Modified Gravity in Metric-Affine and Teleparallel Frameworks

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Lecture 1: MAG: Geometry

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Motivation

- The metric is the geometric structure defining intervals and causal cones
- The linear or affine connection is the structure defining parallel transport
- What if the relation between them is determined by field equations rather than imposed in advance? $\longrightarrow$ Metric-affine gravity
- Gauge formulation of gravity, dark sectors in cosmology
- Linear connection couples to hypermomentum, a matter source accounting for microscopic properties of matter (spin, shear, and dilation)

Stress-energy and hypermomentum

- Matter responses:

$$\Theta_{\mu\nu} := -\frac{2}{\sqrt{-g}}\frac{\delta S_{\rm m}}{\delta g^{\mu\nu}}, \qquad \Delta_\lambda{}^{\mu\nu} := -\frac{2}{\sqrt{-g}}\frac{\delta S_{\rm m}}{\delta \Gamma^\lambda{}_{\mu\nu}}. \tag{1}$$

  Writing $\Delta_{\lambda\mu\nu} := g_{\mu\alpha}g_{\nu\beta}\Delta_\lambda{}^{\alpha\beta}$, we get three independent components of the hypermomentum:

- dilation

$$\Delta_\nu := g^{\lambda\mu}\Delta_{\lambda\mu\nu}, \qquad {}^{(\rm dil)}\Delta_{\lambda\mu\nu} = \frac{1}{4}g_{\lambda\mu}\Delta_\nu. \tag{2}$$

- spin

$$\tau_{\lambda\mu\nu} := \Delta_{[\lambda\mu]\nu}. \tag{3}$$

- shear

$$\Sigma_{\lambda\mu\nu} := \Delta_{(\lambda\mu)\nu} - \frac{1}{4}g_{\lambda\mu}\Delta_\nu, \qquad g^{\lambda\mu}\Sigma_{\lambda\mu\nu} = 0. \tag{4}$$

Physical interpretation of hypermomentum

The hypermomentum describes how the internal structure of matter couples to the independent connection. Its three parts distinguish orientation, scale and shape.

- Spin, $\tau_{\lambda\mu\nu}$: intrinsic angular momentum, such as the spin of a Dirac field. In Einstein–Cartan gravity, this current sources torsion.
- Dilation, Δ_ν: an intrinsic scale current, associated with uniform rescalings of the local frame. It couples to the trace of the connection, associated with Weyl nonmetricity.
- Shear, $\Sigma_{\lambda\mu\nu}$: an intrinsic deformation current, associated with changes of shape that preserve volume. It couples to the symmetric, trace-free part of the connection, associated with traceless nonmetricity.

Conventions

- Metric signature mostly positive $(-,+,+,+)$, $\quad \mu,\nu = 0,1,2,3.$
- Derivative index: last.

$$\nabla_\nu V^\lambda = \partial_\nu V^\lambda + \Gamma^\lambda{}_{\mu\nu} V^\mu$$

- $T^\lambda{}_{\mu\nu} = 2\Gamma^\lambda{}_{[\mu\nu]}, \quad Q_{\rho\mu\nu} = +\nabla_\rho g_{\mu\nu}.$
- (Anti)symmetrization convention

$$A_{(\mu\nu)} = \tfrac{1}{2}(A_{\mu\nu} + A_{\nu\mu}), \qquad A_{[\mu\nu]} = \tfrac{1}{2}(A_{\mu\nu} - A_{\nu\mu})$$

- $\mathring{\Gamma}, \mathring{\nabla}, \mathring{R}$: computed with Levi-Civita connection.

Curvature

- The curvature tensor of the most general linear connection

$$R^{\sigma}{}_{\rho\mu\nu} = \partial_\mu \Gamma^{\sigma}{}_{\rho\nu} - \partial_\nu \Gamma^{\sigma}{}_{\rho\mu} + \Gamma^{\sigma}{}_{\lambda\mu}\Gamma^{\lambda}{}_{\rho\nu} - \Gamma^{\sigma}{}_{\lambda\nu}\Gamma^{\lambda}{}_{\rho\mu} \tag{5}$$

