Introducing Warm Inflation: Dynamics, Microphysics, and Observational Signatures

Lecture 2: Nonequilibrium Quantum Field Theory – Part 1

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13–16 October 2026

Where we are in the minicourse

- **Lecture 1: Introduction and Motivation**
  - Inflation, what for ?
  - Cold versus warm inflation
  - Brief overview of applications
- **Lecture 2: Basics of nonequilibrium QFT - part 1**
  - Dissipation in Quantum Mechanics: the Caldeira-Leggett model
  - Schwinger-Keldysh formalism for non-equilibrium QFT
  - Deriving effective equations of motion

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Today: a derivation, in two languages

First a particle plus oscillators; then fields on a closed time path.

Warm inflation: the effective equation that needs explaining

- Interactions of the inflaton with other d.o.f. $\Rightarrow$ dissipation terms:

$$\dot{\rho}_r + 4H\rho_r = \Upsilon\dot{\phi}^2 \quad \Rightarrow \rho_r \sim \Upsilon\dot{\phi}^2/(4H)$$

$\Rightarrow$ The radiation energy density during inflation can stabilize.

- Production of radiation $\Rightarrow$ friction term in the inflaton equation:

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} + \Upsilon\dot{\phi} = 0$$

- $\Upsilon\dot{\phi}$ (friction term) describes how inflaton's interactions with other fields backreact on the inflaton dynamics.

Questions:

- What interactions determine Υ?
- What was approximated when memory became local friction?
- Where are the fluctuations associated with that friction?

Today we derive the nonlocal structure before making a local approximation.

A system can evolve in a nearly equilibrated bath

system: $q(t)$ or $\phi(x)$

environment: many microscopic modes

- The combined system has unitary microscopic dynamics.
- We follow selected observables and trace over the environment.
- The system may evolve while the bath remains close to equilibrium.

Equilibrium of the bath does not imply equilibrium of the system.

I. A particle plus a bath

The Caldeira–Leggett model

A solvable example of memory, fluctuations, and their relation.

Brownian motion: the familiar effective description

$$M\ddot{q} + \eta\dot{q} + V'(q) = \xi(t).$$

- $\eta\dot{q}$ damps the average motion.
- ξ represents fluctuations of the medium.
- If ξ were absent, every velocity would decay to zero.

The microscopic question

Can the same bath produce both damping and nonzero equilibrium fluctuations?

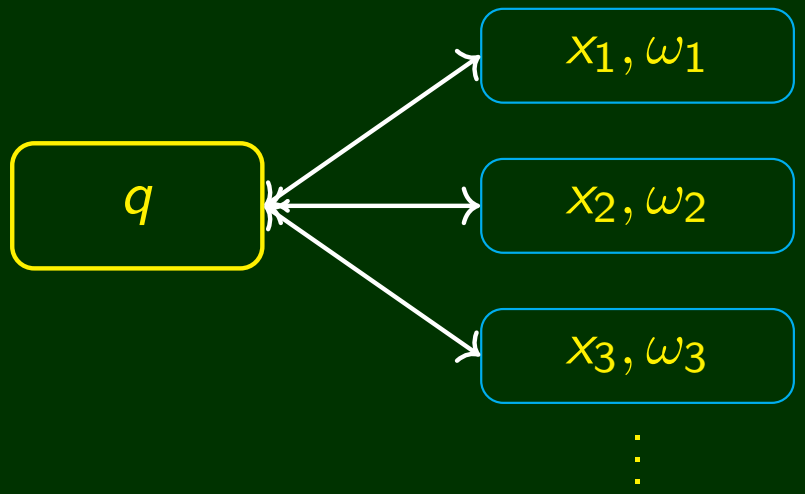
The system + environment Hamiltonian

Systen:

$$H_S = \frac{p^2}{2M} + V(q), \qquad [q, p] = i.$$

Environment: many oscillators

$$H_B = \sum_i \left(\frac{p_i^2}{2m_i} + \frac{1}{2} m_i \omega_i^2 x_i^2 \right), \qquad [x_i, p_j] = i\delta_{ij}.$$



- V may be nonlinear; the bath will remain harmonic.
- Operators obey $\dot{A} = i[H, A]$. Also $[q, p] = i$, $[x_i, p_j] = i\delta_{ij}$.

