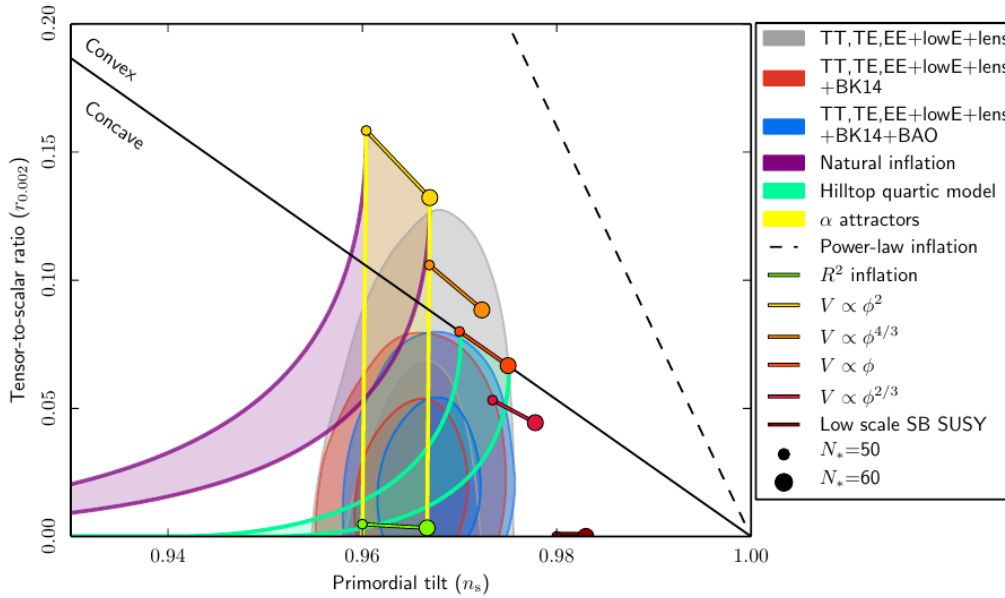
$$\mathcal{P}_s(k) = A_s \left(\frac{k}{k_p}\right)^{n_s - 1}$$

- The tensor PPS is parameterized as

$$\mathcal{P}_t(k) = r A_s \left(\frac{k}{k_p}\right)^{n_t}$$

with $n_t = -r/8$ (consistency relation theo.)

Parameter	TT,TE,EE+lowE+lensing
$\Omega_b h^2$	0.02237 ± 0.00015
$\Omega_c h^2$	0.1200 ± 0.0012
$100\theta_{MC}$	1.04092 ± 0.00031
τ	0.0544 ± 0.0073
$\ln(10^{10} A_s)$	3.044 ± 0.014
n_s	0.9649 ± 0.0042
H_0	67.36 ± 0.54
Ω_m	0.3153 ± 0.0073
σ_8	0.8111 ± 0.0060



Let us look into the details

- The temperature anisotropies of the CMB observed today are described in terms of the coefficients a_{lm} of the multipolar series expansion

$$\frac{\delta T}{T_0}(\theta, \varphi) = \sum_{l,m} a_{lm} Y_{lm}(\theta, \varphi), \tag{26}$$

so

$$a_{lm} = \int d\Omega \, \frac{\delta T}{T_0}(\theta, \varphi) Y^*_{lm}(\theta, \varphi), \tag{27}$$

with θ, φ coordinates on the celestial 2-sphere.

- On the other hand the temperature anisotropies are related to $\mathcal{R}$,

$$\frac{\delta T}{T_0} \propto \mathcal{R} \tag{28}$$

Something is not clear...

- We rewrite the expression for the coefficients a_{lm} in terms of $\mathcal{R}$ (the primor. curv. pert.) in Fourier space.

$$a_{lm} = 4\pi(-i)^l \int \frac{d^3k}{(2\pi)^3} Y^*_{lm}(\hat{k}) \Delta_l(k) \mathcal{R}_{\mathbf{k}}, \qquad (29)$$

with $\Delta_l(k)$ the transfer functions.

Something is not clear...

- But recall that $\mathcal{R}$ is related to the field mode $v_{\mathbf{k}}$ (the MS variable), i.e.

$$\mathcal{R}_{\mathbf{k}} = v_{\mathbf{k}}/z$$

which was obtained using quantum theory.

- How does the identification/transition $\hat{v}_{\mathbf{k}} \to v_{\mathbf{k}}$ takes place?

$$a_{lm} = 4\pi(-i)^l \int \frac{d^3k}{(2\pi)^3} Y^*_{lm}(\hat{k}) \Delta_l(k) \left(\frac{\hat{v}_{\mathbf{k}}}{z}\right) ???, \qquad (30)$$

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- Evidently if one uses $\langle \mathrm{BD}|\hat{v}_{\mathbf{k}}|\mathrm{BD}\rangle = v_{\mathbf{k}} = 0$, then $a_{lm} = 0 \ \forall l, m$. ☹
- The standard argument: It is incorrect to state that $a_{lm} = 0$; the correct statement is $\mathbb{E}\{a_{lm}\} = 0$, i.e. the average over an "ensemble of universes."
- But then, what is the prediction for a_{lm} **within our universe**?