- The only symmetry present: $R^{\sigma}{}_{\rho\mu\nu} = -R^{\sigma}{}_{\rho\nu\mu}$.
- The three independent rank-two metric contractions are the Ricci tensor, the co-Ricci tensor, and the homothetic curvature:

$$R_{\mu\nu} := R^{\lambda}{}_{\mu\lambda\nu} = g^{\alpha\beta} R_{\alpha\mu\beta\nu}. \tag{6}$$

$$\check{R}_{\mu\nu} := g^{\alpha\beta} R_{\mu\alpha\beta\nu}. \tag{7}$$

$$\mathcal{H}_{\mu\nu} := R^{\lambda}{}_{\lambda\mu\nu} = g^{\alpha\beta} R_{\alpha\beta\mu\nu}. \tag{8}$$

Their symmetry properties are

$$\mathcal{H}_{\mu\nu} = -\mathcal{H}_{\nu\mu}, \qquad R_{[\mu\nu]} \neq 0, \qquad \check{R}_{[\mu\nu]} \neq 0 \quad \text{in general.} \tag{9}$$

Curvature

- The remaining single metric contractions are

$$g^{\alpha\nu}R_{\alpha\beta\mu\nu} = -R_{\beta\mu}, \qquad g^{\beta\nu}R_{\alpha\beta\mu\nu} = -\check{R}_{\alpha\mu}, \qquad g^{\mu\nu}R_{\alpha\beta\mu\nu} = 0. \tag{10}$$

- There is only one independent scalar obtained by contracting a single curvature tensor with the metric:

$$R := g^{\mu\nu}R_{\mu\nu} = g^{\alpha\mu}g^{\beta\nu}R_{\alpha\beta\mu\nu}. \tag{11}$$

  The other scalar traces satisfy

$$g^{\mu\nu}\check{R}_{\mu\nu} = -R, \qquad g^{\mu\nu}\mathcal{H}_{\mu\nu} = 0. \tag{12}$$

- With our nonmetricity convention $Q_{\rho\mu\nu} = +\nabla_\rho g_{\mu\nu}$, the homothetic curvature also satisfies

$$\mathcal{H}_{\mu\nu} = 2\partial_{[\mu}\Gamma^{\lambda}{}_{|\lambda|\nu]} = -\partial_{[\mu}Q_{\nu]}, \qquad Q_\mu := Q_{\mu\alpha}{}^{\alpha}. \tag{13}$$