Coupling and the counterterm

$$H = H_S + H_B + H_{SB},$$

$$H_{SB} = -q\sum_i c_i x_i + \frac{q^2}{2}\sum_i \frac{c_i^2}{m_i\omega_i^2}.$$

- The last term cancels the static bath-induced potential shift.
- $V(q)$ is the chosen system potential in this convention.

Coupled Heisenberg equations:

$$M\ddot{q} + V'(q) + q\sum_i \frac{c_i^2}{m_i\omega_i^2} = \sum_i c_i x_i,$$

$$\ddot{x}_i + \omega_i^2 x_i = \frac{c_i}{m_i} q.$$

Strategy: Solve the driven bath oscillator for a given history $q(t)$, then substitute into the system equation

Solve one driven bath oscillator

$$x_i(t) = x_i^h(t) + \frac{c_i}{m_i\omega_i}\int_0^t ds\, \sin[\omega_i(t-s)]q(s),$$

$$x_i^h(t) = x_i(0)\cos\omega_i t + \frac{p_i(0)}{m_i\omega_i}\sin\omega_i t.$$

- x_i^h carries initial bath data.
- The integral describes the response to the system's past motion.

Substitute the bath solution

$$M\ddot{q} + V'(q) + q\sum_i \frac{c_i^2}{m_i\omega_i^2} - \sum_i \frac{c_i^2}{m_i\omega_i}\int_0^t ds\, \sin[\omega_i(t-s)]q(s) = \sum_i c_i x_i^h(t),$$

The bath has disappeared as an independent dynamical variable. Its initial data and response remain in the effective equation.

Integration by parts exposes the memory term:

$$\int_0^t ds\, \sin[\omega(t-s)]q(s) = \frac{q(t) - q(0)\cos\omega t}{\omega} - \frac{1}{\omega}\int_0^t ds\, \cos[\omega(t-s)]\dot{q}(s).$$

- The $q(t)$ term cancels the Hamiltonian counterterm.
- A past-velocity integral remains.
- The lower boundary retains information about preparation.

The exact generalized Langevin equation

$$M\ddot{q} + V'(q) + \int_0^t ds\, \gamma(t-s)\dot{q}(s) = \xi_B(t)$$

$$\gamma(t) = \sum_i \frac{c_i^2}{m_i\omega_i^2} \cos\omega_i t.$$

$$\xi_B(t) \equiv \sum_i c_i x_i^h(t) - \gamma(t)q(0).$$

- Memory: a weighted integral over the past velocity.
- Fluctuating force: initial bath operators in ξ_B.
- Damping describes reduced dynamics and an appropriate bath/time-scale limit.

Encode the bath in a spectral density

Introduce a spectral density of modes in the bath:

$$J(\omega) = \frac{\pi}{2} \sum_i \frac{c_i^2}{m_i \omega_i} \delta(\omega - \omega_i), \qquad \omega > 0.$$

$$\gamma(t) = \frac{2}{\pi} \int_0^\infty d\omega \, \frac{J(\omega)}{\omega} \cos \omega t.$$

- J is a coupling-weighted density of bath modes.
- J encodes the relative fraction of modes with a given frequency present in the bath.
- Low frequencies govern long-time response.
- A physical bath also has an ultraviolet scale or cutoff.

Thermal bath fluctuations

Define the symmetrized force correlator:

$$N_B(t-s) = \tfrac{1}{2}\langle\{\xi_B(t), \xi_B(s)\}\rangle.$$

For thermal harmonic oscillators,

$$N_B(\tau) = \frac{1}{\pi}\int_0^\infty d\omega\, J(\omega)\coth\left(\frac{\beta\omega}{2}\right)\cos\omega\tau.$$

Both kernels are fixed by $J(\omega)$:

$$\gamma(\tau) = \frac{2}{\pi}\int_0^\infty d\omega\,\frac{J(\omega)}{\omega}\cos\omega\tau,$$

$$N_B(\tau) = \frac{1}{\pi}\int_0^\infty d\omega\, J(\omega)\coth\left(\frac{\beta\omega}{2}\right)\cos\omega\tau.$$

- The bath response and fluctuations are linked: fluctuation–dissipation relation
- At $T = 0$, zero-point fluctuations remain.

