Quantum Origins of Cosmic Structure: From Vacuum Fluctuations to Primordial Inhomogeneities

Gabriel León

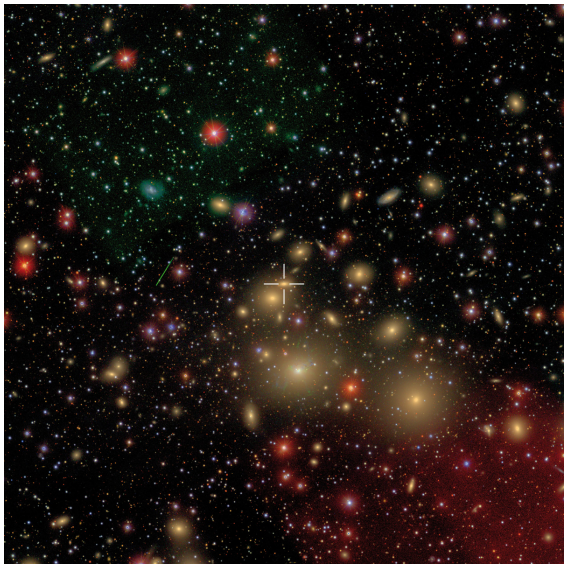
Grupo de Cosmología, Instituto de Astrofísica de La Plata (CONICET)
Universidad Nacional de La Plata , Argentina

1st Arica School of Cosmology and Gravitation



MAIN IDEA OF THE LECTURES

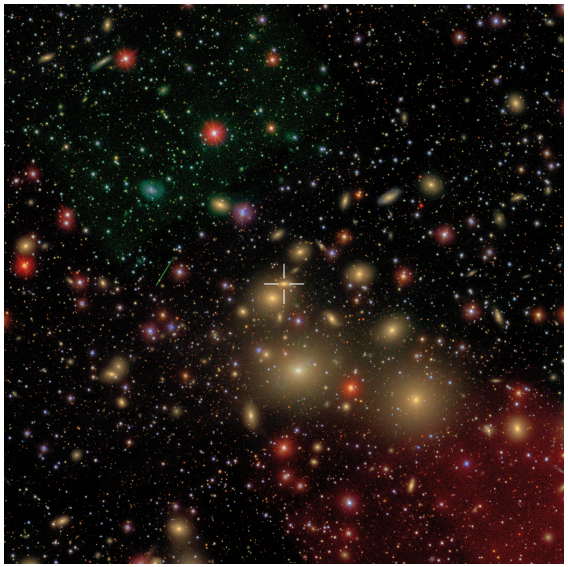
- We look at the sky and ask questions...



- In particular, *where does all this stuff come from?*
  Most accepted answer: Quantum fluctuations of the vacuum during inflation
- *Can we use cosmological data to test this answer?*
  YES!
- *So everything is completely understood and clearly explained?*
  Not really...

MAIN IDEA OF THE LECTURES

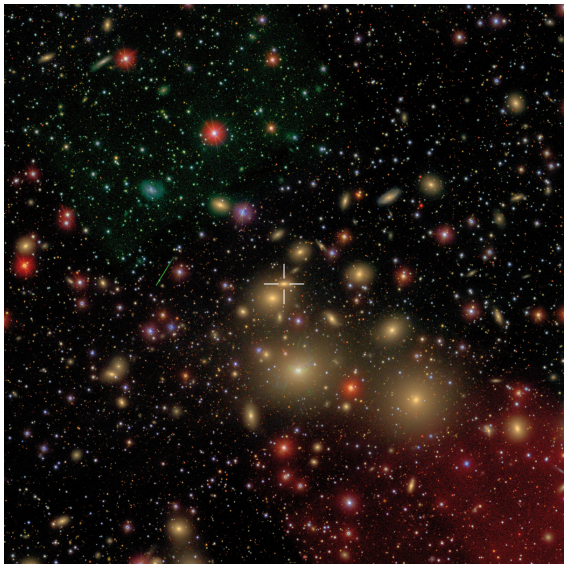
- We look at the sky and ask questions...



- In particular, *where does all this stuff come from?*
  Most accepted answer: Quantum fluctuations of the vacuum during inflation
- *Can we use cosmological data to test this answer?*
  YES!
- *So everything is completely understood and clearly explained?*
  Not really...

MAIN IDEA OF THE LECTURES

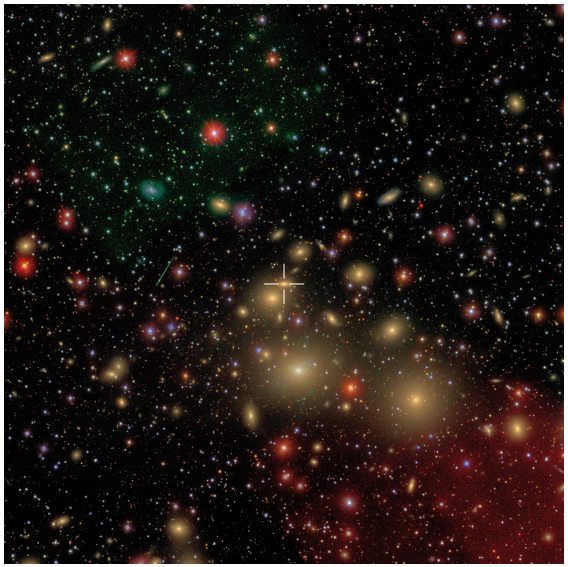
- We look at the sky and ask questions...



- In particular, *where does all this stuff come from?*
  Most accepted answer: Quantum fluctuations of the vacuum during inflation
- *Can we use cosmological data to test this answer?*
  YES!
- *So everything is completely understood and clearly explained?*
  Not really...

MAIN IDEA OF THE LECTURES

- We look at the sky and ask questions...



- In particular, *where does all this stuff come from?*
  Most accepted answer: Quantum fluctuations of the vacuum during inflation
- *Can we use cosmological data to test this answer?*
  YES!
- *So everything is completely understood and clearly explained?*
  Not really...

