

CZ + SK HEP WORKSHOP 2026

Real-Time Instanton Dynamics in a Triple-Well to Double-Well Homotopy

Based on joint work: V. Loubal, T. Sýkora, Š. Koš, “Real-Time Instanton Dynamics in a Triple-Well to Double-Well Homotopy” (2026)

Author: Vojtěch Loubal

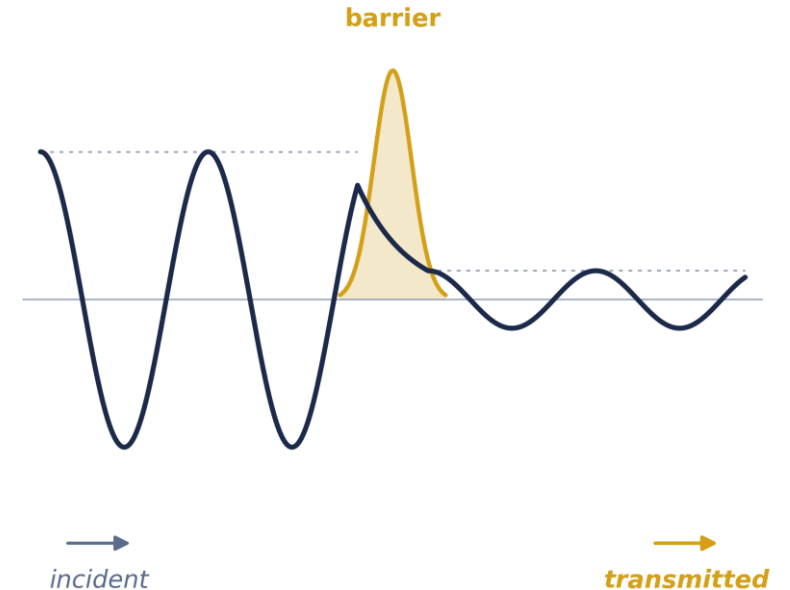
Supervisor: Mgr. Tomáš Sýkora, Ph.D.

Institute of Particle and Nuclear Physics
Faculty of Mathematics and Physics, Charles University, Prague

September 2026

The Real-Time Tunneling Problem

- Real-time (Minkowski) instantons underlie non-perturbative tunneling, scattering, and the real-time path integral
- Cherman & Ünsal (2014): analytically continuing the double-well kink produces an infinite comb of real-time singularities [1408.0012]
- Later work (Behtash et al., 2015) treats these singular saddles as physically meaningful via Picard–Lefschetz theory and resurgence theory [1510.03435]