The classical thermal limit

For the frequencies relevant to the system, $\omega \ll T$:

$$\coth\left(\frac{\omega}{2T}\right) \simeq \frac{2T}{\omega}.$$

$$N_B(\tau) \simeq T\gamma(\tau).$$

What the relation says

A large thermal fluctuating force accompanies strong bath-induced damping.

An Ohmic bath: a preview of Lecture 3

Take a Drude-regulated spectral density (simplest choice $J(\omega) = \eta\omega$ diverges in the UV):

$$J(\omega) = \eta\omega \frac{\Gamma^2}{\omega^2 + \Gamma^2}, \qquad \gamma(\tau) = \eta\Gamma e^{-\Gamma|\tau|}.$$

- Bath memory time: Γ^{-1}.
- Slowly varying velocity, after transients: memory approaches $\eta\dot{q}$.
- In the classical limit: N_B approaches $2\eta T\delta(\tau)$.

Short memory and classical noise are separate approximations.

From the particle to a field

Particle problem	Field problem
$q(t)$	$\phi(x)$
Bath oscillator force	Bath operator correlations
$\gamma(t-s)$	Retarded spacetime kernel
$\xi_B(t)$	Fluctuating field $\xi(x)$
Initial bath state	Initial density matrix

How do we organize these correlations in an interacting quantum field theory?

The Schwinger–Keldysh formalism

The nonequilibrium problem

the general (non-equilibrium) description of a system is determined by the time evolution of the density matrix operator, $\hat{\rho}$:

$$i\hbar \frac{\partial \hat{\rho}(t)}{\partial t} = \left[\hat{H}(t), \hat{\rho}(t)\right]$$

non-equilibrium situation arises whenever the Hamiltonian does not commute with the density matrix.

The formal solution:

$$\hat{\rho}(t) = \hat{U}(t, t_0)\hat{\rho}(t_0)\hat{U}^{-1}(t, t_0)$$

at equilibrium:

$$\hat{U}(t, t_0) = e^{\frac{i}{\hbar}\hat{H}(t-t_0)}$$

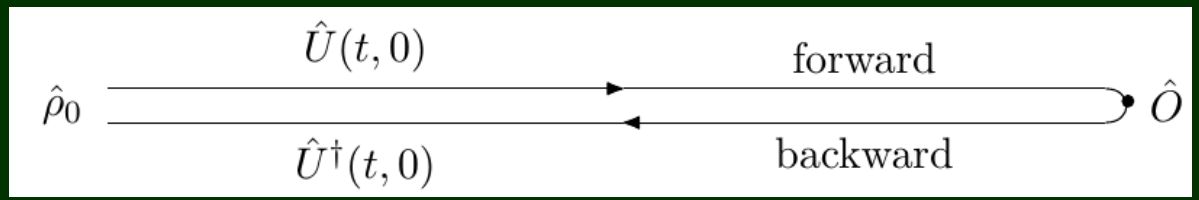
Ensemble averages of operators and correlation functions:

The ensemble average of operators:

$$\langle \hat{\mathcal{O}}(t) \rangle = \frac{1}{\mathrm{tr}\hat{\rho}(t_0)} \mathrm{tr}\left[\hat{\rho}(t_0)\hat{U}^{-1}(t,t_0)\hat{\mathcal{O}}\hat{U}(t,t_0)\right]$$

Recipe:

- (a) take the initial state (or density matrix), evolve it forward in time from t_0 to t,
- (b) insert the operator in question and
- (c) evolve the state backwards in time back to t_0.