Outline

1. Background
2. Scalar perturbations
3. Quantization

Ingredients for Standard Big Bang cosmology

- Our tool: General Relativity
- Based on the cosmological principle: At large scales (~ 100 Mpc) the Universe is Homogeneous & Isotropic.
- The Friedmann-Lemaître-Robertson-Walker metric

$$ds^2 = -dt^2 + a(t)^2\left[\frac{dr^2}{1-Kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right]$$

,

- Einstein Equations: $G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$
- Perfect Fluid: $T_{\mu\nu} = p g_{\mu\nu} + (\rho + p) u_\mu u_\nu$ where p and ρ are the pressure and energy density of the fluid
- Equivalently:

$$G_{\mu\nu} = 8\pi G(T_{\mu\nu} + T^{(\Lambda)}_{\mu\nu}) \qquad T^{(\Lambda)}_{\mu\nu} = -\frac{\Lambda}{8\pi G} g_{\mu\nu} \equiv -\rho_\Lambda g_{\mu\nu}$$

Ingredients for Standard Big Bang cosmology

- Our tool: General Relativity
- Based on the cosmological principle: At large scales (~ 100 Mpc) the Universe is Homogeneous & Isotropic.
- The Friedmann-Lemaître-Robertson-Walker metric

$$ds^2 = -dt^2 + a(t)^2\left[\frac{dr^2}{1-Kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right]$$

,

- Einstein Equations: $G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$
- Perfect Fluid: $T_{\mu\nu} = p g_{\mu\nu} + (\rho + p) u_\mu u_\nu$ where p and ρ are the pressure and energy density of the fluid
- Equivalently:

$$G_{\mu\nu} = 8\pi G(T_{\mu\nu} + T^{(\Lambda)}_{\mu\nu}) \qquad T^{(\Lambda)}_{\mu\nu} = -\frac{\Lambda}{8\pi G} g_{\mu\nu} \equiv -\rho_\Lambda g_{\mu\nu}$$

Ingredients for Standard Big Bang cosmology

- Our tool: General Relativity
- Based on the cosmological principle: At large scales (~ 100 Mpc) the Universe is Homogeneous & Isotropic.
- The Friedmann-Lemaître-Robertson-Walker metric

$$ds^2 = -dt^2 + a(t)^2\left[\frac{dr^2}{1-Kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right]$$

,

- Einstein Equations: $G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$
- Perfect Fluid: $T_{\mu\nu} = p g_{\mu\nu} + (\rho + p) u_\mu u_\nu$ where p and ρ are the pressure and energy density of the fluid
- Equivalently:

$$G_{\mu\nu} = 8\pi G(T_{\mu\nu} + T^{(\Lambda)}_{\mu\nu}) \qquad T^{(\Lambda)}_{\mu\nu} = -\frac{\Lambda}{8\pi G} g_{\mu\nu} \equiv -\rho_\Lambda g_{\mu\nu}$$

- Then EE yield *Friedmann's Equations*

$$H^2 + \frac{K}{a^2} = \frac{8\pi G}{3}\rho; \qquad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \tag{1}$$

where $H \equiv \frac{\dot{a}}{a}$

- There is also the continuity equation $\nabla_\mu T^{\mu\nu} = 0$, i.e.

$$\dot{\rho} + 3H(\rho + p) = 0 \tag{2}$$

But this is not and independent equation (it can be obtained from Friedmann's Equations)

- So we have 2 indep. equations, and 3 unknowns $a(t), \rho(t), p(t)$. We need the *Equation of State*:

(i) $p = \rho/3$ radiation, i.e. relativistic matter

(ii) $p = 0$ matter, i.e. dust or dark matter.

(iii) $p = -\rho$ dark energy (negative pressure)

- If $K = 0$ (spatially flat Universe), consistent with observations, the solutions for $a(t)$ are:

$$a(t) \propto t^{1/2}, \quad \rho \propto a^{-4}, \quad \text{radiation dominated} \tag{3}$$

$$a(t) \propto t^{2/3}, \quad \rho \propto a^{-3}, \quad \text{matter dominated} \tag{4}$$

  In these simple cases **the Universe expands deceleratedly**

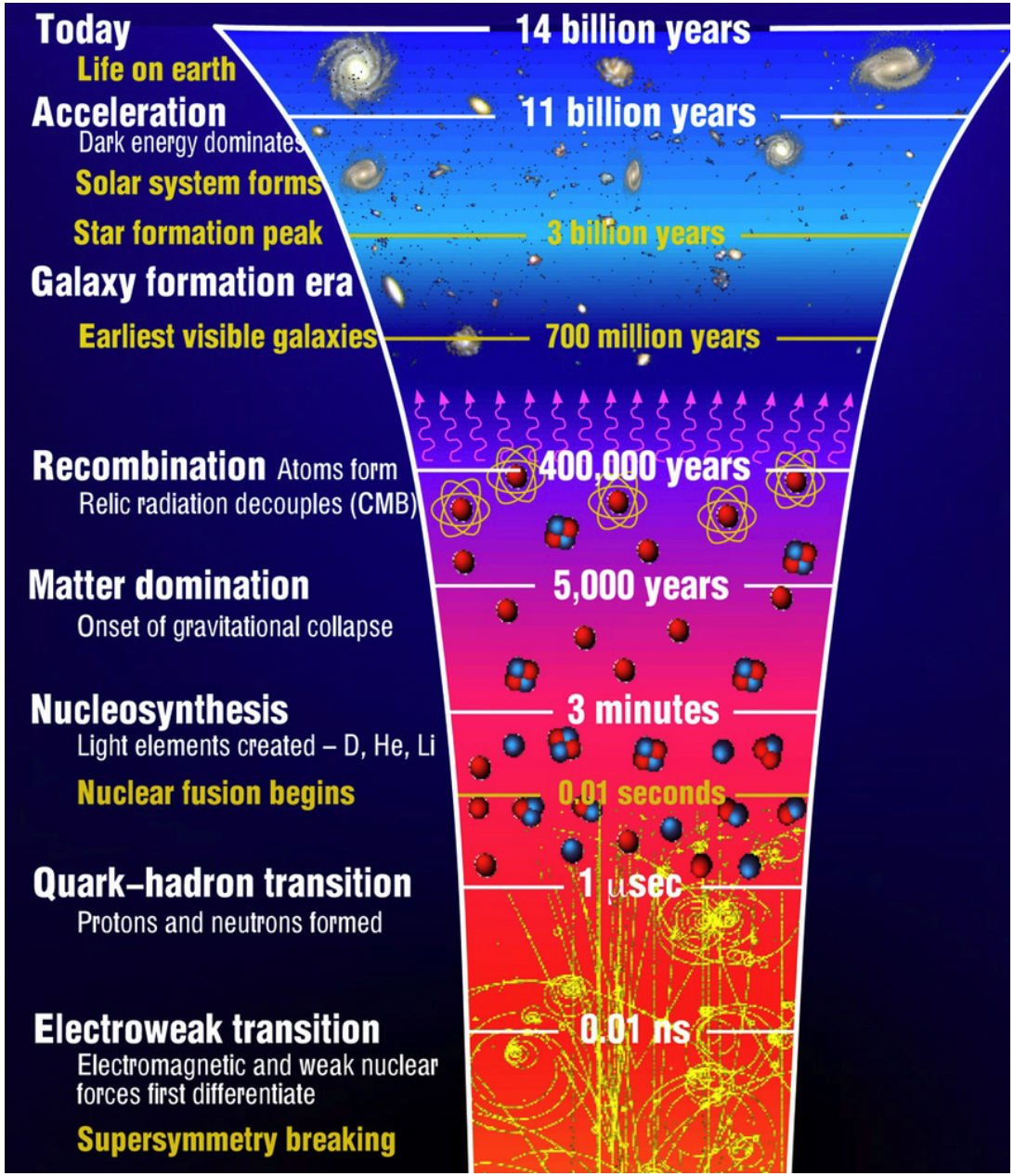
- However, for dark energy or pure cosmological constant