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- **Official Answer:** The classic field $v_{\mathbf{k}}$ that is being used to calculate a_{lm} is related with the "quantum fluctuations" of the vacuum, this is

$$v_{\mathbf{k}} = \sqrt{\langle \mathrm{BD}|\hat{v}^2_{\mathbf{k}}|\mathrm{BD}\rangle}\, e^{i\alpha_k} \tag{31}$$

- Thus, *the amplitude of the classical quantity* $v_{\mathbf{k}}$ *is equal to the quantum uncertainty* $\sqrt{\langle BD|\hat{v}^2_{\mathbf{k}}|BD\rangle}$ and α_k corresponds to a random phase.

Something is not clear...

- We have a similar situation with the angular spectrum,

$$C_l = 4\pi \int \frac{dk}{k} \Delta_l^2(k) \mathcal{P}_{\mathcal{R}}(k) \tag{32}$$

where

$$\mathbb{E}\{\mathcal{R}_{\mathbf{k}} \mathcal{R}_{\mathbf{k}'}\} = (2\pi^2/k^3) \mathcal{P}_{\mathcal{R}}(k) \delta^{(3)}(\mathbf{k} + \mathbf{k}') \tag{33}$$

and $\mathbb{E}\{...\}$ denotes an ensemble average of the **classical field** $\mathcal{R}_k$

Something is not clear...

- For the *primordial power spectrum* we ask: How does the identification

$$\mathbb{E}\{\mathcal{R}^{\rm cl}_{\mathbf{k}}\mathcal{R}^{\rm cl}_{\mathbf{k}'}\} = \langle {\rm BD}|\hat{\mathcal{R}}_{\mathbf{k}}\hat{\mathcal{R}}_{\mathbf{k}'}|{\rm BD}\rangle \tag{34}$$

takes place ? Remember $\mathcal{R}_{\mathbf{k}} = v_{\mathbf{k}}/z$

- **Justification 0:** Because it works or *Shut up and calculate!*
- **Justification 1:** We present the most popular approaches:

(i) *Non-commutativity 'fades' during inflation*

(ii) *Quantum squeezing as feature of the emergence of classical behavior*

(iii) *Decoherence*

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"Fading" of non-commutativity

We consider a canonical pair with $\langle\hat{q}\rangle = \langle\hat{p}\rangle = 0$ and $[\hat{q}, \hat{p}] = i\hbar$. Define the symmetrized covariance

$$C_{qp} \equiv \frac{1}{2}\langle\{\hat{q}, \hat{p}\}\rangle. \tag{35}$$

The two operator orderings give

$$\langle\hat{q}\hat{p}\rangle = C_{qp} + \frac{i\hbar}{2}, \qquad \langle\hat{p}\hat{q}\rangle = C_{qp} - \frac{i\hbar}{2}. \tag{36}$$

A first criterion for "'classicality"

If

$$R_{qp} \equiv \frac{|\langle[\hat{q}, \hat{p}]\rangle|}{|\langle\{\hat{q}, \hat{p}\}\rangle|} = \frac{\hbar}{2|C_{qp}|} \ll 1, \tag{37}$$

then $\langle\hat{q}\hat{p}\rangle \simeq \langle\hat{p}\hat{q}\rangle$. It is similar to its classical counter parts.

The canonical commutator itself remains exactly $i\hbar$.

Ashtekar, Corichi and Kesavan (2020), Sec. III.

The covariance has a classical counterpart

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The anti-commutator $\frac{1}{2}\{\hat{q},\hat{p}\}$ is Hermitian, so C_{qp} is Real
For classical random variables with probability density $P(q,p)$,

$$C_{qp}^{\rm cl} = \mathbb{E}[qp] = \mathbb{E}[pq] = \int dq\, dp\, P(q,p)\, qp \qquad (\bar{q} = \bar{p} = 0). \tag{38}$$

The quantum C_{qp} has 3 cases:

- $C_{qp} > 0$: position and momentum deviations tend to have the same sign, positive correlated.
- $C_{qp} < 0$: the deviations tend to have opposite signs, negative correlated.
- $C_{qp} = 0$: no linear correlation, not necessarily independence.

For nonzero means, use $\Delta\hat{q} = \hat{q} - \langle\hat{q}\rangle$ and $\Delta\hat{p} = \hat{p} - \langle\hat{p}\rangle$ in the covariance.