For the correlation function ($t_1 > t_2$),

$$\langle \hat{\mathcal{O}}(t_1)\hat{\mathcal{O}}(t_2)\rangle = \frac{1}{\mathrm{tr}\hat{\rho}(t_0)}\mathrm{tr}\left[\hat{\rho}(t_0)\hat{U}^{-1}(t_1,t_0)\hat{\mathcal{O}}\hat{U}(t_1,t_2)\hat{\mathcal{O}}\hat{U}(t_2,t_0)\right]$$

recipe:

- (a) evolve the initial state to t_2,
- (b) insert the operator,
- (c) evolve it further to t_1,
- (d) insert the second operator, and finally
- (e) evolve the state backwards to t_0.

$$\Downarrow$$

Correlation functions and ensemble averages can be obtained by considering evolution forward in time (from $t_0 \to -\infty$), insertion of operators and backwards to the original time

The Closed Time Path formalism avoids reference to the state at $t = +\infty$: the finite state is exactly the initial one:

$$\Downarrow$$

the quantum system is let to evolve first in the forward direction in time and then to "rewind" its evolution back, playing the movie in the backward direction.

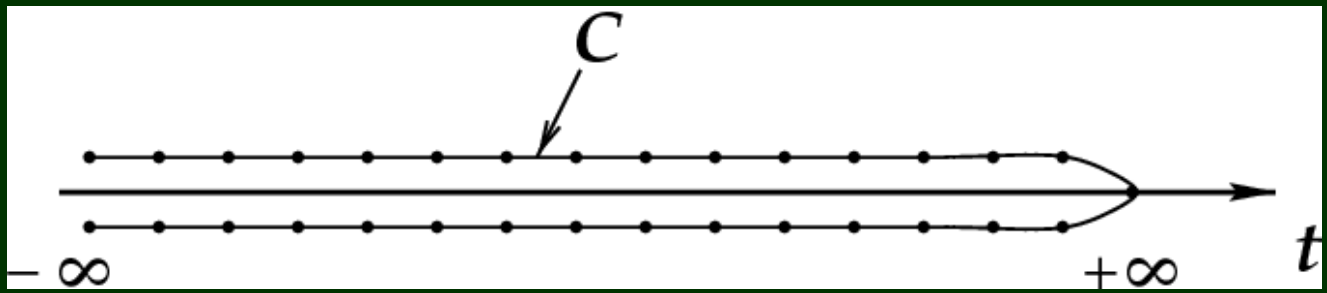


Figure: The closed time contour contour $\mathcal{C}$. Dots on the forward and the backward branches of the contour denote the discreticized time points.

In–out and in–in answer different questions

In–out

$\langle \mathrm{out}|U|\mathrm{in}\rangle$

- Transition amplitudes.
- Specified initial and final states.

In–in

$\mathrm{Tr}[\rho_0 O_H(t)]$

- Time-dependent expectation values.
- Specified initial state.

We need in–in observables for cosmological evolution.

Along the closed contour $\mathcal{C}$:

$$Z = \frac{1}{\mathrm{tr}\{\rho_0\}} \int \mathcal{D}_{\mathcal{C}}\phi \, \exp\{iS[\phi]\}, \quad S[\phi] = \int_{\mathcal{C}} dt \int d^3x [\frac{1}{2}\phi \mathcal{D}^{-1}\phi - V(\phi)]$$

and $\mathcal{D}^{-1} = [-\partial^2 - m^2]$. Divide the bosonic field ϕ onto the two components ϕ_+ and ϕ_- which reside on the forward and the backward parts of the time contour correspondingly:
$\Rightarrow S \equiv S[\phi_+] - S[\phi_-]$
For $\mathcal{L}_{\mathrm{int}} = -\lambda\phi^4/4!$:

$$+ \text{ vertex} : -i\lambda, \qquad - \text{ vertex} : +i\lambda.$$

- Propagators carry two branch labels.
- The interaction is the same physical interaction on both branches.

Four contour components

Wightman functions:

$$G^{>}(x,y) = \langle \hat{\phi}(x)\hat{\phi}(y)\rangle \equiv \langle \phi_-(x)\phi_+(y)\rangle,$$
$$G^{<}(x,y) = \langle \hat{\phi}(y)\hat{\phi}(x)\rangle \equiv \langle \phi_-(y)\phi_+(x)\rangle$$

These two functions build the contour propagators:

$$G^{++} = \langle T_+[\phi_+(x)\phi_+(x')]\rangle \equiv G^{>}\theta(x^0-y^0) + G^{<}\theta(y^0-x^0),$$
$$G^{--} = \langle T_-[\phi_-(x)\phi_-(x')]\rangle \equiv G^{>}\theta(y^0-x^0) + G^{<}\theta(x^0-y^0),$$
$$G^{+-} = G^{<}, \qquad G^{-+} = G^{>}.$$

- $++$: time ordered; $--$: anti-time ordered.
- Mixed branches: Wightman functions.