$$a(t) \propto \exp(\sqrt{\Lambda/3}\, t) \tag{5}$$

  The Universe expands acceleratedly

The Big Bang model

Background
Scalar perturbations
Quantization



Flatness Problem

- The first Friedmann Eq. can be written as

$$\Omega - 1 = \frac{8\pi G\rho}{3H^2} - 1 = \frac{K}{a^2H^2} \tag{6}$$

  For RD and MD universe we have $H^2 \propto \rho$ and $\rho \propto a^{-m}$ with $m = 3, 4$, so $|K|/(a^2H^2) \sim a^n$ with $n = 2, 1$

- Consequently $|\Omega - 1|$ increases with the expansion.
- Current observational data (CMB + BAO) indicate $\Omega_0 = 0.9979 \pm 0.0011$. Thus, at the beginning of the Universe $|K|/(a^2H^2)$ must have been extremely small. This seems a fine tuning issue (you can calculate it quantitatively, using e.g. $a_{nuc} = 10^{-10}a_0$ and $a_{eq} = 10^{-4}a_0$)

Flatness Problem

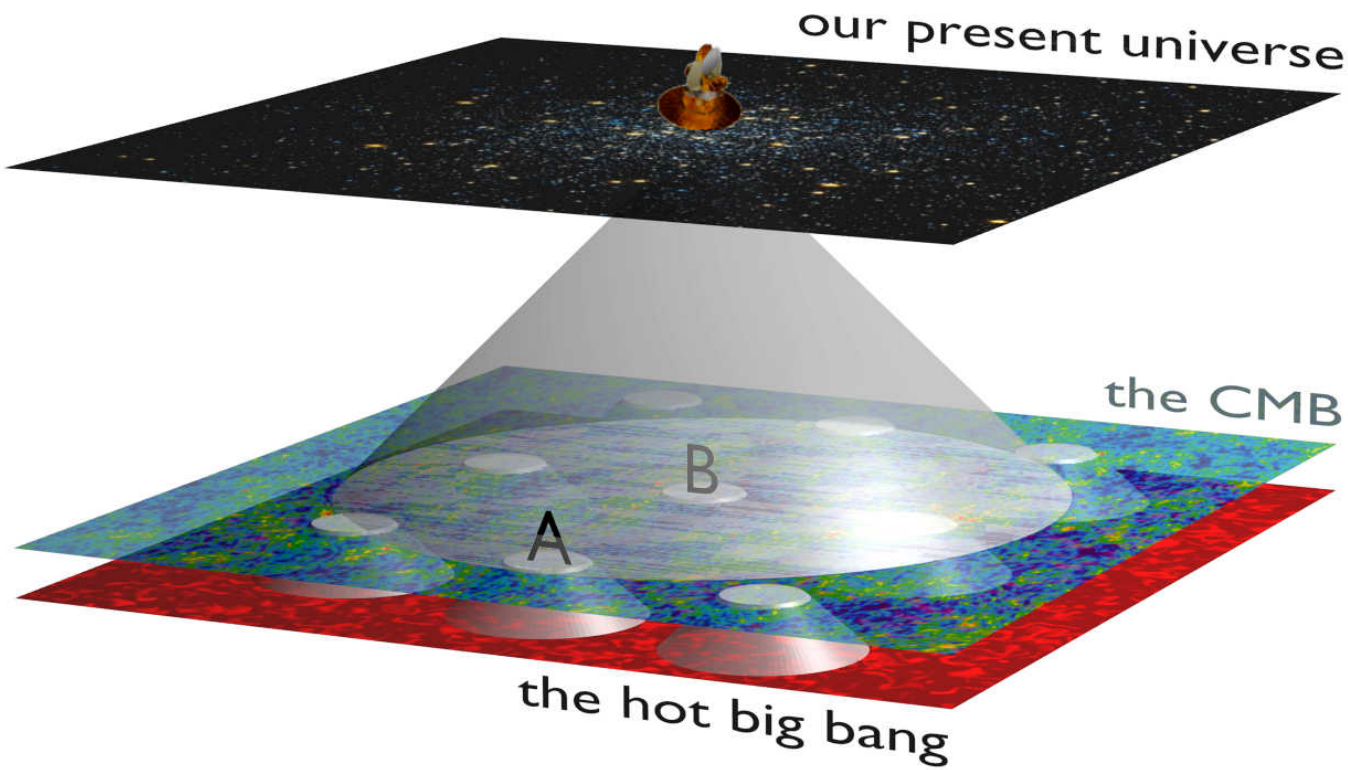
- The first Friedmann Eq. can be written as