Curvature

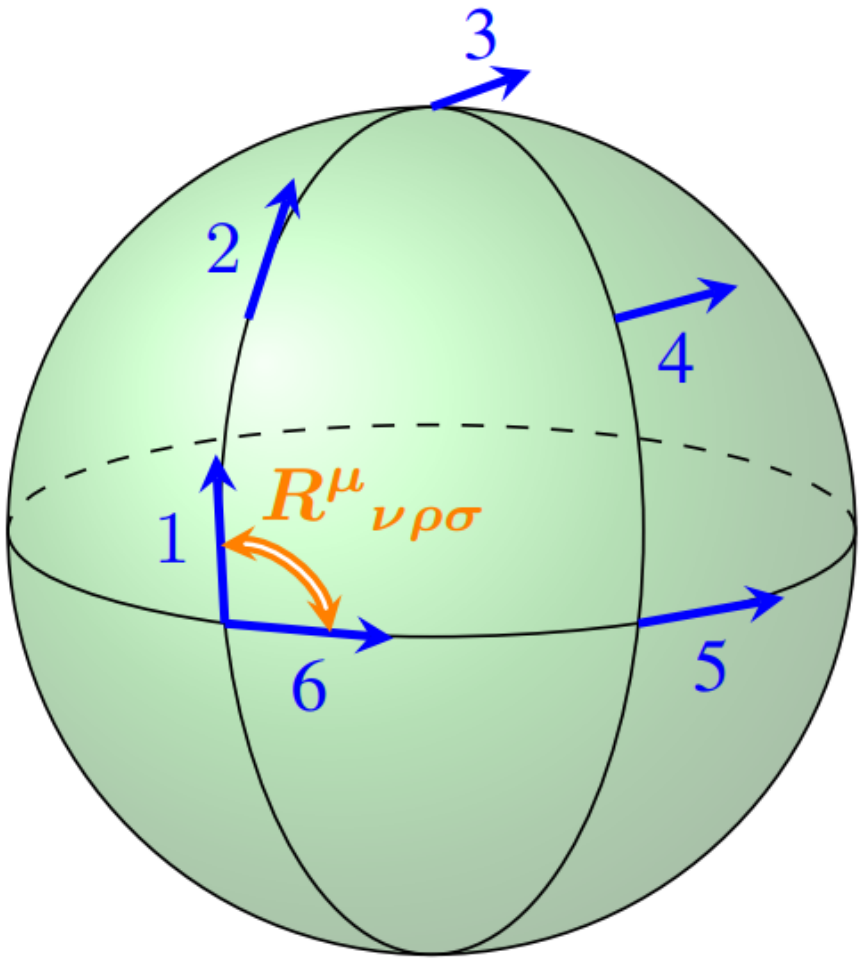


Figure: 3D representation of a 4D spacetime with only curvature.
Source: Laur Järv, Manuel Hohmann, Margus Saal, ("Laiendatud geomeetrilised gravitatsiooniteooriad," Eesti Vabariigi preemiad)

Torsion

- Our convention for the antisymmetric part of the linear connection will be

$$T^{\lambda}{}_{\mu\nu} := \Gamma^{\lambda}{}_{\mu\nu} - \Gamma^{\lambda}{}_{\nu\mu}$$

- We can work with the torsion tensor as

$$T_{\lambda\mu\nu} = g_{\lambda\alpha} T^{\alpha}{}_{\mu\nu}, \qquad T_{\lambda\mu\nu} = -T_{\lambda\nu\mu}$$

- There is only one independent trace

$$T_{\mu} := T^{\lambda}{}_{\mu\lambda}$$

- Due to its antisymmetry in last two indices, it has $4 \times 6 = 24$ independent components.

Torsion

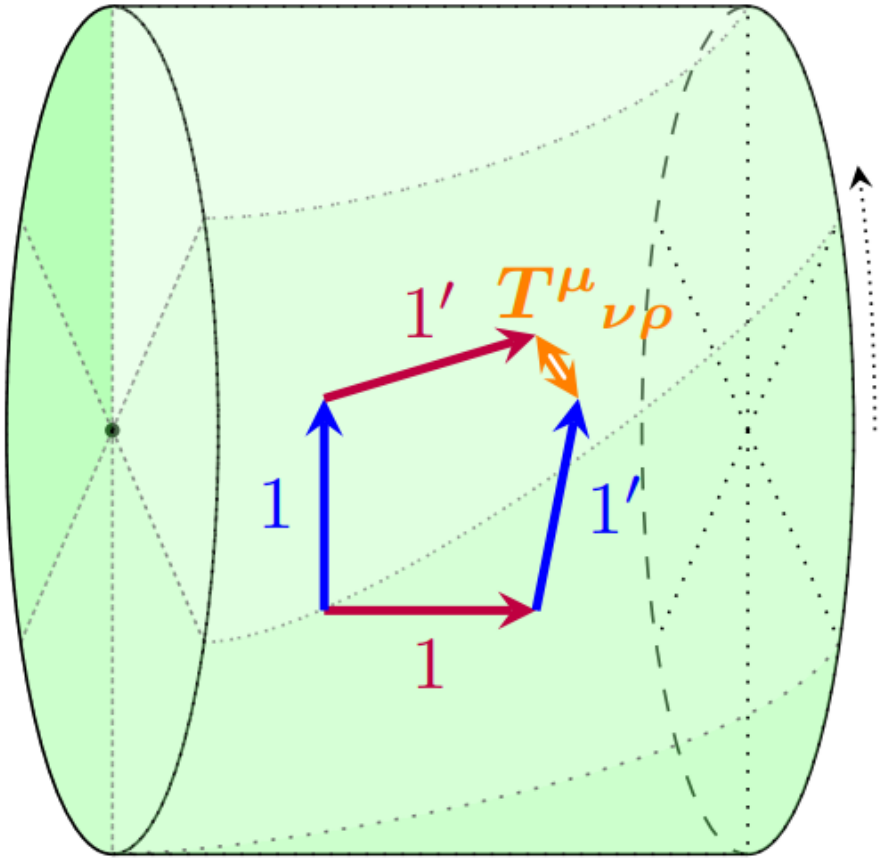


Figure: 3D representation of a 4D spacetime with only torsion.
Source: Laur Järv, Manuel Hohmann, Margus Saal, (“Laiendatud geomeetrilised gravitatsiooniteooriad,” Eesti Vabariigi preemiad)

Nonmetricity

- The nonmetricity tensor is defined as

$$Q_{\rho\mu\nu} := \nabla_\rho g_{\mu\nu} = \partial_\rho g_{\mu\nu} - \Gamma^\beta{}_{\mu\rho} g_{\beta\nu} - \Gamma^\beta{}_{\nu\rho} g_{\mu\beta} \tag{14}$$

- Due to the symmetry $Q_{\rho\mu\nu} = Q_{\rho\nu\mu}$, it has $4 \times 10 = 40$ independent components.
- It has two independent traces

$$Q_\mu := Q_{\mu\nu}{}^\nu, \qquad \tilde{Q}_\mu := Q_{\nu\mu}{}^\nu \tag{15}$$

- $Q_{\rho\mu\nu} = 0$: metric compatibility.