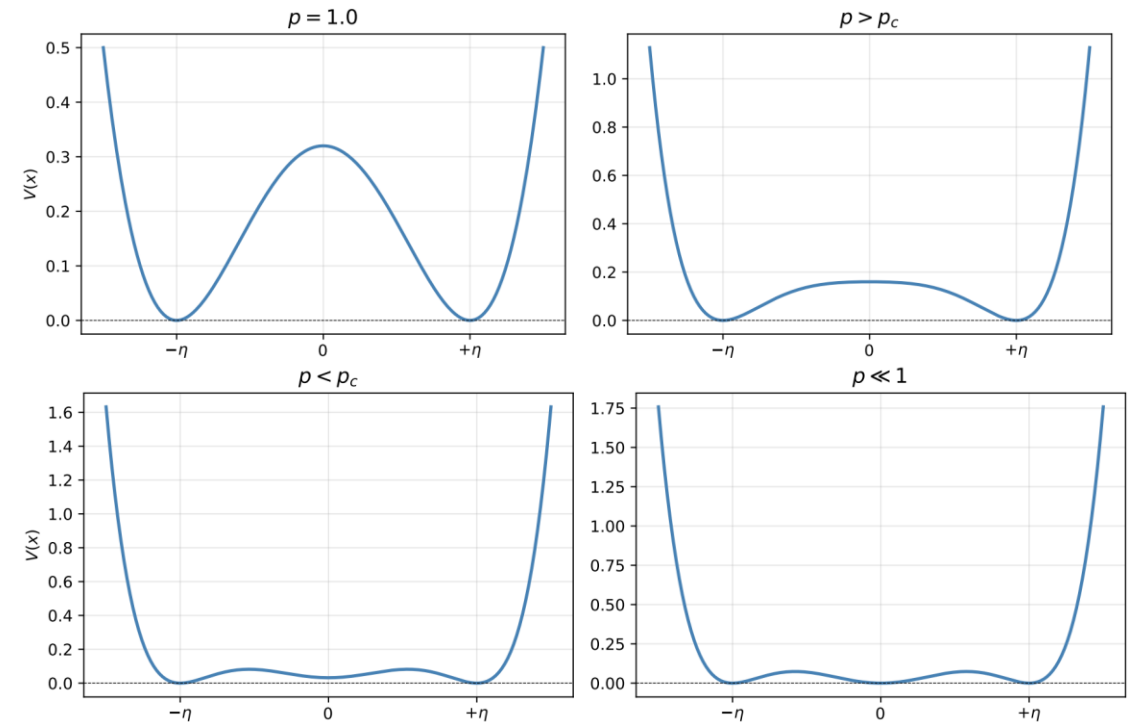


Open question: are these singularities an inherent feature of real-time tunneling or an artifact of the double-well limit specifically?

A Triple-Well $\leftrightarrow$ Double-Well Homotopy

$$V_p(z) = \lambda(z^2 - \eta^2)^2[(1 - p)z^2 + \xi^2 p], \quad p \in [0, 1]$$

- The unique linear homotopy connecting potentials with consecutive well-counts (triple-well $\rightarrow$ double-well)
- Critical value p_c : the central well flattens, fixed purely by the geometry η/ξ
- At $p = 1$: double well • $p < p_c$: triple well • $p \ll 1$: deep triple well