Identity: $\boxed{G^{++} + G^{--} = G^{+-} + G^{-+}}$

A quick check: For $x^0 > y^0$, substitute the time-ordering expressions and verify the identity

Defining the statistical and spectral functions:

$$F(x,y) = \tfrac{1}{2}\langle\{\hat{\phi}(x), \hat{\phi}(y)\}\rangle,$$

$$\rho(x,y) = i\langle[\hat{\phi}(x), \hat{\phi}(y)]\rangle.$$

$$G^{>} = F - \tfrac{i}{2}\rho, \qquad G^{<} = F + \tfrac{i}{2}\rho.$$

- F is symmetric: occupation and fluctuations.
- ρ is antisymmetric: commutator and response.
- $\mathcal{R}(x,y) = \theta(x^0 - y^0)\rho(x,y)$.
- The commutator determines linear response.
- The step function makes causality explicit.

One free mode makes the distinction concrete

For a unit-mass oscillator of frequency ω_0 in a stationary state,

$$F(\tau) = \frac{n + \frac{1}{2}}{\omega_0} \cos \omega_0 \tau, \qquad \rho(\tau) = \frac{1}{\omega_0} \sin \omega_0 \tau.$$

- The occupation n changes F.
- The free commutator does not depend on n.
- In an interacting theory, the spectral function can depend on the state.

Thermal equilibrium: the KMS relation

For $\rho_{\rm th} = Z^{-1} e^{-\beta H}$ and a stationary state,

$$G^{>}(t) = G^{<}(t + i\beta).$$

With our Fourier convention,

$$G^{>}(\omega) = e^{\beta\omega} G^{<}(\omega).$$

- It follows from trace cyclicity and thermal weighting.
- It is an equilibrium relation, not a general nonequilibrium identity.

KMS relates statistical fluctuations to response

Define a real frequency-space spectral weight:

$$\mathcal{A}(\omega) \equiv G^{>}(\omega) - G^{<}(\omega) = -i\rho(\omega).$$

$$F(\omega) = \tfrac{1}{2} \coth\left(\frac{\beta\omega}{2}\right) \mathcal{A}(\omega).$$

$$\mathcal{A}(\omega) = 2\,\mathrm{Im}\,\mathcal{R}(\omega).$$

At equilibrium, response and fluctuations are linked. Away from equilibrium, F and ρ must generally be evolved independently.

Separate fluctuations from response: Keldysh rotation

Recall: $\boxed{G^{++} + G^{--} = G^{+-} + G^{-+}}$

The Keldysh rotation makes the effective dynamics easier to read:

$$\phi_c = \frac{1}{2}(\phi_+ + \phi_-), \qquad \phi_\Delta = \phi_+ - \phi_-.$$

$$\phi_+ = \phi_c + \frac{1}{2}\phi_\Delta, \qquad \phi_- = \phi_c - \frac{1}{2}\phi_\Delta.$$

$$J_+\phi_+ - J_-\phi_- = J_\Delta\phi_c + J_c\phi_\Delta.$$

- Average field: ϕ_c.
- Difference/response field: ϕ_Δ.

The rotated correlation matrix

With $a, b \in \{c, \Delta\}$ and contour-ordered correlations,

$$\langle \phi_a(x)\phi_b(y)\rangle_C = \begin{pmatrix} F(x,y) & -i\mathcal{R}(x,y) \\ -i\mathcal{R}_A(x,y) & 0 \end{pmatrix}.$$

$$\mathcal{R}_A(x,y) = \mathcal{R}(y,x).$$

- cc: statistical fluctuations.
- Mixed entries: causal response.
- $\Delta\Delta = 0$: the contour-component identity.

A scalar action in the rotated variables

For a real scalar with quartic coupling,

$$S[\phi] = \int_x \left[\tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \right].$$

After integration by parts,

$$S[\phi_+] - S[\phi_-] = -\int_x \phi_\Delta(\Box + m^2)\phi_c - \frac{\lambda}{4!}\int_x (4\phi_\Delta\phi_c^3 + \phi_\Delta^3\phi_c).$$

The term linear in ϕ_Δ contains the classical equation.

