Quantum Cosmology

1. Canonical Quantization of GR

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Outline of Lectures

1. Canonical quantization of general relativity;
2. Boundary proposals;
3. Loop quantum cosmology and modified versions;
4. de Broglie-Bohm quantum cosmology.

Outline

1. Hamiltonian Formulation of General Relativity
2. Canonical Quantization of General Relativity
3. Wheeler–DeWitt equation in minisuperspace
4. Internal clocks and quantum evolution
5. Semiclassical limit
6. Conclusions and open questions

Why Quantum Cosmology?

- Primordial universe $\Rightarrow$ high curvature/density $\Rightarrow$ classical spacetime description breaks down;
- Quantum description seems necessary;
- Quantum theory $\Rightarrow$ universe as a whole $\Rightarrow$ emergence of classical spacetime;
- Quantum-gravitational effects $\Rightarrow$ may modify classical Big Bang models;
- Central question$\Rightarrow$ quantum effects avoid classical singularity?

Hamiltonian Formulation of General Relativity

General Relativity

Fundamental variable of GR $\Rightarrow$ spacetime metric

$$g_{\mu\nu}(x), \qquad \mu, \nu = 0, 1, 2, 3.$$

Einstein–Hilbert (EH) action:

$$\boxed{S_{\rm EH} = \frac{1}{2\kappa} \int_{\mathcal{M}} d^4x \sqrt{-g}\, {}^{(4)}R}$$

Boundary contribution $\Rightarrow$ Gibbons–Hawking–York term:

$$\boxed{S_{\rm GHY} = \frac{1}{\kappa} \int_{\partial\mathcal{M}} d^3y \sqrt{|q|}\, K}$$

K trace of *extrinsic curvature* $\Rightarrow$ $K_{ab} = q_a{}^\mu q_b{}^\nu \nabla_\mu n_\nu,$

$n^\mu \Rightarrow$ normal vector.

Complete gravitational action:

$$\boxed{S_{\rm grav} = S_{\rm EH} + S_{\rm GHY}.} \quad \Rightarrow \quad \delta S_{\rm grav} = 0 \quad \Rightarrow \quad \boxed{G_{\mu\nu} = 0}.$$

Foliation of spacetime

Arnowitt–Deser–Misner (ADM) formalism.

Separation:

$$x^\mu = (t, x^i), \qquad i = 1, 2, 3.$$

t labels the spatial hypersurfaces $\Sigma_t \Rightarrow$ choice of **foliation** of spacetime.

$$\mathcal{M} = \bigcup_t \Sigma_t.$$

Need to describe evolution:

$$\Sigma_t \quad \longrightarrow \quad \Sigma_{t+dt}.$$

Spatial metric: $g_{\mu\nu} \Rightarrow$ induces three-dimensional metric on $\Sigma_t \Rightarrow$ intrinsic metric:

$$\boxed{h_{ij}(t, \mathbf{x})}.$$

$$\dot{h}_{ij} = \partial_t h_{ij}?$$

P. A. M. Dirac, Proc. Roy. Soc. A **246**, 333 (1958); R. Arnowitt, S. Deser and C. W. Misner, Phys. Rev. **116**, 1322 (1959).

Lapse and shift

Coordinates (t, x^i).

Vector t^μ connecting *successive leaves* $\Rightarrow$*same spatial* coordinates:

$$\boxed{t^\mu = N n^\mu + N^\mu} \quad , \quad n_\mu N^\mu = 0.$$

Directional derivative along t^μ:

$$t^\mu \partial_\mu = N n^\mu \partial_\mu + N^\mu \partial_\mu \quad \Rightarrow \quad \boxed{\partial_t = N n + N^i \partial_i.}$$

$N \Rightarrow$ **lapse function** $\Rightarrow$ advance in normal direction between two leaves.

$N^i \Rightarrow$ **shift vector** $\Rightarrow$ coordinate shift between successive leaves.

Metric, Action and Hamiltonian

Line element:

$$\boxed{ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt)} \quad , \quad \boxed{g_{\mu\nu} = \begin{pmatrix} -N^2 + N_i N^i & N_j \\ N_i & h_{ij} \end{pmatrix}.}$$

Extrinsic curvature:

$$\boxed{K_{ij} = \frac{1}{2N}\left(\dot{h}_{ij} - D_i N_j - D_j N_i\right)}$$

Einstein–Hilbert:

$$\boxed{S = \frac{1}{2\kappa}\int dt\, d^3x\, N\sqrt{h}\left[{}^{(3)}R + K_{ij}K^{ij} - K^2\right]} \quad , \quad \boxed{\sqrt{-g} = N\sqrt{h}.}$$

Conjugate momentum:

$$\boxed{\pi^{ij} \equiv \frac{\partial \mathcal{L}}{\partial \dot{h}_{ij}} = \frac{\sqrt{h}}{2\kappa}\left(K^{ij} - K h^{ij}\right).} \Rightarrow \boxed{K_{ij}K^{ij} - K^2 = \frac{4\kappa^2}{h}\left(\pi_{ij}\pi^{ij} - \frac{1}{2}\pi^2\right).}$$

Hamiltonian density:

$$\mathscr{H} = \pi^{ij}\dot{h}_{ij} - \mathcal{L} = \underbrace{\boldsymbol{N}\left[\frac{2\kappa}{\sqrt{h}}\left(\pi_{ij}\pi^{ij} - \frac{1}{2}\pi^2\right) - \frac{\sqrt{h}}{2\kappa}{}^{(3)}R\right]}_{\mathcal{H}} + \boldsymbol{N}^i \underbrace{[-2(D_j\pi_i^j)]}_{\mathcal{H}_i}$$