Shape of the homotopy potential

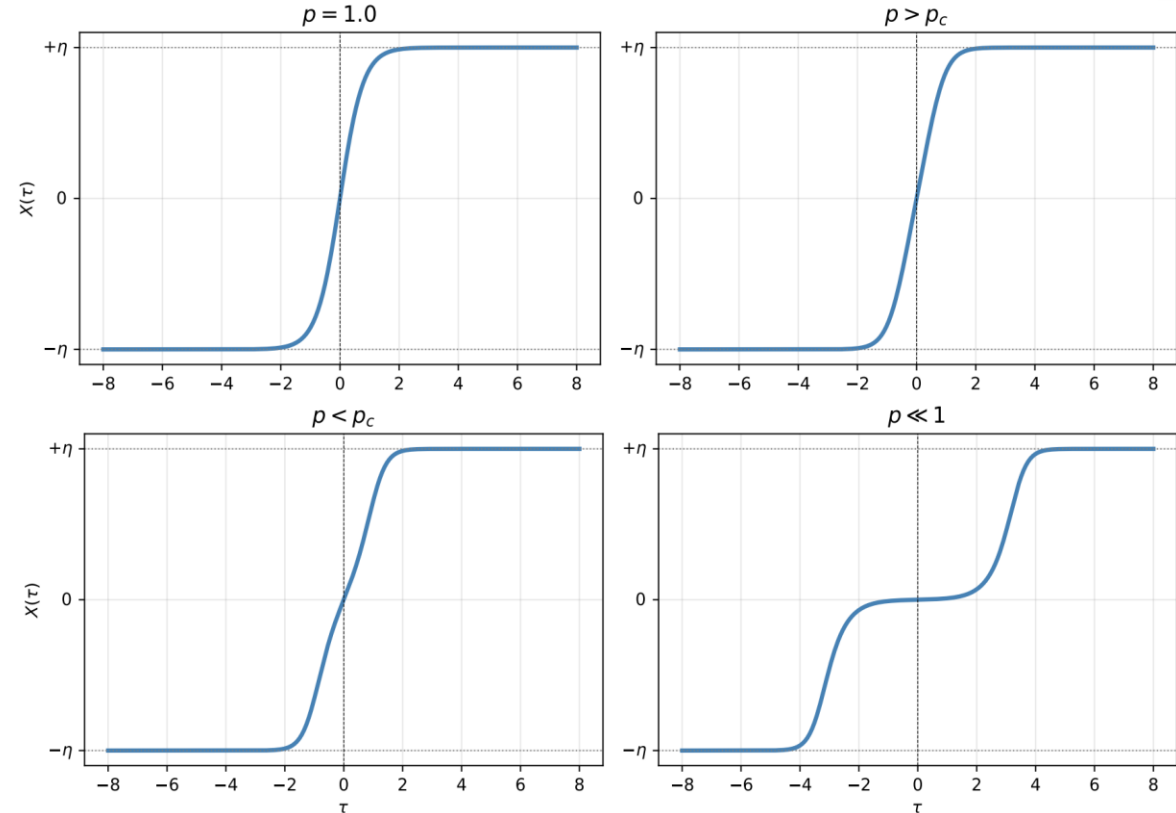
Euclidean Instantons in Closed Form

- BPS equation solved exactly:

$$X_p(\tau) = \pm \frac{\eta \sinh(u)}{\sqrt{\cosh^2(u) + b^2}}$$

- Entire family controlled by one dimensionless ratio $b^2 = (\eta/\xi)^2 \frac{1-p}{p}$
- For $p \leq p_c$ the instanton splits into two half-instantons — tunneling via the middle well
- Closed-form action $S_E^0(b)$ interpolates smoothly between known double-well and triple-well limits including instanton interactions

$$S_E^0(b) = \sqrt{2\lambda}\eta^4\xi \left[\frac{(2b^2 - 1)\sqrt{1 + b^2}}{4b^2\sqrt{\eta^2 + \xi^2 b^2}} + \frac{4b^2 + 1}{4b^3\sqrt{\eta^2 + \xi^2 b^2}} \operatorname{arctanh} \sqrt{\frac{b^2}{1 + b^2}} \right]$$



Behaviour of the exact instanton solution

Real-Time Continuation: Regularity Is Generic

1

For any $p \in (0,1)$

the real-time ($\alpha = 0$) solution is bounded:
 $X_p \in [-i\eta/b, +i\eta/b]$. No divergences.

2

For each $\alpha \in (0, \pi/2)$

a countably infinite family $p_{s,k}(\alpha)$ exists, each
giving exactly two singularities and never more.