Nonmetricity

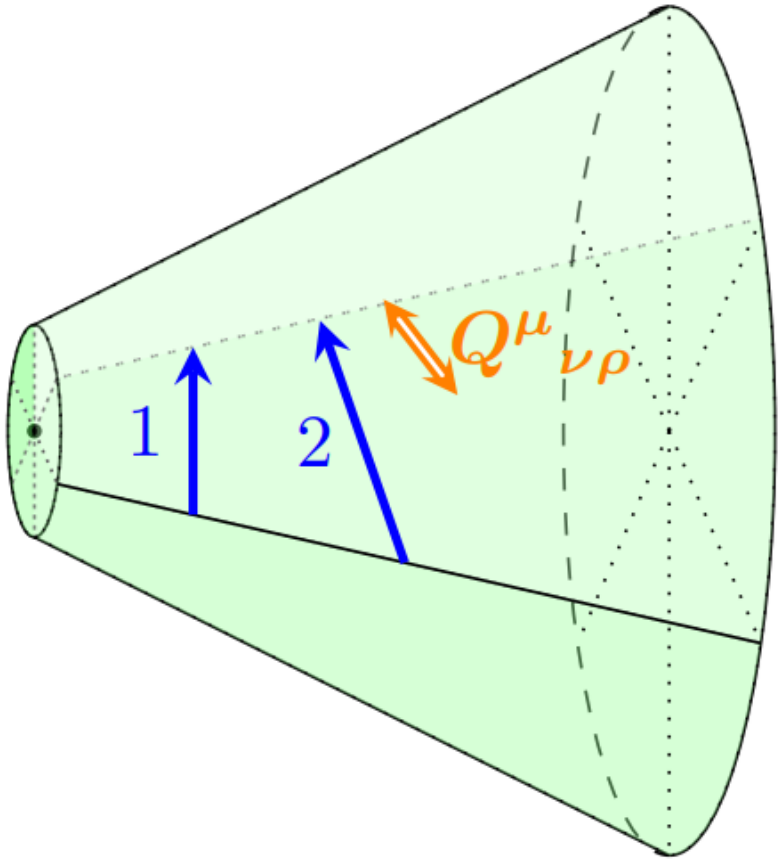


Figure: 3D representation of a 4D spacetime with only nonmetricity
Source: Laur Järv, Manuel Hohmann, Margus Saal, (“Laiendatud geomeetrilised gravitatsiooniteooriad,” Eesti Vabariigi preemiad)

Connection decomposition

- The most general linear connection can be decomposed as

$$\Gamma^{\lambda}{}_{\mu\nu} = \mathring{\Gamma}^{\lambda}{}_{\mu\nu} + K^{\lambda}{}_{\mu\nu} + L^{\lambda}{}_{\mu\nu}$$

- Levi-Civita part:

$$\mathring{\Gamma}^{\lambda}{}_{\mu\nu} = \frac{1}{2} g^{\lambda\beta} \left(\partial_{\mu} g_{\beta\nu} + \partial_{\nu} g_{\beta\mu} - \partial_{\beta} g_{\mu\nu} \right)$$

- Distortion:

$$N^{\lambda}{}_{\mu\nu} := \Gamma^{\lambda}{}_{\mu\nu} - \mathring{\Gamma}^{\lambda}{}_{\mu\nu} = K^{\lambda}{}_{\mu\nu} + L^{\lambda}{}_{\mu\nu}$$

- K and L are so-called contortion and disformation tensors, respectively
- N or K, L tensors span $24 + 40 = 64$ independent components, as expected.