ADM Hamiltonian and momentum constraints

Hamiltonian

$$\boxed{H = \int d^3x \left(N\mathcal{H} + N^i \mathcal{H}_i \right)},$$

Lagrangian density:

$$\boxed{\mathcal{L}_{\rm ADM} = \pi^{ij}\dot{h}_{ij} - N\mathcal{H} - N^i\mathcal{H}_i.}$$

$\Rightarrow$ contains neither $\dot{N}$ nor $\dot{N}^i$ $\Rightarrow$ N and N^i are Lagrange multipliers:

$$\frac{\delta S}{\delta N} = 0 \quad \Rightarrow \quad \boxed{\mathcal{H} = \frac{2\kappa}{\sqrt{h}}\left(\pi_{ij}\pi^{ij} - \frac{1}{2}\pi^2\right) - \frac{\sqrt{h}}{2\kappa}\,{}^{(3)}R \approx 0,}$$

$$\frac{\delta S}{\delta N^i} = 0 \quad \Rightarrow \quad \boxed{\mathcal{H}_i = -2D_j\pi^j{}_i \approx 0.}$$

- $\mathcal{H}$: **Hamiltonian constraint** $\Rightarrow$ generates normal deformations of Σ_t.
- $\mathcal{H}_i$: **momentum constraint** $\Rightarrow$ generates spatial diffeomorphisms on Σ_t.
- ≈ 0: vanishes only on the constraint surface in phase space.

Including: Λ+ matter

Hamiltonian constraint:

$$\boxed{\mathcal{H} = \frac{2\kappa}{\sqrt{h}}\left(\pi_{ij}\pi^{ij} - \frac{1}{2}\pi^2\right) - \frac{\sqrt{h}}{2\kappa}\,{}^{(3)}R + \sqrt{h}\,\rho + \frac{\sqrt{h}}{\kappa}\Lambda \approx 0.}$$

Momentum constraint:

$$\boxed{\mathcal{H}_i = -2D_j\pi^j{}_i - \sqrt{h}\,j_i \approx 0.}$$

For an FLRW universe,

$$\boxed{\mathcal{H}_i \equiv 0.}$$

Hamilton's equations

Phase-space: $\boxed{(h_{ij}(\mathbf{x}), \pi^{ij}(\mathbf{x})).}$

Hamilton's equations:

$$\boxed{\dot{h}_{ij} = \{h_{ij}, H\} = \frac{\delta H}{\delta \pi^{ij}}} \quad , \quad \boxed{\dot{\pi}^{ij} = \{\pi^{ij}, H\} = -\frac{\delta H}{\delta h_{ij}}}$$

First equation:

$$\boxed{\dot{h}_{ij} = \underbrace{2NK_{ij}}_{\text{normal deformation (lapse)}} + \underbrace{D_i N_j + D_j N_i}_{\text{tangential deformation (shift)}}}$$

DeWitt supermetric

DeWitt supermetric G_{ijkl}:

$$\boxed{\frac{1}{\sqrt{h}}\left(\pi_{ij}\pi^{ij}-\frac{1}{2}\pi^{2}\right)=G_{ijkl}\pi^{ij}\pi^{kl}.}$$

Metric on the space of spatial metrics.

$$\boxed{G_{ijkl}=\frac{1}{2\sqrt{h}}\left(h_{ik}h_{jl}+h_{il}h_{jk}-h_{ij}h_{kl}\right).}$$

Relevant for quantization.

DeWitt, Phys. Rev. **160**, 1113 (1967).

Superspace and Minisuperspace

Superspace: space of physical three-geometries:

$$\mathscr{S}(\Sigma) = \frac{\mathrm{Riem}(\Sigma)}{\mathrm{Diff}(\Sigma)} \quad \Rightarrow \quad \text{equivalence classes.}$$

Metrics related by spatial diffeomorphisms $\Rightarrow$ same three-geometry.

$$\mathrm{Riem}(\Sigma): \text{ spatial metrics,} \qquad \mathrm{Diff}(\Sigma): \text{ spatial coordinate changes.}$$

Minisuperspace: spatial symmetries $\Rightarrow$ homogeneity $\Rightarrow$ finite configuration variables:

$$\boxed{h_{ij}(\mathbf{x}, t) \quad \xrightarrow{\text{symmetry reduction}} \quad q^A(t), \qquad A = 1, \ldots, n,}$$

C. W. Misner, "Minisuperspace," in *Magic Without Magic*, pp. 441–473 (1972).

FLRW minisuperspace

Line element

$$\boxed{ds^2 = -N^2(t)\,dt^2 + h_{ij}(t,\mathbf{x})(dx^i + N^i dt)(dx^j + N^j dt).}$$

No preferred spatial direction $\Rightarrow$ shift can be chosen as

$$\boxed{N^i = 0}.$$

Reduced spatial metric $\Rightarrow$ homogeneity and isotropy:

$$h_{ij}(t,r,\theta) = a^2(t)\ \mathrm{diag}\left(\frac{1}{1-kr^2},\ r^2,\ r^2\sin^2\theta\right).$$

For $k = 0$,

$$h_{ij}(t) = a^2(t)\delta_{ij}.$$

FLRW momentum and Hamiltonian

Reduced action (set $V_0 = 1$):

$$\boxed{S_g = \frac{1}{\kappa}\int dt\left(-\frac{3a\dot{a}^2}{N} + 3kNa\right).}$$