3

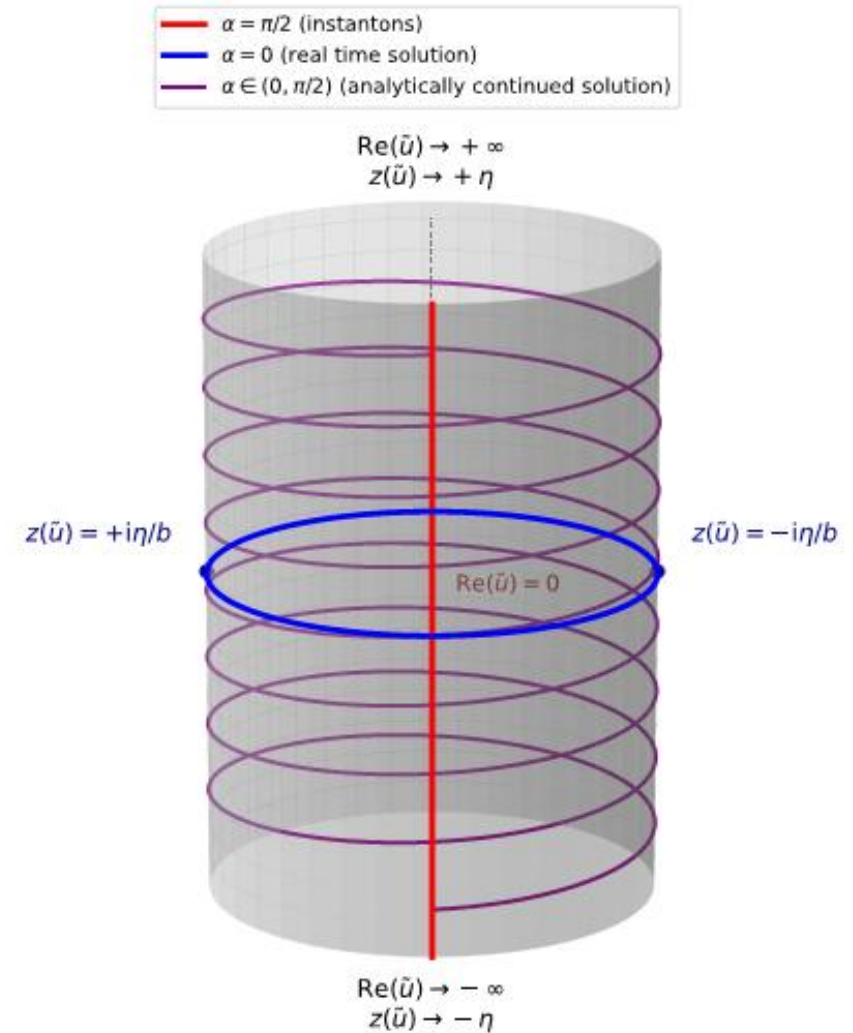
Double-well limit only

at $(p, \alpha) = (1,0)$ all singular pairs collapse into
Cherman–Ünsal's infinite comb. This is a single
degenerate point.

$$t = e^{-i\alpha} \tau$$

Weierstrass Elliptic Geometry

- Full complexified solution described by the Weierstrass elliptic function
- At complex energy $C = 0$ the complex solution tracks a geodesic on a cylinder
- A generic Wick-rotation ray ($\alpha \neq 0$) crosses at most two poles before escaping
- The point $\alpha = 0$ uniquely aligns with the periodic direction. Crossing the entire row gives the Dirac comb only at $p = 1$



Geometric interpretation of present singularities

Singularity structure of the homotopy

- 1 The Cherman–Ünsal singular comb is a degenerate phenomenon of a single point (p, α) , not a generic feature of real-time tunneling
- 2 Singular potentials form a discrete, measure-zero set for any fixed $\alpha \neq 0$. Regularity is generic and present in any complex time
- 3 Open direction: higher-genus ($n \geq 5$) homotopies, explicit Picard–Lefschetz thimble analysis and resurgent analysis through perturbation theory and WKB analysis

Acknowledgments

- This work was supported by the **Studentský fakultní grant 2025**, Faculty of Mathematics and Physics, Charles University
- My supervisor and co-author, **Mgr. Tomáš Sýkora, Ph.D.** — Institute of Particle and Nuclear Physics, Charles University
- Co-author **doc. Mgr. Šimon Koš, Ph.D.** — Department of Physics, Faculty of Applied Sciences, University of West Bohemia, Pilsen
- For providing valuable suggestions: **doc. Klaus Bering Larsen, Ph.D.** — Institute of Theoretical physics and Astrophysics, Masaryk University in Brno

The background features a series of concentric circles in a light blue-grey color, centered on the slide. The circles vary in size, creating a subtle geometric pattern.