A simple example

- Consider a classical cloud of noninteracting particles of mass m, with initial q_0 and p_0 independent and

$$\mathbb{E}\{q_0\} = \mathbb{E}\{p_0\} = 0, \qquad \mathbb{E}\{q_0 p_0\} = 0 \qquad \mathbb{E}\{p_0^2\} = \sigma_p^2 \tag{39}$$

  The eqs. of motion are

$$q(t) = q_0 + \frac{t}{m} p_0, \qquad p(t) = p_0, \qquad \sigma_p^2 = \langle \hat{p}_0^2 \rangle. \tag{40}$$

- Therefore

$$\begin{aligned} C_{qp}^{\rm cl}(t) &= \mathbb{E}[q(t)p(t)] \\ &= \mathbb{E}\{q_0 p_0\} + \frac{t}{m}\mathbb{E}\{p_0^2\} \\ &= \frac{t}{m}\sigma_p^2 \end{aligned} \tag{41}$$

A simple example

- Take an initially centered quantum Gaussian packet with $C_{qp}(0) = 0$. Then in the Heissenberg picture

$$\hat{q}(t) = \hat{q}_0 + \frac{t}{m}\hat{p}_0, \qquad \hat{p}(t) = \hat{p}_0, \qquad \sigma_p^2 = \langle \hat{p}_0^2 \rangle. \tag{42}$$

$$\langle \hat{q}(t)\hat{p}(t) \rangle = \frac{t}{m}\sigma_p^2 + \frac{i\hbar}{2}, \tag{43}$$

- Consequently,

$$C_{qp}(t) = \frac{t}{m}\sigma_p^2 = C_{qp}^{\rm cl}(t) = \mathbb{E}[q(t)p(t)] \tag{44}$$

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- The classical initial distribution can be chosen Gaussian with the same position and momentum variances as the quantum packet. The comparison concerns ensemble moments, not individual quantum trajectories.

$$\boxed{R_{qp}(t) = \frac{m\hbar}{2t\sigma_p^2}} \quad (t > 0). \tag{45}$$

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Return to slow roll inflation

For a canonical single inflaton, at quadratic order,

- *Mukhanov–Sasaki variable:* For $v = z\mathcal{R}$, the action at second order is

$$S_v = \frac{1}{2}\int d\eta\, d^3x\, [v'^2 - (\nabla v)^2 + (z''/z)v^2]$$

  with canonical momentum $p = v'$.
- Expressing the quantum field in Fourier modes

$$\hat{v}(\eta, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} \hat{v}_{\mathbf{k}}(\eta) e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{46}$$

  Because the field is real $\hat{v}^\dagger_{\mathbf{k}} = \hat{v}_{-\mathbf{k}}$

Return to slow roll inflation

- The field mode expansion and the canonical momentum $\hat{p} = \hat{v}'$ are

$$\hat{v}_{\mathbf{k}}(\eta) = f_k(\eta)\hat{a}_{\mathbf{k}} + f_k(\eta)^* \hat{a}^\dagger_{-\mathbf{k}} \tag{47}$$

$$\hat{p}_{\mathbf{k}}(\eta) = f_k'(\eta)\hat{a}_{\mathbf{k}} + f_k^{*'}(\eta)\hat{a}^\dagger_{-\mathbf{k}} \tag{48}$$

  where $[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}] = \delta^{(3)}(\mathbf{k} - \mathbf{k}')$.
- Using the normalization

$$f_k f_k^{*'} - f_k^* f_k' = i \tag{49}$$

  we obtain

$$[\hat{v}_{\mathbf{k}}, \hat{p}_{\mathbf{k}}] = i\delta^{(3)}(\mathbf{k} + \mathbf{k}') \tag{50}$$

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- As a consequence

$$\frac{1}{2}\langle\{\hat{v}_{\mathbf{k}}, \hat{p}_{\mathbf{k}'}\}\rangle = \mathrm{Re}(f_k f_k^{*'})\delta^{(3)}(\mathbf{k}+\mathbf{k}') \tag{51}$$

- We can define the **covariance per mode**, $C_{vp}(k,\eta)$, as the coefficient of this delta function:

$$C_{vp}(k,\eta) = \mathrm{Re}(f_k f_k^{*'}) \tag{52}$$

- The corresponding ratio of the commutator and anticommutator coefficients is:

$$R_k(\eta) = \frac{1}{2|C_{vp}(k,\eta)|} = \frac{1}{2|\mathrm{Re}(f_k f_k^{*'})|} \tag{53}$$

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- For the BD vacuum in de Sitter approximation, we have

$$f_k(\eta) = \frac{1}{\sqrt{2k}}\left(1 - \frac{i}{k\eta}\right)e^{-ik\eta} \tag{54}$$

hence

$$\mathrm{Re}(f_k f_k^{*\prime}) = -\frac{1}{2k^3\eta^3} \tag{55}$$

Thus

$$R_k(\eta) = |k\eta|^3 \simeq \left(\frac{k}{aH}\right)^3$$

becomes very small $\ll 1$ for super-Hubble modes $k \ll aH$.

Some problems with the first criterion

- $R_k(\eta)$ does not always decrease with η, e.g. in a **radiation filled universe**. So in the RD era the universe is quantum?
- Evidently $[\hat{S}_z, \hat{S}^2] = 0$, so a spin 1/2 particle is classical?
- From the example: it seems that a quantum Gaussian packet becomes classical simply by waiting enough time.

Thus

The first criterion is not a necessary and sufficient condition for classicality.

eofL2