Contortion and disformation tensors

- We define the contortion tensor as

$$K^{\lambda}{}_{\mu\nu} = \frac{1}{2} g^{\lambda\beta} \left(-T_{\mu\beta\nu} - T_{\nu\beta\mu} + T_{\beta\mu\nu} \right)$$

- It satisfies the properties $K_{\lambda\mu\nu} = -K_{\mu\lambda\nu}$ and $T^{\lambda}{}_{\mu\nu} = 2K^{\lambda}{}_{[\mu\nu]}$, also $K^{\lambda}{}_{(\mu\nu)} \neq 0$ in general.
- The disformation tensor is defined as

$$L^{\lambda}{}_{\mu\nu} = \frac{1}{2} g^{\lambda\beta} \left(-Q_{\mu\beta\nu} - Q_{\nu\beta\mu} + Q_{\beta\mu\nu} \right).$$

- It satisfies $Q_{\rho\mu\nu} = -2L_{(\mu\nu)\rho} = -2N_{(\mu\nu)\rho}$ and $L^{\lambda}{}_{\mu\nu} = L^{\lambda}{}_{\nu\mu},$, then $L^{\lambda}{}_{[\mu\nu]} = 0$.

Identities

- For a metric-compatible connection,

$$Q_{\rho\mu\nu} = 0 \quad \Longrightarrow \quad \mathcal{H}_{\mu\nu} = 0, \qquad \check{R}_{\mu\nu} = -R_{\mu\nu}. \tag{16}$$

- For a torsion-free connection, the first Bianchi identity gives

$$T^{\lambda}{}_{\mu\nu} = 0 \quad \Longrightarrow \quad \mathcal{H}_{\mu\nu} = 2R_{[\mu\nu]}. \tag{17}$$

- For the Levi-Civita connection,

$$R_{\mu\nu} = \mathring{R}_{\mu\nu} = \mathring{R}_{\nu\mu}, \qquad \check{R}_{\mu\nu} = -\mathring{R}_{\mu\nu}, \qquad \mathcal{H}_{\mu\nu} = 0. \tag{18}$$

Geometric sectors

- The Levi-Civita connection is recovered when $T^{\lambda}{}_{\mu\nu} = 0$, $Q_{\rho\mu\nu} = 0$. This describes standard Riemannian geometry.
- Riemann–Cartan geometry has curvature and torsion, therefore there $Q_{\rho\mu\nu} = 0$.
- Metric teleparallel gravity satisfies

$$R^{\alpha}{}_{\beta\mu\nu} = 0, \qquad Q_{\rho\mu\nu} = 0$$

- Symmetric teleparallel satisfies

$$R^{\alpha}{}_{\beta\mu\nu} = 0, \qquad T^{\lambda}{}_{\mu\nu} = 0$$

- The most general curvature tensor $R^{\alpha}{}_{\beta\mu\nu} = 0$ allows $\mathring{R}^{\alpha}{}_{\beta\mu\nu} \neq 0$.