Thank You

Questions & Discussion

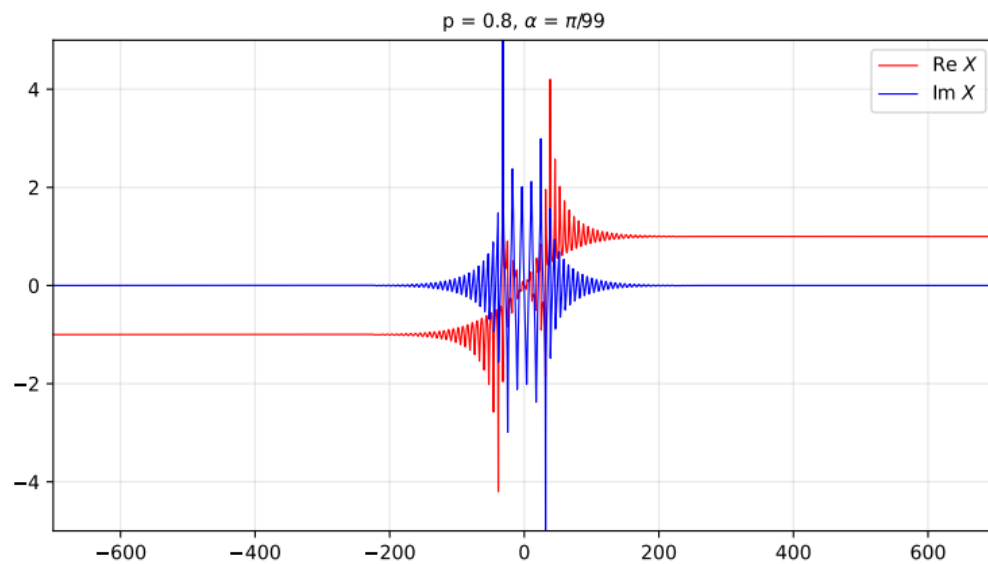
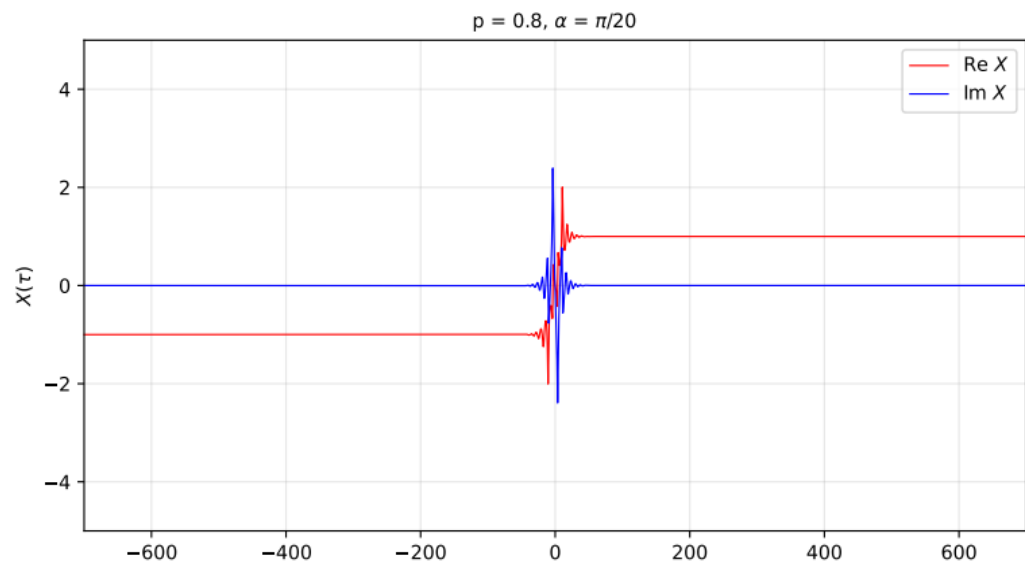
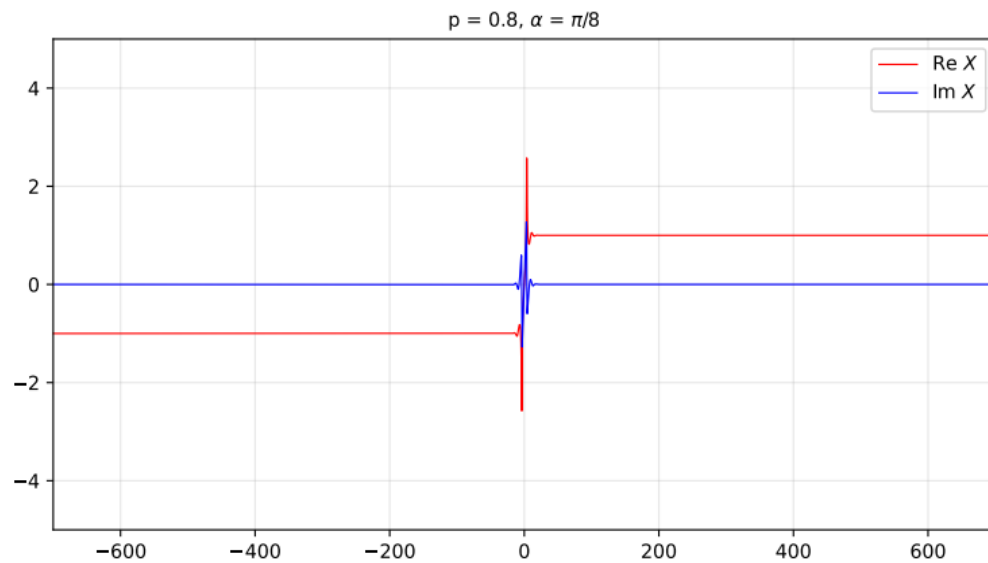