$$\Omega - 1 = \frac{8\pi G\rho}{3H^2} - 1 = \frac{K}{a^2H^2} \tag{6}$$

  For RD and MD universe we have $H^2 \propto \rho$ and $\rho \propto a^{-m}$ with $m = 3, 4$, so $|K|/(a^2H^2) \sim a^n$ with $n = 2, 1$
- Consequently $|\Omega - 1|$ increases with the expansion.
- Current observational data (CMB + BAO) indicate $\Omega_0 = 0.9979 \pm 0.0011$. Thus, at the beginning of the Universe $|K|/(a^2H^2)$ must have been extremely small. This seems a fine tuning issue (you can calculate it quantitatively, using e.g. $a_{nuc} = 10^{-10}a_0$ and $a_{eq} = 10^{-4}a_0$)

Horizon Problem

The CMB has practically the same temperature everywhere (deviations of 10^{-5}). However, many regions are causally disconnected.



Cosmic Inflation



- A. Guth (1980) proposed a solution to these problems: **Cosmic Inflation**. An accelerated expansion phase in the early Universe

$$\ddot{a} > 0 \tag{7}$$

- If $\ddot{a} > 0$ implies $d/dt[|\Omega - 1|] < 0$. For instance, if $H =$ constant then $|K|/(a^2 H^2)$ decreases exponentially during the expansion. Solves the flatness problem

Background
Scalar perturbations
Quantization



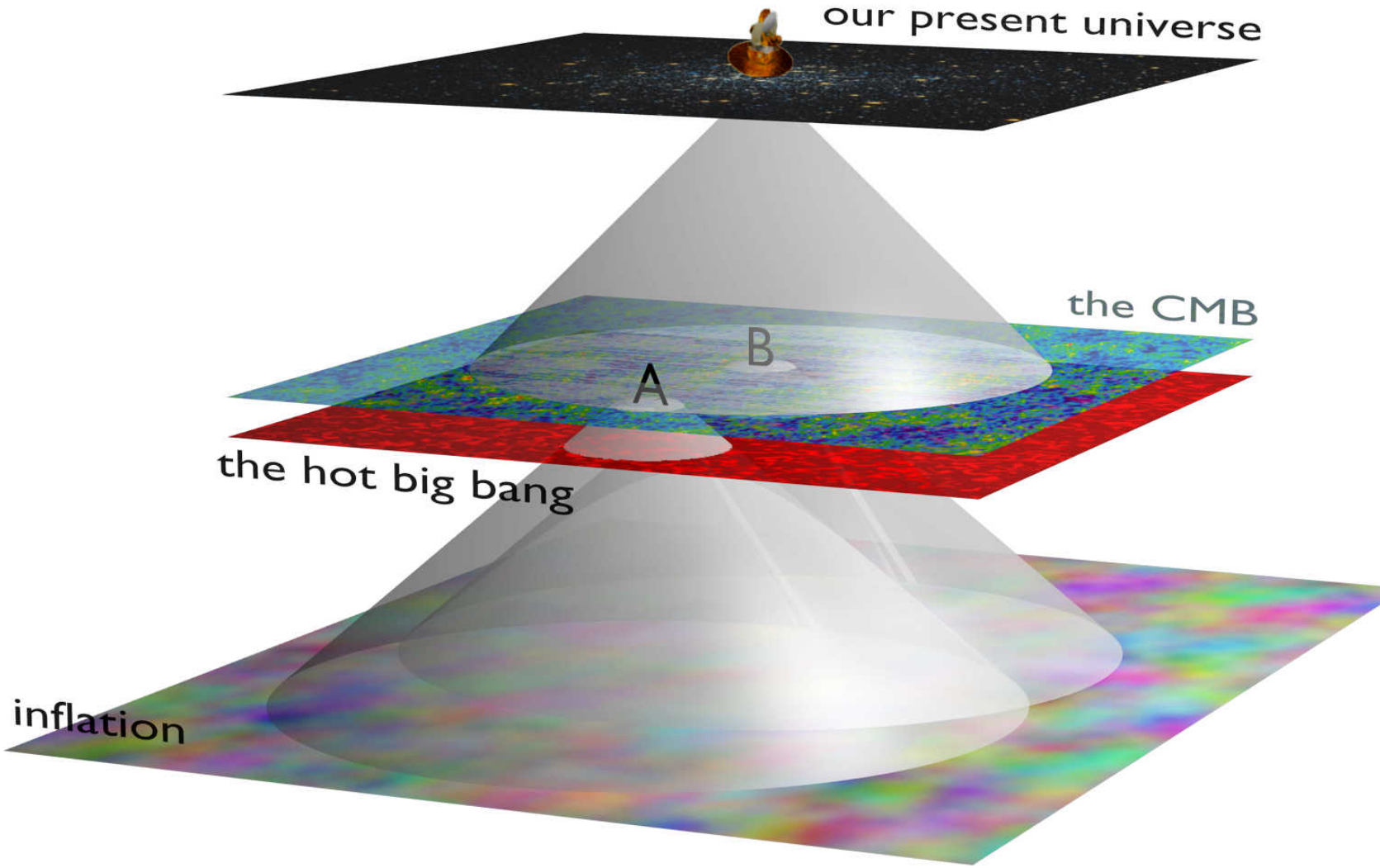
- A. Guth (1980) proposed a solution to these problems: **Cosmic Inflation**. An accelerated expansion phase in the early Universe

$$\ddot{a} > 0 \tag{7}$$

- If $\ddot{a} > 0$ implies $d/dt[|\Omega - 1|] < 0$. For instance, if $H =$ constant then $|K|/(a^2 H^2)$ decreases exponentially during the expansion. Solves the flatness problem