Geometric sectors

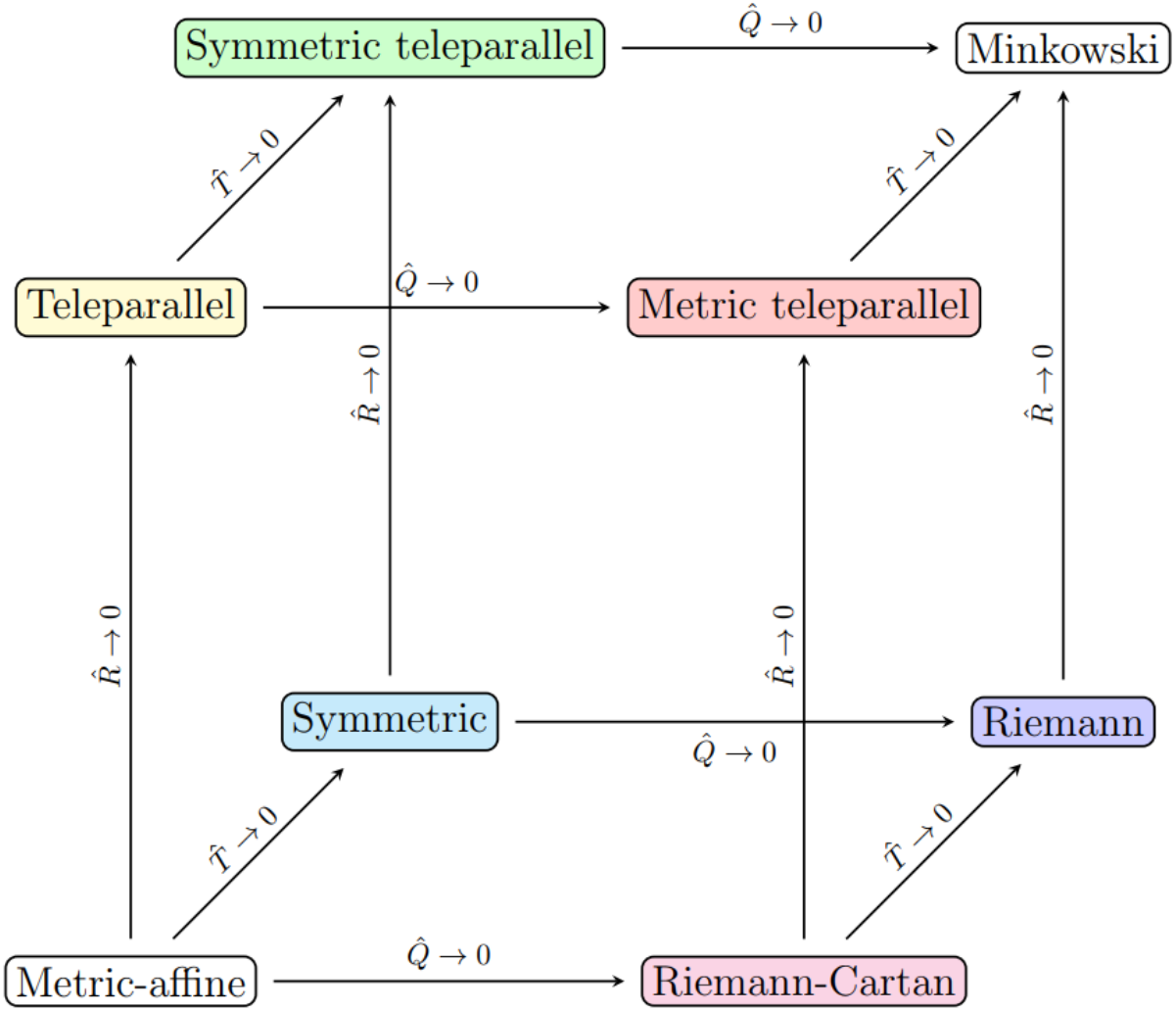


Figure: Possible geometric sectors based on values of R, T and Q. Taken from 2106.13793

Curvature tensor and distortion

- If the linear connection is split in terms of distortion N as

$$\Gamma^{\rho}{}_{\sigma\nu} = \mathring{\Gamma}^{\rho}{}_{\sigma\nu} + N^{\rho}{}_{\sigma\nu}$$

  then its associated curvature tensor is

$$R^{\rho}{}_{\sigma\mu\nu} = \mathring{R}^{\rho}{}_{\sigma\mu\nu} + \mathring{\nabla}_{\mu} N^{\rho}{}_{\sigma\nu} - \mathring{\nabla}_{\nu} N^{\rho}{}_{\sigma\mu} + N^{\rho}{}_{\lambda\mu} N^{\lambda}{}_{\sigma\nu} - N^{\rho}{}_{\lambda\nu} N^{\lambda}{}_{\sigma\mu}$$

- This can also be written as

$$R^{\rho}{}_{\sigma\mu\nu} = \mathring{R}^{\rho}{}_{\sigma\mu\nu} + 2\mathring{\nabla}_{[\mu} N^{\rho}{}_{|\sigma|\nu]} + 2N^{\rho}{}_{\lambda[\mu} N^{\lambda}{}_{|\sigma|\nu]}$$

Ricci and scalar curvature contractions

- The first and third contraction of the curvature tensor gives the Ricci curvature tensor

$$R_{\mu\nu} := R^{\lambda}{}_{\mu\lambda\nu}$$

- It therefore corresponds to

$$R_{\mu\nu} = \mathring{R}_{\mu\nu} + \mathring{\nabla}_{\lambda} N^{\lambda}{}_{\mu\nu} - \mathring{\nabla}_{\nu} N^{\lambda}{}_{\mu\lambda} + N^{\lambda}{}_{\rho\lambda} N^{\rho}{}_{\mu\nu} - N^{\lambda}{}_{\rho\nu} N^{\rho}{}_{\mu\lambda}$$