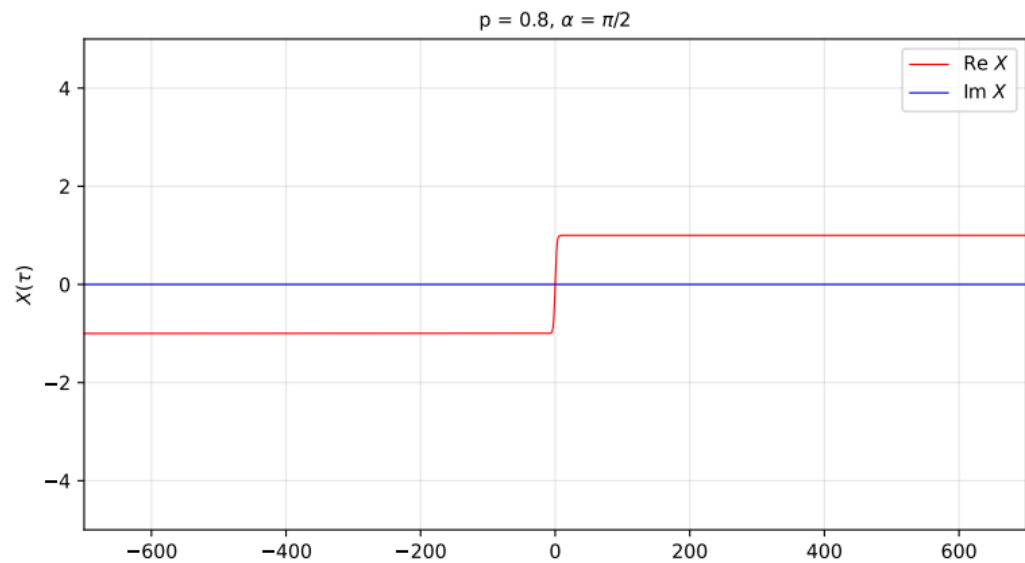
Institute of Particle and Nuclear Physics, Faculty of Mathematics and Physics, Charles University