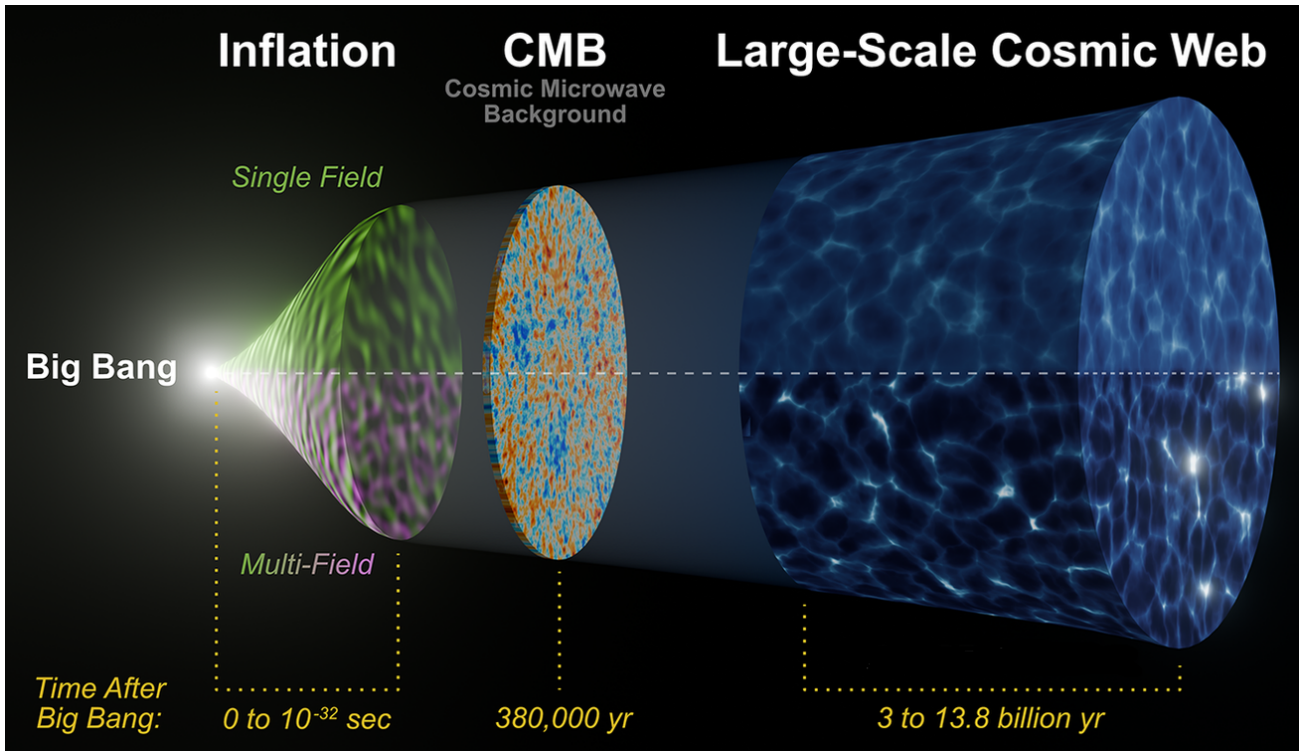
Background
Scalar perturbations
Quantization

Also solves the horizon problem



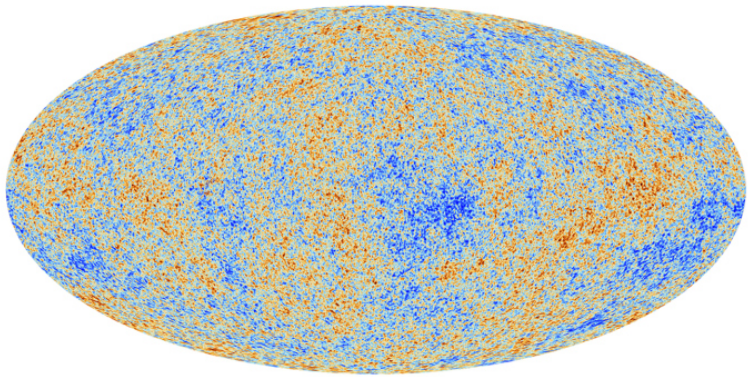
The Inflationary Paradigm

- **Inflation:** an early cosmological epoch characterized by an accelerated (exponential) expansion of the Universe. It is added to the standard ΛCDM cosmological model, which is constructed within GR.



The Inflationary Paradigm

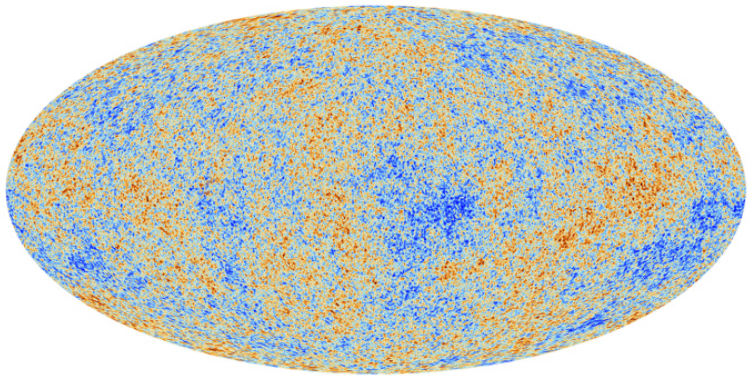
- Historically, it was proposed to solve the problems of the Big Bang model (Horizon, Flatness, etc.)
- *Main success today:* The most accepted model which "explains" the origin of primordial cosmic seeds (inhomogeneities).
  "Explanation:" **Quantum Fluctuations of the inflaton** results in theoretical predictions of the **spectrum of primordial inhomogeneities**.
- Observational Data (CMB) in agreement with such predictions



The Inflationary Paradigm

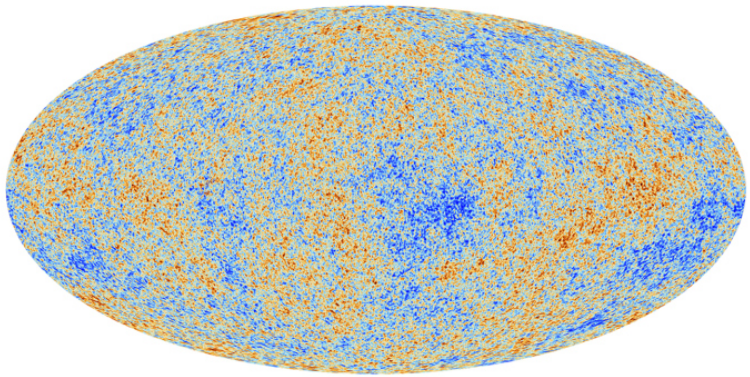
- Historically, it was proposed to solve the problems of the Big Bang model (Horizon, Flatness, etc.)
- *Main success today:* The most accepted model which ”explains” the origin of primordial cosmic seeds (inhomogeneities).