From the reduced Lagrangian,

$$p_a = \frac{\partial L_g}{\partial \dot{a}} = -\frac{6a\dot{a}}{\kappa N}, \qquad \dot{a} = -\frac{\kappa N}{6a}p_a.$$

The Legendre transformation gives

$$H_g = p_a\dot{a} - L_g = N\mathcal{H}_g,$$

with

$$\boxed{\mathcal{H}_g = -\frac{\kappa p_a^2}{12a} - \frac{3ka}{\kappa}.}$$

Including matter,

$$H = N\left(\mathcal{H}_g + \mathcal{H}_{\text{matter}}\right). \quad \Rightarrow \quad \boxed{\frac{\delta S}{\delta N} = 0 \quad \Longrightarrow \quad \mathcal{H}_g + \mathcal{H}_{\text{matter}} \approx 0.}$$

FLRW minisuperspace with a scalar field

Minisuperspace variables and momenta:

$$q^A = (a, \phi), \qquad p_a = -\frac{6a\dot{a}}{\kappa N}, \qquad p_\phi = \frac{a^3\dot{\phi}}{N}.$$

Hamiltonian constraint:

$$\boxed{\mathcal{H} = -\frac{\kappa p_a^2}{12a} - \frac{3ka}{\kappa} + \frac{p_\phi^2}{2a^3} + a^3 V(\phi) \approx 0.}$$

With $H = \frac{\dot{a}}{Na}$,

$$\boxed{\frac{1}{N}\frac{d}{dt}\left(\frac{\dot{\phi}}{N}\right) + 3H\frac{\dot{\phi}}{N} + V_{,\phi} = 0.}, \quad \boxed{\left(\frac{\dot{a}}{Na}\right)^2 + \frac{k}{a^2} = \frac{\kappa}{3}\rho_\phi, \quad , \quad \rho_\phi = \frac{\dot{\phi}^2}{2N^2} + V(\phi).}$$

Schutz fluid in FLRW

For a barotropic perfect fluid,

$$P = w\rho.$$

Hamiltonian:

$$\boxed{H_f = N\frac{e^S p_\epsilon^{1+w}}{a^{3w}}.}$$

S: specific entropy; ϵ: fluid velocity potential. p_S, p_ϵ: conjugate momenta.
Canonical transformation

$$p_T = e^S p_\epsilon^{1+w}, \qquad T = -p_S e^{-S} p_\epsilon^{-(1+w)}, \qquad \{T, p_T\} = 1,$$

gives

$$H_f = N\frac{p_T}{a^{3w}}.$$

Hamiltonian constraint is

$$H = N\mathcal{H}, \qquad \boxed{\mathcal{H} = -\frac{\kappa p_a^2}{12a} - \frac{3ka}{\kappa} + \frac{p_T}{a^{3w}} \approx 0.}$$

Then,

$$\boxed{\left(\frac{\dot{a}}{Na}\right)^2 + \frac{k}{a^2} = \frac{\kappa}{3}\rho, \qquad \dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0.} \quad , \quad \rho = \frac{p_T}{a^{3(1+w)}}$$

Schutz, Phys. Rev. D 2, 2762 (1970); Lapchinskii and Rubakov, Theor. Math. Phys. 33, 1076 (1977).

Canonical Quantization of General Relativity

Dirac quantization

Classical phase space and constraints:

$$\boxed{(h_{ij}, \pi^{ij}), \qquad \mathcal{H}(\mathbf{x}) \approx 0, \qquad \mathcal{H}_i(\mathbf{x}) \approx 0}$$

Quantization:

$$h_{ij}(\mathbf{x}) \Rightarrow \hat{h}_{ij}(\mathbf{x}), \qquad \pi^{ij}(\mathbf{x}) \Rightarrow \hat{\pi}^{ij}(\mathbf{x}).$$

$$\boxed{\left[\hat{h}_{ij}(\mathbf{x}), \hat{\pi}^{kl}(\mathbf{y})\right] = i\hbar\, \delta^{k}_{(i}\delta^{l}_{j)}\, \delta^{(3)}(\mathbf{x}-\mathbf{y})}$$

Metric representation:

$$\boxed{\hat{h}_{ij}(\mathbf{x})\Psi = h_{ij}(\mathbf{x})\Psi, \qquad \hat{\pi}^{ij}(\mathbf{x})\Psi = -i\hbar \frac{\delta\Psi}{\delta h_{ij}(\mathbf{x})}}$$

$$\boxed{\Psi = \Psi[h_{ij}(\mathbf{x}), \phi(\mathbf{x}), \ldots]}$$

Wheeler–DeWitt equation

Hamiltonian and momentum constraints $\Rightarrow$ conditions on the quantum state:

$$\boxed{\hat{\mathcal{H}}(x)\Psi \approx 0, \qquad \hat{\mathcal{H}}_i(x)\Psi \approx 0.}$$

The first gives the formal **Wheeler–DeWitt equation**:

$$\left[-2\kappa\hbar^2 G_{ijkl}\frac{\delta^2}{\delta h_{ij}\delta h_{kl}} - \frac{\sqrt{h}}{2\kappa}({}^{(3)}R - 2\Lambda) + \hat{\mathcal{H}}_{\text{matter}}\right]\Psi[h,\phi] \approx 0.$$

$\Psi[h,\phi] \Rightarrow$ amplitude for a spatial geometry and matter configuration.