Vary before setting the difference field to zero

$$\left.\frac{\delta S_C}{\delta \phi_\Delta(x)}\right|_{\phi_\Delta=0} = 0$$

$$\Longrightarrow \quad (\Box + m^2)\phi_c + \frac{\lambda}{3!}\phi_c^3 = 0.$$

- Setting $\phi_\Delta = 0$ in S_C first gives zero action.
- Varying first extracts the physical equation.
- Higher powers of ϕ_Δ encode additional quantum structure.

Integrating out the environment

The effective action and noise

The same bath correlations generate response and fluctuations.

Choose a system–environment interaction:

$$S[\phi,\chi] = S_S[\phi] + S_B[\chi] - g\int_x \phi(x)\mathcal{O}_B(x).$$

- $\mathcal{O}_B$ is a Hermitian bath operator.
- It may be elementary or composite, e.g. χ^2.

The two bath kernels

$$\mathcal{R}_B(x,y) = i\theta(x^0 - y^0)\langle[\mathcal{O}_B(x), \mathcal{O}_B(y)]\rangle,$$
$$N(x,y) = \frac{g^2}{2}\langle\{\mathcal{O}_B(x), \mathcal{O}_B(y)\}\rangle_c,$$
$$\Sigma_R(x,y) \equiv -g^2\mathcal{R}_B(x,y).$$

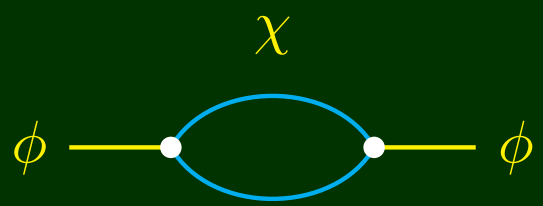
- Σ_R: causal bath backreaction in our equation convention.
- N: symmetrized fluctuations of the bath force.

Both are computed from the same microscopic bath.

One field-theory example: a composite bath operator

Let $\mathcal{O}_B = \chi^2 - \langle \chi^2 \rangle$ for a zero-mean Gaussian bath field. Wick's theorem gives

$$C_B^>(x,y) \equiv \langle \mathcal{O}_B(x)\mathcal{O}_B(y)\rangle = 2[G_\chi^>(x,y)]^2.$$



The connected two-point bath correlator becomes a loop kernel.

Read the quadratic effective action

$$S_{\rm eff} = -\int_x \phi_\Delta(x)\mathcal{E}[\phi_c](x) - \int_{x,y} \phi_\Delta(x)\Sigma_R(x,y)\phi_c(y) + \frac{i}{2}\int_{x,y} \phi_\Delta(x)N(x,y)\phi_\Delta(y) + \cdots .$$

$$\mathcal{E}[\phi_c] = \Box\phi_c + V'_{\rm ren}(\phi_c).$$

- Real response term: deterministic backreaction.
- Imaginary quadratic term: fluctuations and decoherence.

First extract the deterministic equation

$$\left.\frac{\delta S_{\rm eff}}{\delta \phi_\Delta(x)}\right|_{\phi_\Delta=0} = 0$$

$$\Longrightarrow \quad \Box\phi_c + V'_{\rm ren}(\phi_c) + \int_y \Sigma_R(x,y)\phi_c(y) = 0.$$

- The retarded kernel only samples earlier times.
- Its static part can modify the potential.
- Its dynamical part can contain dispersive and dissipative response.

Turn the imaginary quadratic term into noise

Use the normalized Gaussian identity:

$$e^{-\frac{1}{2}\phi_\Delta N \phi_\Delta} = \int \mathcal{D}\xi \, P[\xi] e^{i\int_x \xi \phi_\Delta},$$

$$P[\xi] = \mathcal{N} \exp\left[-\frac{1}{2}\int_{x,y} \xi(x) N^{-1}(x,y)\xi(y)\right].$$

- This is a Hubbard–Stratonovich transformation.
- The quadratic form N is a noise covariance.
- The inverse is taken on its supported subspace.