  ”Explanation:” **Quantum Fluctuations of the inflaton** results in theoretical predictions of the **spectrum of primordial inhomogeneities**.
- Observational Data (CMB) in agreement with such predictions



The Inflationary Paradigm

- Historically, it was proposed to solve the problems of the Big Bang model (Horizon, Flatness, etc.)
- *Main success today:* The most accepted model which ”explains” the origin of primordial cosmic seeds (inhomogeneities).
  ”Explanation:” **Quantum Fluctuations of the inflaton** results in theoretical predictions of the **spectrum of primordial inhomogeneities**.
- Observational Data (CMB) in agreement with such predictions



Accelerated expansion

- One starts with a (flat) FRLW metric.

$$ds^2 = -dt^2 + a(t)^2 \delta_{ij} dx^i dx^j ,$$

and the energy-momentum tensor $T_{\mu\nu}$of a perfect fluid.

- From Friedmann's eqs we have

$$H^2 = \frac{\rho}{3M_P^2}$$

$$\dot{H} + H^2 = \frac{\ddot{a}}{a} = -\frac{1}{6M_P^2}(\rho + 3p)$$

with $M_P^2 = 1/(8\pi G)$ the reduced Planck mass.

- If $p \leq -\rho/3$ then $\ddot{a} > 0 \Rightarrow$ **Inflation occurs**

Accelerated expansion

- One starts with a (flat) FRLW metric.

$$ds^2 = -dt^2 + a(t)^2 \delta_{ij} dx^i dx^j ,$$

and the energy-momentum tensor $T_{\mu\nu}$of a perfect fluid.

- From Friedmann's eqs we have

$$H^2 = \frac{\rho}{3M_P^2}$$

$$\dot{H} + H^2 = \frac{\ddot{a}}{a} = -\frac{1}{6M_P^2}(\rho + 3p)$$

with $M_P^2 = 1/(8\pi G)$ the reduced Planck mass.

- If $p \leq -\rho/3$ then $\ddot{a} > 0 \Rightarrow$ **Inflation occurs**
But, what is exactly a fluid with negative pressure?

Simplest model: Slow-roll Inflation

- We consider single scalar field ϕ (the dominant type of matter)

$$S = \int d^4x\sqrt{-g}\left[\frac{M_P^2}{2}R - \frac{g^{\mu\nu}}{2}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right]$$

- ϕ is highly homogeneous; $\phi \simeq \phi_0(t) + \delta\phi(\mathbf{x}, t)$, with $\delta\phi(\mathbf{x}, t) \ll 1$
- The dynamical equations for the background are

$$H^2 = \frac{1}{3M_P^2}\underbrace{\left(\frac{\dot{\phi}^2}{2} + V\right)}_{\rho_\phi} \tag{8}$$

$$\ddot{\phi} + 3H\dot{\phi} + \partial_\phi V = 0 \tag{9}$$

- Also $p_\phi = \dot{\phi}^2/2 - V$.

Simplest model: Slow-roll Inflation

- Inflation is guaranteed if the field *rolls slowly* down the potential such that

$$\frac{\dot{\phi}^2}{2} \ll V. \tag{10}$$

It behaves as a fluid with EOS $p \simeq -\rho$. Moreover, inflation last long enough provided

$$\ddot{\phi} \ll 3H\dot{\phi} \tag{11}$$

- Those two conditions lead to **slow roll inflation**.

Simplest model: Slow-roll Inflation

- Another convenient way to describe slow roll inflation is via the Hubble flow functions ϵ_n or "slow roll parameters", defined as

$$\epsilon_{n+1} \equiv \frac{d \ln \epsilon_n}{dN}, \qquad \epsilon_0 \equiv \frac{H_{\rm ini}}{H} \tag{12}$$

where $N \equiv \ln(a/a_{\rm ini})$ is the number of e-folds from the beginning of inflation. Equivalently, using $dN = H dt$,

$$\dot{\epsilon}_n = H \epsilon_n \epsilon_{n+1} \tag{13}$$

- From above $\epsilon_1 = -\dot{H}/H^2$. **If $\epsilon_1 < 1$ inflation is guaranteed.**
- **The slow roll approximation** assumes that all these parameters are small during inflation $|\epsilon_n| \ll 1$

Simplest model: Slow-roll Inflation

- In terms of the first two HFF, the dynamical equations for the background are

$$H^2 = \frac{V}{M_P^2(3-\epsilon_1)} \qquad \text{Friedmann Eq.} \qquad (14)$$

$$3H\dot{\phi}\left(1-\frac{\epsilon_1}{3}+\frac{\epsilon_2}{6}\right) = -\partial_\phi V \qquad \text{KG Eq.} \qquad (15)$$

  These equations are exact.

- If the slow roll aproximation is valid, $\epsilon_{1,2} < 1$ then

$$H^2 \simeq \frac{V}{3M_P^2} \qquad 3H\dot{\phi} \simeq -\partial_\phi V \qquad (16)$$

- Thus for a given V, the background is completely determined within the slow roll approximation.

Simplest model: Slow-roll Inflation

- In terms of the first two HFF, the dynamical equations for the background are

$$H^2 = \frac{V}{M_P^2(3-\epsilon_1)} \qquad \text{Friedmann Eq.} \qquad (14)$$

$$3H\dot{\phi}\left(1-\frac{\epsilon_1}{3}+\frac{\epsilon_2}{6}\right) = -\partial_\phi V \qquad \text{KG Eq.} \qquad (15)$$

  These equations are exact.
- If the slow roll aproximation is valid, $\epsilon_{1,2} < 1$ then

$$H^2 \simeq \frac{V}{3M_P^2} \qquad 3H\dot{\phi} \simeq -\partial_\phi V \qquad (16)$$

- Thus for a given V, the background is completely determined within the slow roll approximation.