- We know that $\mathring{R}_{[\mu\nu]} = 0$, but $R_{[\mu\nu]} \neq 0$ in general.
- The scalar curvature is $R = g^{\mu\nu} R_{\mu\nu}$, which is explicitly

$$R = \mathring{R} + \mathring{\nabla}_{\lambda}\left(g^{\mu\nu} N^{\lambda}{}_{\mu\nu} - g^{\lambda\mu} N^{\nu}{}_{\mu\nu}\right) + g^{\mu\nu} N^{\lambda}{}_{\rho\lambda} N^{\rho}{}_{\mu\nu} - g^{\mu\nu} N^{\lambda}{}_{\rho\nu} N^{\rho}{}_{\mu\lambda}$$

Identities for the distortion

- Two selected traces for the distortion are

$$A^\mu := g^{\rho\nu} N^\mu{}_{\rho\nu}, \qquad B^\mu := g^{\mu\rho} N^\lambda{}_{\rho\lambda}$$

- They are related to torsion and nonmetricity traces as

$$A_\mu = -\hat{Q}_\mu + \frac{1}{2} Q_\mu - T_\mu, \qquad B_\mu = -\frac{1}{2} Q_\mu + T_\mu$$

- Therefore,

$$A^\mu - B^\mu = Q^\mu - \tilde{Q}^\mu - 2T^\mu$$

- The curvature scalar can then be alternatively written as

$$R = \mathring{R} + \mathring{\nabla}_\mu (A^\mu - B^\mu) + B_\mu A^\mu - N_{\alpha\mu\nu} N^{\mu\nu\alpha}$$

Quadratic distortion

- If we define a function of the distortion as

$$\mathcal{U}(N) := B_\mu A^\mu - N_{\alpha\mu\nu} N^{\mu\nu\alpha}$$

and remember that distortion is contortion plus disformation ($N = K + L$), then we recover the well-known torsion and nonmetricity scalars $\mathcal{T}$ and $\mathcal{Q}$ as

$$\mathcal{U}(K) = \mathcal{T}, \qquad \mathcal{U}(L) = -Q \tag{19}$$

- Then the distortion scalar corresponds to

$$\mathcal{U}(K + L) = \mathcal{T} - Q + \mathcal{C}$$

where

$$\mathcal{C} = -Q_{\alpha\mu\nu} T^{\nu\alpha\mu} + T_\mu (Q^\mu - \tilde{Q}^\mu)$$

is a term explicitly mixing torsion and nonmetricity. It is clear that if $T^\lambda{}_{\mu\nu} = 0$ or $Q_{\rho\mu\nu} = 0$, then $\mathcal{C} = 0$.

Torsion, nonmetricity and general scalars

- Meanwhile, the torsion scalar is

$$\mathcal{T} := \frac{1}{4}T_{\lambda\mu\nu}T^{\lambda\mu\nu} + \frac{1}{2}T_{\lambda\mu\nu}T^{\nu\mu\lambda} - T_\mu T^\mu$$

- The nonmetricity scalar is

$$Q := -\frac{1}{4}Q_{\lambda\mu\nu}Q^{\lambda\mu\nu} + \frac{1}{2}Q_{\lambda\mu\nu}Q^{\mu\nu\lambda} + \frac{1}{4}Q_\mu Q^\mu - \frac{1}{2}Q_\mu \tilde{Q}^\mu$$

- General curvature scalar

$$R = \mathring{R} + \mathcal{T} - Q + \mathcal{C} + \mathring{\nabla}_\mu(Q^\mu - \tilde{Q}^\mu - 2T^\mu)$$

or

$$R = \mathring{R} + \mathcal{T} - Q + \mathcal{C} - B_T - B_Q$$

after defining the boundary terms $B_T := 2\mathring{\nabla}_\mu T^\mu$, $B_Q := \mathring{\nabla}_\mu(\tilde{Q}^\mu - Q^\mu)$.

Boundary terms

- The vectorial parts of torsion and nonmetricty form part of a boundary term

$$V^\mu := 2T^\mu + \tilde{Q}^\mu - Q^\mu, \qquad B_T + B_Q = \mathring{\nabla}_\mu V^\mu$$

- They enter the action as a boundary term due to the identity

$$\sqrt{-g}(B_T + B_Q) = \partial_\mu(\sqrt{-g}V^\mu)$$

- Therefore, once integrating out

$$\int_{\mathcal{M}} d^4x\sqrt{-g}(B_T + B_Q) = \int_{\partial\mathcal{M}} d\Sigma_\mu V^\mu$$

- For compactly supported variations then there is no bulk contribution.