Classically, two normal deformations generate a spatial deformation:

$$\boxed{\{H[N], H[M]\} = D[h^{ij}(N\partial_j M - M\partial_j N)].}$$

$$H[N] = \int d^3x\, N\mathcal{H}, \qquad D[N^i] = \int d^3x\, N^i\mathcal{H}_i.$$

N, M: lapse functions. D: smeared momentum constraint.

Quantization must preserve this relation $\Rightarrow$ an extra term is an anomaly.

DeWitt, Phys. Rev. **160**, 1113 (1967); Teitelboim, Ann. Phys. **79**, 542 (1973).

Wheeler–DeWitt equation in minisuperspace

Minisuperspace quantization

First reduce, then quantize: spatial symmetries $\Rightarrow$ finite variables $q^A(t)$.

$$L = \frac{1}{2N}\mathcal{G}_{AB}\dot{q}^A\dot{q}^B - NU(q) \quad \Rightarrow \quad p_A = \frac{\mathcal{G}_{AB}\dot{q}^B}{N}.$$

$\mathcal{G}_{AB}$: kinetic metric of minisuperspace. U: potential.

$$\boxed{\mathcal{H} = \frac{1}{2}\mathcal{G}^{AB}p_Ap_B + U \approx 0.}$$

Dirac quantization:

$$\hat{q}^A\Psi = q^A\Psi, \qquad \hat{p}_A\Psi = -i\hbar\partial_A\Psi, \qquad \mathcal{H} \approx 0 \Rightarrow \hat{\mathcal{H}}\Psi = 0.$$

Choose, e.g.:

$$\boxed{\left[-\frac{\hbar^2}{2}\mathcal{G}^{AB}(q)\partial_A\partial_B + U(q)\right]\Psi = 0.}$$

Klein–Gordon-like equation.

Misner, *Minisuperspace*, in *Magic Without Magic* (1972); Hawking and Page, Nucl. Phys. B **264**, 185 (1986).

Wheeler–DeWitt equation: problems

- **Ordering and measure**:
  - *Ordering ambiguity*[1]: depedence $G^{AB}(h)$:

$$G^{AB}p_Ap_B \Rightarrow -\hbar^2 G^{AB}\partial_A\partial_B \quad \neq \quad -\hbar^2\partial_A(G^{AB}\partial_B).$$

  - *Measure*[2]: chosen ordering $\Rightarrow$ seek compatible measure $\Rightarrow$ inner product, weight, normalization.
  - *Self-adjointness*: chosen measure $\Rightarrow \langle\Phi, \hat{H}\Psi\rangle_\mu = \langle\hat{H}\Phi, \Psi\rangle_\mu$.
  - *Path integral*[3]: no chosen ordering $\Rightarrow$ choose measure $\Rightarrow$ **restricts** possible orderings.

  Open: a unique prescription.

- **Time**[4]: no external time $\Rightarrow$ seek internal clocks.

  Open: global validity, equivalence of clocks, observables, probabilities.

[1] DeWitt, PR **160**, 1113 (1967); Hawking and Page, NPB **264**, 185 (1986); Ryan and Turbiner, PLA **333**, 30 (2004); [2] Kaimakkamis and Sil, arXiv:2407.03077 (2024); [3] Halliwell, PRD **38**, 2468 (1988); [4] Isham, arXiv:gr-qc/9210011 (1992).

Wheeler–DeWitt equation: the problem of time

Canonical quantization: $\boxed{\hat{\mathcal{H}}\Psi = 0}$ instead of $i\hbar\partial_t\Psi = \hat{H}\Psi$.

Coordinate time is gauge in GR $\Rightarrow$ no external time in WDW.

Attempts:

- **Internal clock / relational dynamics**[1,2]: choose dynamical variable T.

  $O_{A|T=\tau}$ = value of A at $T = \tau$.

  When constraint deparametrized,

  $p_T + H_{\rm phys} \approx 0 \Rightarrow i\hbar\partial_T\psi = \hat{H}_{\rm phys}\psi$.

- **Semiclassical time**[3]: WKB geometry + quantum matter.

  $\Psi \simeq Ae^{iMS_0/\hbar}\psi \Rightarrow i\hbar\partial_{t_{\rm WKB}}\psi \simeq \hat{H}_m\psi$.

Remaining questions:

- Does clock cover whole evolution?
- Different clocks $\Rightarrow$ equivalent theories?
- Probabilities/observables defined?
- When semiclassical time approximation is valid?

[1] Brown and Kuchař, Phys. Rev. D **51**, 5600 (1995); [2] Page and Wootters, Phys. Rev. D **27**, 2885 (1983); Rovelli, Phys. Rev. D **43**, 442 (1991); Dittrich, Class. Quantum Grav. **23**, 6155 (2006); [3] Kiefer and Wichmann, Gen. Relativ. Gravit. **50**, 66 (2018).

FLRW universe with a scalar field

Hamiltonian constraint: $q^A = (a, \phi)$.

$$\boxed{\mathcal{H} = -\frac{\kappa}{12a} p_a^2 + \frac{p_\phi^2}{2a^3} - \frac{3ka}{\kappa} + a^3 V(\phi) \approx 0.}$$

Quantization $\Rightarrow$ introduce the ordering parameter p:

$$\boxed{\left[\frac{\kappa\hbar^2}{12a}\left(\partial_a^2 + \frac{p}{a}\partial_a\right) - \frac{\hbar^2}{2a^3}\partial_\phi^2 - \frac{3ka}{\kappa} + a^3 V(\phi)\right]\Psi \approx 0.}$$

$$\boxed{\text{Laplace–Beltrami ordering: } p = 1.}$$

$$\boxed{\Delta_{\mathcal{G}} = \frac{1}{\sqrt{|\mathcal{G}|}}\partial_A\left(\sqrt{|\mathcal{G}|}\,\mathcal{G}^{AB}\partial_B\right).}$$

Blyth and Isham, Phys. Rev. D **11**, 768 (1975); Hawking and Page, Nucl. Phys. B **264**, 185 (1986).