The auxiliary field has definite statistics

$$\langle \xi(x) \rangle_\xi = 0, \qquad \langle \xi(x)\xi(y) \rangle_\xi = N(x,y).$$

For a real test function h,

$$\int_{x,y} h(x) N(x,y) h(y) = g^2 \left\langle \left(\int_x h(x) \delta\mathcal{O}_B(x) \right)^2 \right\rangle \geq 0.$$

Physical meaning

The stochastic field reproduces the bath's symmetrized force fluctuations at this level of the effective description.

The nonlocal stochastic field equation

$$\boxed{\Box\phi_c + V'_{\rm ren}(\phi_c) + \int_y \Sigma_R(x,y)\phi_c(y) = \xi(x)}\,.$$

$$\langle \xi(x)\xi(y)\rangle_\xi = N(x,y).$$

- Retarded response: causality and memory.
- Noise covariance: environmental fluctuations.
- Gaussian stochastic form: the stated effective truncation.

No local friction coefficient has yet been assumed.

A nonlinear system coupling gives multiplicative noise

Replace $g\phi\mathcal{O}_B$ by $f(\phi)\mathcal{O}_B$. To first order in the difference field,

$$f(\phi_+) - f(\phi_-) \simeq f'(\phi_c)\phi_\Delta.$$

$$\text{noise force} \sim f'(\phi_c)\xi_B.$$

- $f = g\phi$: additive force.
- $f \propto \phi^2$: field-dependent force and response.
- A later white-noise limit must preserve its microscopic prescription.

The field-theory fluctuation–dissipation relation

For a stationary thermal bath, KMS implies

$$N(\omega, \mathbf{k}) = -\coth\left(\frac{\omega}{2T}\right) \operatorname{Im} \Sigma_R(\omega, \mathbf{k}).$$

If a local limit later gives $\Sigma_R \supset -i\Upsilon\omega$, then

$$N(\omega, \mathbf{0}) \simeq \Upsilon\omega \coth\left(\frac{\omega}{2T}\right) \longrightarrow 2\Upsilon T \quad (|\omega| \ll T).$$

- Finite response and thermal fluctuations are linked.
- Quantum noise generally remains frequency dependent.

Suppose $\Sigma_R(x,y) = -\partial_{y^0} D(x,y)$. Suppressing spatial integrals,

$$\int_{t_0}^{t} ds\, \Sigma_R(t,s)\phi(s) = -[D(t,s)\phi(s)]_{t_0}^{t} + \int_{t_0}^{t} ds\, D(t,s)\dot{\phi}(s).$$

For the scalar field theory, we obtain[1]

$$\partial^2\phi_c(x) + V'_{\rm eff}(\phi_c) + \int d^4x' D(x,x')\dot{\phi}_c(x') = \xi(x)$$

and the noise kernel and the dissipation kernel $\mathcal{D}(x,x')$ are related by the fluctuation-dissipation relation

$$N(\mathbf{p},\omega) = 2\omega\left[n(\omega) + \frac{1}{2}\right] D(\mathbf{p},\omega)$$

$\Rightarrow$ the effective equation of motion for the field adquires the form of a (nonlocal) Langevin equation !

[1] M. Gleiser and R.O.R., Phys. Rev. **D50**, 2441 (1994)

The inflaton effective equation in an expanding Universe

In spatially flat FLRW,

$$\Box_g \phi = \ddot{\phi} + 3H\dot{\phi} - \frac{\nabla^2 \phi}{a^2}.$$

$$\Box_g \phi_c + V'_{\rm ren}(\phi_c) + \int d^4y \sqrt{-g(y)}\, \Sigma_R(x,y)\phi_c(y) = \xi(x).$$

- Expansion supplies the $3H\dot{\phi}$ term.
- Bath correlations supply additional response and noise.
- Curved-space kernels and bath evolution need consistent approximations.

Four things to carry forward

1. Integrating out a bath leaves both response and fluctuations.
2. The closed time path computes causal in–in observables.
3. The average/difference basis exposes the physical equation and noise.
4. A local warm-inflation equation is a controlled limit of a nonlocal theory (Lecture 3).

Reading and source material

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