Simplest model: Slow-roll Inflation

- In terms of the first two HFF, the dynamical equations for the background are

$$H^2 = \frac{V}{M_P^2(3-\epsilon_1)} \qquad \text{Friedmann Eq.} \qquad (14)$$

$$3H\dot{\phi}\left(1-\frac{\epsilon_1}{3}+\frac{\epsilon_2}{6}\right) = -\partial_\phi V \qquad \text{KG Eq.} \qquad (15)$$

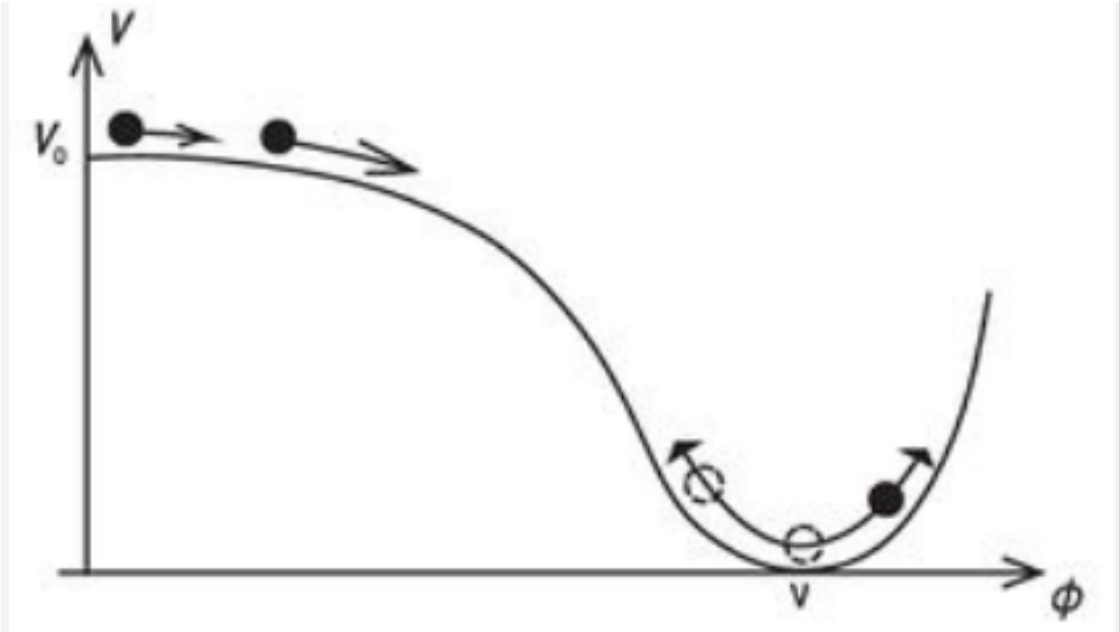
  These equations are exact.
- If the slow roll aproximation is valid, $\epsilon_{1,2} < 1$ then

$$H^2 \simeq \frac{V}{3M_P^2} \qquad 3H\dot{\phi} \simeq -\partial_\phi V \qquad (16)$$

- Thus for a given V, the background is completely determined within the slow roll approximation.

Simplest model: Slow-roll Inflation

- **Inflation ends** when $\frac{\dot{\phi}^2}{2} \simeq V$; when the field oscillates around the potential's minimum.



- Note that we have presented a "classical" analysis

Where are the “quantum fluctuations”?

- The scalar field can be separated in a homogeneous part plus *small inhomogeneities*

$$\phi(\mathbf{x}, t) = \phi_0(t) + \delta\phi(\mathbf{x}, t)$$

- The **quantum fluctuations** of $\hat{\delta\phi}(\mathbf{x}, t)$ generates, via Einstein Equations $\delta g_{ab} \propto \delta T_{ab}$, metric perturbations (scalar Ψ, Φ, tensor h_{ij}).
- Metric perturbations are related to $\mathcal{R}(\mathbf{x}, t)$ called the **primordial curvature perturbation**. This is, in a specific gauge, $\mathcal{R}$ corresponds to the Ricci scalar associated to spatial 3D-hypersurfaces,
- $\mathcal{R}$ is directly related to $(\delta T/T)_{\text{CMB}}$

Scalar perturbations contain gauge freedom

Let us switch to conformal time

$$dt = a(\eta)\, d\eta$$

A general scalar perturbation of a flat FLRW metric can be written as

$$\begin{aligned} ds^2 &= a^2(\eta)\Big\{ -(1+2A)d\eta^2 + 2\partial_i B\, dx^i d\eta \\ &\quad + [(1-2\psi)\delta_{ij} + 2\partial_i\partial_j E]\, dx^i dx^j \Big\}. \end{aligned} \tag{17}$$

The fields A, B, ψ, E, and $\delta\phi$ are not separately observable.

- Two scalars are removed by "coordinate transf." (δx and δt) or *gauge choice* .
- Constraint equations remove two more.
- One physical scalar mode remains.

Comoving curvature is gauge invariant

For a single scalar field, the physical scalar mode can be represented by

$$\mathcal{R} \equiv \psi + \frac{\mathcal{H}}{\phi_0'}\delta\phi. \tag{18}$$

In comoving gauge, $\delta\phi = 0$, so $\mathcal{R} = \psi$.

- $\mathcal{R}$ measures the intrinsic curvature perturbation on comoving hypersurfaces.
- For adiabatic perturbations, $\mathcal{R}$ is conserved on super-Hubble scales up to gradient corrections.
- It provides the primordial initial condition used in late-time cosmology.

See Mukhanov, Feldman and Brandenberger, Phys. Rep. 215, 203 (1992).