Different measures/orderings: equivalent quantum theories?

Recent result: flat minisuperspace metric $\Rightarrow$ class of measures $\Rightarrow$ equivalent quantum theories.[1,2]

Measure $\Rightarrow$ introduces positive weight $J(q)$ $\Rightarrow$ similarity $\hat{\mathcal{H}}_J = J^{-1}\hat{\mathcal{H}}_0 J$:

$$\hat{\mathcal{H}}_J\psi = J^{-1}\hat{\mathcal{H}}_0(J\psi) = 0, \qquad \hat{\mathcal{H}}_0 = -\frac{\hbar^2}{2}\Delta_{\mathcal{G}} + U.$$

Change wavefunction and measure together:

$$\boxed{\Phi = J\psi} \quad \Rightarrow \quad \boxed{\hat{\mathcal{H}}_0\Phi = 0},$$

$$\underbrace{\int dq\,\sqrt{|\mathcal{G}|}\,J^2|\psi|^2}_{\text{original description}} = \underbrace{\int dq\,\sqrt{|\mathcal{G}|}\,|\Phi|^2}_{\text{new description}}.$$

J disappears $\Rightarrow$ equivalent descriptions within this class.

[1] Kaimakkamis, Partouche, Sil and Toumbas, Class. Quantum Grav. **42**, 115003 (2025); [2] Franken, Kaimakkamis, Partouche and Toumbas, Class. Quantum Grav. **43**, 105023 (2026).

Current and inner product

From WDW $\Rightarrow$ **Conserved current**:

$$\boxed{J^A = \frac{\hbar}{2i}\mathcal{G}^{AB}\left(\Psi^*\partial_B\Psi - \Psi\partial_B\Psi^*\right)}, \qquad \boxed{\nabla_A J^A = 0}.$$

But flux can have either sign:

Klein–Gordon-like equation $\Rightarrow$ J^0 not automatically probability density.

$\Rightarrow$ J^0 is not necessarily positive
$\Rightarrow$ Seek Schrödinger description using internal clock.
$\Rightarrow$ Find positive inner product
$\Rightarrow$ Probabilities at a given clock reading.

Next example: recent construction of Hilbert space in solvable model $\Rightarrow\Rightarrow\Rightarrow$

Halliwell, Phys. Rev. D **80**, 124032 (2009); Craig and Singh, Phys. Rev. D **82**, 123526 (2010); Gielen, Class. Quantum Grav. **43**, 195001 (2026).

Exactly solvable WDW equation (1/2)

Model: closed FLRW ($k = 1$). No-roll approximation:

$$\phi \simeq \phi_*, \qquad \dot{\phi} \simeq 0, \qquad \boxed{\lambda = V(\phi_*) > 0}.$$

$\Rightarrow$ effective cosmological constant.

In paper's ordering,

$$\left[\frac{1}{24\pi^2 a}\frac{d}{da}\left(\frac{1}{a}\frac{d}{da}\right) - 6\pi^2 + 2\pi^2\lambda a^2\right]\Psi = 0.$$

Define $z = (18\pi^2/\lambda)^{2/3}(1 - \lambda a^2/3)$:

$$\frac{d^2\Psi}{dz^2} - z\Psi = 0 \quad \Rightarrow \quad \boxed{\Psi = c_1\,\mathrm{Ai}(z) + c_2\,\mathrm{Bi}(z).}$$

Ai and Bi: the two independent Airy solutions.

Solved: the WDW differential equation. **Still needed:** which results define physical states?

Gielen, Class. Quantum Grav. **43**, 195001 (2026), arXiv:2606.17163.

Allowed states (2/2)

Idea: allow different values of λ.

$$\hat{H}_S \Psi_\lambda = \lambda \Psi_\lambda, \qquad \hat{H}_S = -\frac{1}{48\pi^4 a^3} \frac{d}{da}\left(\frac{1}{a}\frac{d}{da}\right) + \frac{3}{a^2}.$$

Inner product: $\boxed{\langle \Psi, \Phi \rangle = \int_0^\infty da\, \boldsymbol{a^3} \Psi^* \Phi.}$, $\boxed{\mu(a) = a^3.}$

Integration by parts $\Rightarrow$ boundary term

$$\left[\frac{\Psi'^* \Phi - \Psi^* \Phi'}{a}\right]_0^\infty.$$

Paper proves $\Rightarrow$ self-adjoint domains specified by

$$\boxed{\Psi(0) = \gamma \lim_{a\to 0} \frac{\Psi'(a)}{a}}.$$

Meaning: fix γ $\Rightarrow$ restrict Airy combinations $\Rightarrow$ superpose allowed λ's $\Rightarrow$ infinite-dimensional Hilbert space.

Limit: only for closed ($k = 1$), cosmological constant and chosen ordering.

Gielen, Class. Quantum Grav. **43**, 195001 (2026), arXiv:2606.17163.

Internal clocks and quantum evolution

Free massless scalar field as an internal clock

Why can ϕ be a clock?

$$\dot{\phi} = \frac{N p_\phi}{a^3}, \qquad \dot{p}_\phi = -N a^3 V_{,\phi}.$$

Free field:

$$V = 0 \quad \Rightarrow \quad p_\phi = \text{constant}.$$

For $p_\phi \neq 0$,

$$\boxed{\phi \text{ is monotonic}}$$

along each classical branch.

WDW equation

For $k = 0$, define

$$\alpha = \ln a, \qquad p_\alpha = a p_a, \qquad c^2 = \kappa/6.$$

$$\boxed{p_\phi^2 - c^2 p_\alpha^2 \approx 0.}$$

Quantize with the previous ordering:

$$\boxed{\partial_\phi^2 \Psi = -\Theta \Psi}, \qquad \Theta = -c^2 \partial_\alpha^2.$$

$\phi \Rightarrow$ clock candidate.

$$\boxed{\left(-i\partial_\phi + \sqrt{\Theta}\right)\left(i\partial_\phi + \sqrt{\Theta}\right)\Psi = 0}$$

Fourier modes $\Psi_k(\alpha, \phi) = f_k(\phi) e^{ik\alpha}$,

$$\Theta \Psi_k = -c^2 \partial_\alpha^2 \Psi_k \Rightarrow \boxed{\frac{d^2 f_k}{d\phi^2} + \boldsymbol{c^2 k^2} f_k = 0.}$$

Craig and Singh, Phys. Rev. D **82**, 123526 (2010).

Frequency sectors and unitary evolution

$$\omega_k = c|k|, \qquad f_k(\phi) = A(k)e^{-i\omega_k\phi} + B(k)e^{+i\omega_k\phi}.$$

$$\Theta\Psi_k = c^2k^2\Psi_k = \omega_k^2\Psi_k \Rightarrow \boxed{\sqrt{\Theta}\,\Psi_k = \omega_k\Psi_k, \qquad \omega_k = c|k|.}$$

General solution:

$$\Psi = \int dk\,[A(k)e^{-i\omega_k\phi} + B(k)e^{i\omega_k\phi}]e^{ik\alpha}, \qquad \omega_k = c|k|.$$

Positive frequency: $i\partial_\phi e^{-i\omega_k\phi} = +\omega_k e^{-i\omega_k\phi} \Rightarrow$ choose $B(k) = 0 \Rightarrow \boxed{i\partial_\phi\Psi = \sqrt{\Theta}\Psi}$.

Θ is self-adjoint:

$$\Theta e^{ik\alpha} = c^2k^2e^{ik\alpha}, \qquad \langle\psi, \Theta\psi\rangle = c^2\int d\alpha\,|\partial_\alpha\psi|^2 \geq 0.$$

Spectral theorem: $\sqrt{\Theta}$ is self-adjoint:

$$\boxed{\Psi(\phi) = e^{-i\sqrt{\Theta}(\phi-\phi_0)}\Psi(\phi_0)}, \qquad U^\dagger U = 1.$$

Craig and Singh, Phys. Rev. D 82, 123526 (2010).

Evolution in scalar field time

Choose positive frequency and an expanding packet:

$$B(k) = 0, \qquad \boxed{A(k) = \mathcal{N} e^{-\frac{(k-k_0)^2}{4\sigma_k^2}} e^{ik\alpha_0}}.$$

Initial state at $\phi = \phi_0$ and evolution:

$$\Psi(\alpha, \phi_0) = \int dk\, A(k) e^{-ik\alpha} = F(\alpha - \alpha_0).$$

$$c = \sqrt{\frac{\kappa}{6}} \Rightarrow \boxed{\Psi(\alpha, \phi) = e^{-i\sqrt{\kappa/6}k(\phi-\phi_0)}\Psi(\alpha, \phi_0) = F\left[\alpha - \alpha_0 - \sqrt{\kappa/6}(\phi - \phi_0)\right]}$$

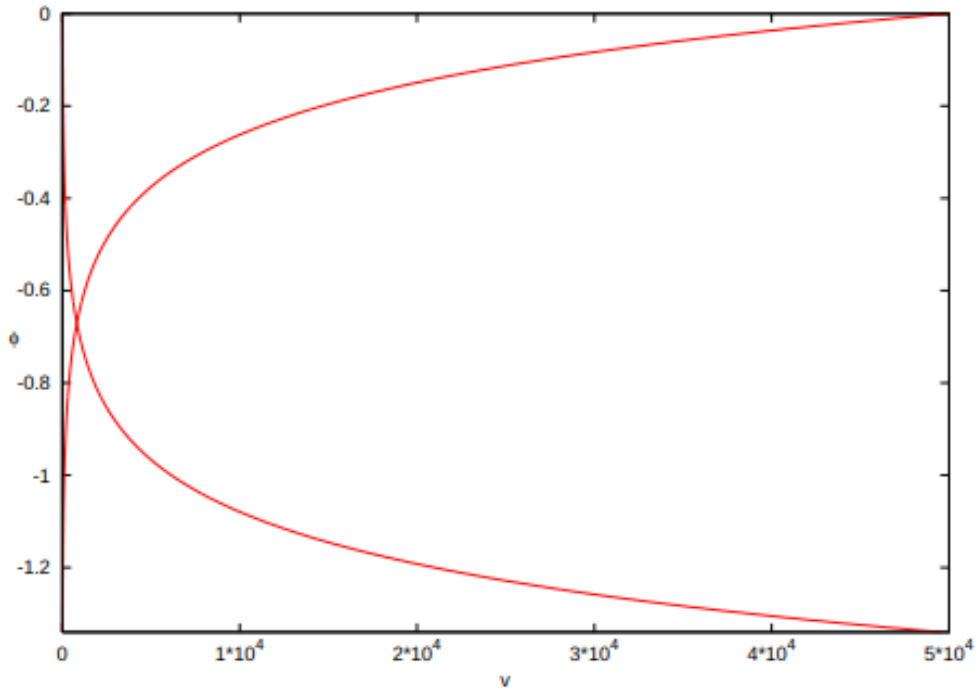
Peak at $\alpha - \alpha_0 - \sqrt{\kappa/6}(\phi - \phi_0) = 0$:

$$\boxed{\langle a \rangle_\phi = \langle a \rangle_{\phi_0} e^{\sqrt{\kappa/6}\,(\phi-\phi_0)}} \quad \Rightarrow \quad \text{expanding classical branch.}$$