QFT theory in a de Sitter spacetime

- The QFT of the perturbations can be described by the quantization of the *Mukhanov-Sasaki variable*

$$v \equiv a\delta\phi + \phi_0' a^2/a'\Psi$$

in a quasi-de Sitter space–time background. In fact

$$\mathcal{R} = \frac{v}{z} \qquad z \equiv \frac{a\phi_0'}{\mathcal{H}} \tag{19}$$

- Expanding the action to second order, v has a canonical kinetic term. It is therefore the natural variable to quantize.
- Quantization of v implies **quantization of $\delta\phi$ and Ψ,** i.e. $\hat{\delta\phi}$ and $\hat{\Psi}$.

The action contains a time-dependent mass

The second-order action for scalar perturbations is

$$S^{(2)} = \frac{1}{2}\int d\eta\, d^3x \left[(v')^2 - (\nabla v)^2 + \frac{z''}{z}v^2\right]. \tag{20}$$

The canonical momentum is

$$\pi_v = \frac{\partial \mathcal{L}}{\partial v'} = v'. \tag{21}$$

All background dependence enters through the effective mass term $-z''/z$.
At linear order, the cosmological problem becomes a collection of decoupled, time-dependent harmonic oscillators.

Fourier modes decouple at quadratic order

We use the convention

$$v(\eta, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} v_{\mathbf{k}}(\eta) e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{22}$$

Because v is real,

$$v_{-\mathbf{k}} = v^*_{\mathbf{k}}. \tag{23}$$

At quadratic order, different wavevectors do not interact. Rotational invariance further implies that the mode functions depend only on $k = |\mathbf{k}|$.

Scalar modes obey an oscillator equation

Varying the quadratic action gives

$$v_k'' + \left(k^2 - \frac{z''}{z}\right) v_k = 0. \tag{24}$$

For slowly varying Hubble-flow parameters,

$$\frac{z''}{z} \simeq \frac{\nu_s^2 - 1/4}{\eta^2}, \qquad \nu_s \simeq \frac{3}{2} + \epsilon_1 + \frac{\epsilon_2}{2}. \tag{25}$$

Two regimes are already visible:

- **Sub-Hubble modes** $k \gg aH$ or $k^2 \gg z''/z$: oscillatory, approximately Minkowskian behavior;
- **Super-Hubble modes** $k \ll aH$ or $k^2 \ll z''/z$: growing and decaying solutions replace oscillations.

EOFL1

Quantizing v produces an operator

We impose the equal-time commutators

$$[\hat{v}(\eta, \mathbf{x}), \hat{\pi}_v(\eta, \mathbf{y})] = i\delta^{(3)}(\mathbf{x} - \mathbf{y}), \tag{26}$$

and

$$[\hat{v}(\eta, \mathbf{x}), \hat{v}(\eta, \mathbf{y})] = [\hat{\pi}_v(\eta, \mathbf{x}), \hat{\pi}_v(\eta, \mathbf{y})] = 0. \tag{27}$$

The background $a(\eta)$ and $\phi_0(\eta)$ are still treated classically.

Modes evolve; operators encode the algebra

The field operator can be expanded as

$$\hat{v}(\eta, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \left[v_k(\eta) \hat{a}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} + v_k^*(\eta) \hat{a}_{\mathbf{k}}^\dagger e^{-i\mathbf{k}\cdot\mathbf{x}} \right]. \tag{28}$$

With

$$[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}^\dagger] = (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{k}'), \tag{29}$$

canonical normalization requires

$$v_k v_k^{*\prime} - v_k^* v_k' = i. \tag{30}$$

Each mode is a quantum oscillator

Schematically, the Hamiltonian is

$$\hat{H} = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\left[\hat{\pi}_{\mathbf{k}}\hat{\pi}_{-\mathbf{k}} + \omega_k^2(\eta)\hat{v}_{\mathbf{k}}\hat{v}_{-\mathbf{k}}\right], \tag{31}$$

with

$$\omega_k^2(\eta) = k^2 - \frac{z''}{z}. \tag{32}$$

The Bunch–Davies condition fixes the vacuum

At sufficiently early times, observable modes satisfy $k \gg aH$ and should behave as positive-frequency Minkowski modes:

$$v_k(\eta) \longrightarrow \frac{e^{-ik\eta}}{\sqrt{2k}} \qquad \text{as} \qquad -k\eta \longrightarrow \infty. \tag{33}$$

This defines the Bunch–Davies vacuum,

$$\hat{a}_{\mathbf{k}}|0_{\rm BD}\rangle = 0 \qquad \text{for all } \mathbf{k}. \tag{34}$$

Unitary evolution changes the initial zero particle state

$$|\Psi(\eta)\rangle_{\mathbf{k},\mathbf{k}'} = \hat{\mathcal{U}}_{\mathbf{k}}(\eta)|0_{\rm BD}\rangle = \sum_{n=0}^{\infty} \mathcal{F}[r_k(\eta), \alpha_k(\eta)]|n_{\mathbf{k}}, n_{\mathbf{k}'}\rangle \tag{35}$$

However, unitary evolution does not break the symmetries of the initial BD vacuum

The de Sitter mode joins two regimes

For exact de Sitter, $\nu_s = 3/2$, the normalized Bunch–Davies solution is

$$v_k(\eta) = \frac{1}{\sqrt{2k}}\left(1 - \frac{i}{k\eta}\right)e^{-ik\eta}. \tag{36}$$

For quasi-de Sitter expansion, the corresponding solution is a Hankel function,

$$v_k(\eta) = \frac{\sqrt{-\pi\eta}}{2}e^{i(2\nu_s+1)\pi/4}H^{(1)}_{\nu_s}(-k\eta). \tag{37}$$

The slow evolution of H and ϵ_1 slightly changes the index ν_s and produces the scalar tilt.

Curvature modes freeze after Hubble crossing

For sub-Hubble modes, $-k\eta \gg 1$,

$$v_k \simeq \frac{e^{-ik\eta}}{\sqrt{2k}}. \tag{38}$$

For super-Hubble $-k\eta \ll 1$ in de Sitter,

$$|v_k| \simeq \frac{1}{\sqrt{2k^3}|\eta|}. \tag{39}$$

Since $z \propto a \propto (-\eta)^{-1}$ at leading order,

$$\mathcal{R}_k = \frac{v_k}{z} \longrightarrow \text{constant} \tag{40}$$

for the growing mode. The decaying mode rapidly becomes negligible. EOFL1