Analogously,

$$\boxed{\langle a \rangle_\phi = \langle a \rangle_{\phi_0} e^{-\sqrt{\kappa/6}\,(\phi-\phi_0)}} \quad \Rightarrow \quad \text{contracting classical branch.}$$

Where is the bounce?



Craig and Singh, Phys. Rev. D **82**, 123526 (2010). Figure adapted from Ashtekar and Singh, Class. Quantum Grav. **28**, 213001 (2011), Fig. 2(a).

Where is the quantum behavior?

$$\boxed{\Psi = F(\alpha - c\phi) + G(\alpha + c\phi).}$$

Superposition $\Rightarrow$ the wavefunction contains both branches.

$$|\Psi|^2 = |F|^2 + |G|^2 + 2\,\mathrm{Re}(F^* G).$$

Interference can occur where the packets overlap.

But $\Rightarrow$ Free evolution preserves each branch $\Rightarrow$ interference *alone* does not transfer one packet into the other[1].

A bounce requires an additional argument: Examples:

- Change/add clock[2], change boundary condition[2,3], add V[4], add a perfect fluid[3], etc.
- LQC[5] and de Broglie–Bohm[6] trajectories will be discussed later.

1. Craig and Singh, Phys. Rev. D **82**, 123526 (2010); 2. Gielen and Menéndez-Pidal, CQG **37**, 205018 (2020); 3. Lemos, J. Math. Phys. **37**, 1449 (1996); 4. Giovannetti, Maione and Montani, Universe **9**, 373 (2023); 5. Ashtekar, Pawlowski and Singh, Phys. Rev. D **74**, 084003 (2006); 6. Pinto-Neto et al., Phys. Rev. D **86**, 063504 (2012).

Beyond the free scalar field: nontrivial $V(\phi)$

Classical problem: the field can turn around.

$$\dot{\phi} = \frac{N p_\phi}{a^3}, \qquad \dot{p}_\phi = -N a^3 V_{,\phi} \quad \Rightarrow \quad p_\phi \text{ may change sign.}$$

Quantum problem: for flat FLRW,

$$\boxed{[\partial_\phi^2 + \Theta(\phi)]\Psi = 0}, \qquad \Theta(\phi) = -c^2 \partial_\alpha^2 - \frac{2e^{6\alpha}}{\hbar^2} V(\phi), \quad c^2 = \kappa/6.$$

Progress:

- $V(\phi) = \frac{\lambda}{2} e^{-n\phi} \Rightarrow$ nonzero contraction-to-expansion amplitude in WDW[1].
- $V(\phi) = V_* e^{-\lambda c \phi} \Rightarrow$ exact WDW modes in terms of Bessel functions[2].

Limits:

- Global ϕ-time is not guaranteed.
- Exact modes alone do not establish physical probabilities.
- Transition amplitude $\neq$ proof of a trajectory with $a_{\min} > 0$.

[1] Giovannetti, Maione and Montani, Universe **9**, 373 (2023); [2] Paliathanasis, MPLA **32**, 1750206 (2017).

Second scalar field as a clock

Keep ϕ with its potential, and add a free homogeneous field χ. For flat FLRW,

$$S = \int dt \left[-\frac{3a\dot{a}^2}{\kappa N} + \frac{a^3}{2N}(\dot{\phi}^2 + \dot{\chi}^2) - N a^3 V(\phi) \right],$$

$$\boxed{\mathcal{H} = -\frac{\kappa p_a^2}{12a} + \frac{p_\phi^2 + p_\chi^2}{2a^3} + a^3 V(\phi) \approx 0.}$$

Since χ is free,

$$\dot{p}_\chi = 0, \qquad \dot{\chi} = \frac{N p_\chi}{a^3}.$$

For $p_\chi \neq 0$ and fixed-sign N:

$$\boxed{\chi \text{ remains monotonic even when } \dot{\phi} = 0.}$$

$\rho_\chi = p_\chi^2/(2a^6) \Rightarrow$ its energy cannot simply be ignored.

Comment: did not find any result for WDW, but usual, e.g., in LQC[1].

1. Gupta, Sharma, Vicente, Ramos and Wang, Phys. Rev. D **114**, 043503 (2026).

Schutz fluid as an internal clock

Fluid momentum enters linearly.[1,2] $\Rightarrow$ FLRW with perfect fluid $P = w\rho$:

$$\mathcal{H} = -\frac{\kappa p_a^2}{12a} - \frac{3ka}{\kappa} + \frac{p_T}{a^{3w}} \approx 0.$$

Choose $p_a^2 \to -\hbar^2\partial_a^2$, $p_T \to -i\hbar\partial_T$:

$$\boxed{i\hbar\partial_T\Psi = \left[\frac{\kappa\hbar^2}{12}a^{3w-1}\partial_a^2 - \frac{3k}{\kappa}a^{3w+1}\right]\Psi.}$$

Example: flat radiation model ($k = 0$, $w = 1/3$), $a > 0$.[3]

$$\boxed{\langle a\rangle_T = a_{\min}\sqrt{1 + \left(\frac{T}{T_0}\right)^2}.}$$

Result: contraction $\Rightarrow$ $\langle a\rangle_{\min} > 0$ $\Rightarrow$ expansion $\Rightarrow$ bounce of the expectation value.

Limit: packet, ordering and boundary condition.

[1] Schutz, PRD **2**, 2762 (1970); [2] Lapchinskii and Rubakov, Theor. Math. Phys. **33**, 1076 (1977); [3] Lemos, J. Math. Phys. **37**, 1449 (1996).

Free scalar field + Schutz fluid: Wheeler–DeWitt

Flat FLRW: massless scalar with $V(\phi) = 0$ + radiation fluid, $P = \rho/3$.

$$H = N\mathcal{H}, \qquad \boxed{\mathcal{H} = -\frac{\kappa p_a^2}{12a} + \frac{p_\phi^2}{2a^3} + \frac{p_T}{a} \approx 0.}$$

For $\alpha = \ln a$, $p_\alpha = a p_a$, $c^2 = \kappa/6$,

$$\boxed{\partial_T \Psi(\alpha, \phi, T) = \frac{\hbar}{2i} e^{-2\alpha} \left[c^2 \partial_\alpha^2 - \partial_\phi^2\right] \Psi(\alpha, \phi, T).}$$

Results and limits:

- WDW admits bouncing solutions in this model.
- Singularity resolution depends on the clock, inner product and boundary conditions.[1]

[1] Gielen and Menéndez-Pidal, CQG **37**, 205018 (2020).

Semiclassical limit

Semiclassical limit: recovering classical trajectories

WKB ansatz: slowly varying amplitude A and rapidly varying phase S.

$$\boxed{\Psi(q) = A(q)e^{iS(q)/\hbar}.}$$

Insert into WDW equation $\Rightarrow$ leading order $\Rightarrow$ **Hamilton–Jacobi equation**:

$$\boxed{\frac{1}{2}\mathcal{G}^{AB}\partial_A S\,\partial_B S + U = 0.}$$

$\boxed{p_A = \partial_A S}$ $\Rightarrow$ classical trajectories in minisuperspace.

Quantum state $\Rightarrow$ approximately classical geometry in a WKB regime.

Open: how does quantum matter evolve on this emergent classical background?

Halliwell, *Introductory Lectures on Quantum Cosmology* (1990), arXiv:0909.2566; Kiefer and Wichmann, Gen. Relativ. Gravit. **50**, 66 (2018).

Semiclassical time: the matter remains quantum

Separate the background geometry q from matter variables ξ:

$$\boxed{\Psi(q,\xi) \simeq A(q) e^{iMS_0(q)/\hbar}\psi(q,\xi).}$$

M: expansion scale. S_0 determines classical background trajectory.

Time along this trajectory:

$$\partial_{t_{\rm WKB}} = g^{AB}\partial_A S_0\, \partial_B.$$

Next order of WDW gives

$$\boxed{i\hbar\frac{\partial\psi}{\partial t_{\rm WKB}} = \hat{H}_{\rm matter}\psi + O(M^{-1}).}$$

- Geometry $\Rightarrow$ approximately classical background.
- Matter and perturbations $\Rightarrow$ quantum state ψ on that background.

Current work:

- Control backreaction and unitarity beyond the leading order.
- Determine which quantum states admit a semiclassical spacetime.

Kiefer and Wichmann, Gen. Relativ. Gravit. **50**, 66 (2018), arXiv:1802.01422.

WDW and cosmological perturbation theory

Question: WDW recover quantum fluctuations by perturbing classical universe?
Choose same classical background and clock:

- **WDW:** remove WKB phase $\Rightarrow$ remaining ψ.
- **Perturbation theory:** quantize small deviations from background $\Rightarrow$ wavefunction ψ_P.

WDW $\Rightarrow$ probability weight w.

On one WKB branch (fixed clock reading) $\Rightarrow$ matching:

$$\underbrace{|\psi_P|^2}_{\text{perturbation-theory density}} \simeq \underbrace{w|\psi|^2}_{\text{WDW probability density}} .$$

Results:

- Matching: verified for *exponential potential* and *local slow-roll model*[1];
- Connection with ordinary perturbation theory.

Limits:

- homogeneous fluctuations only;
- Higher-order WDW terms $\Rightarrow$ possible deviations from classical motion.

[1] Piazza and Vareilles, arXiv:2412.19782v2 (2025).

Conclusions and open questions

Conclusions and open questions

Progress and ongoing work:

For some specific models:

- WDW admits bouncing states;
- Explicit solutions $\Rightarrow$ physical inner products and unitary clock evolution;

- WKB $\Rightarrow$ ongoing work on backreaction and corrections to primordial spectra.

Open questions:

- Mechanism which connects contraction/expansion.
- Unique prescriptions for ordering, measure, etc.
- Different clocks $\Rightarrow$ how affects probabilities and observables $\Rightarrow$ beyond solvable models.
- Conditions in which WDW admit semiclassical spacetime description $\Rightarrow$ where WKB fail?
- How results change $\Rightarrow$ under changes of state, ordering, etc?

Thank you!

Selected References: Hamiltonian GR and Quantum Cosmology

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