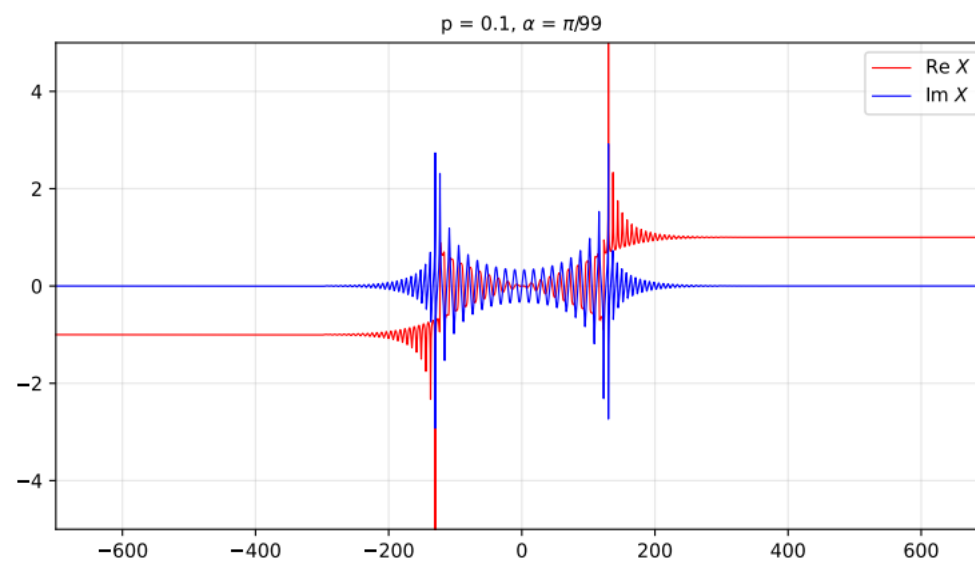
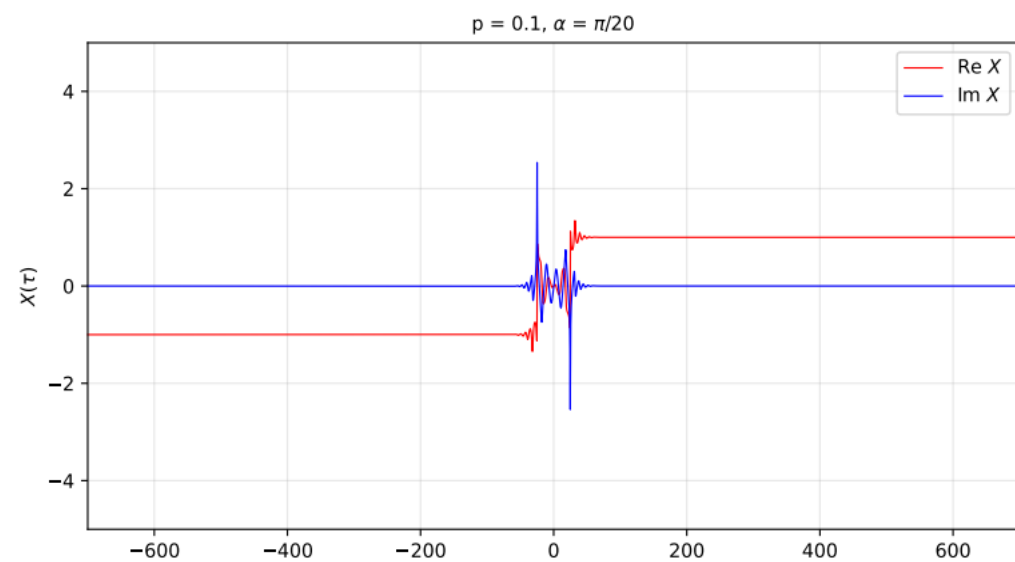
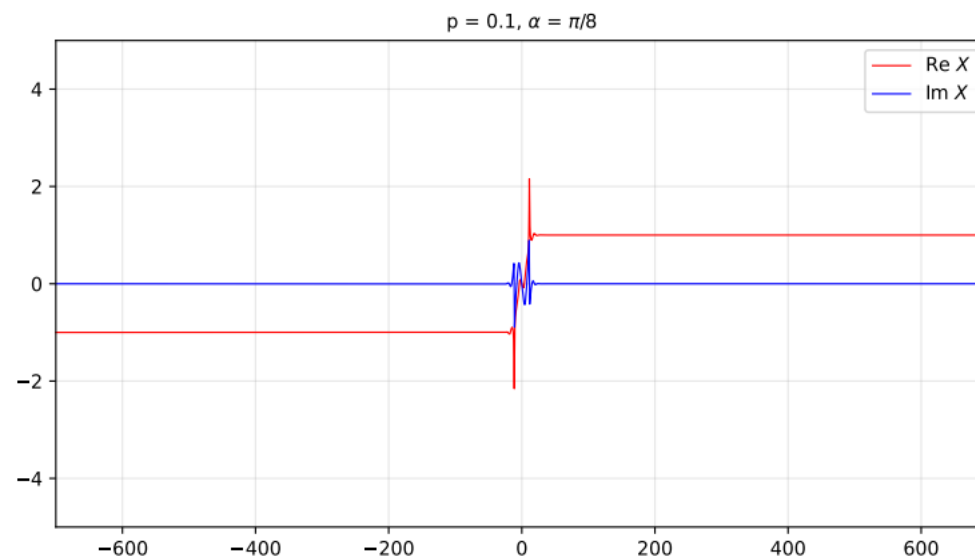
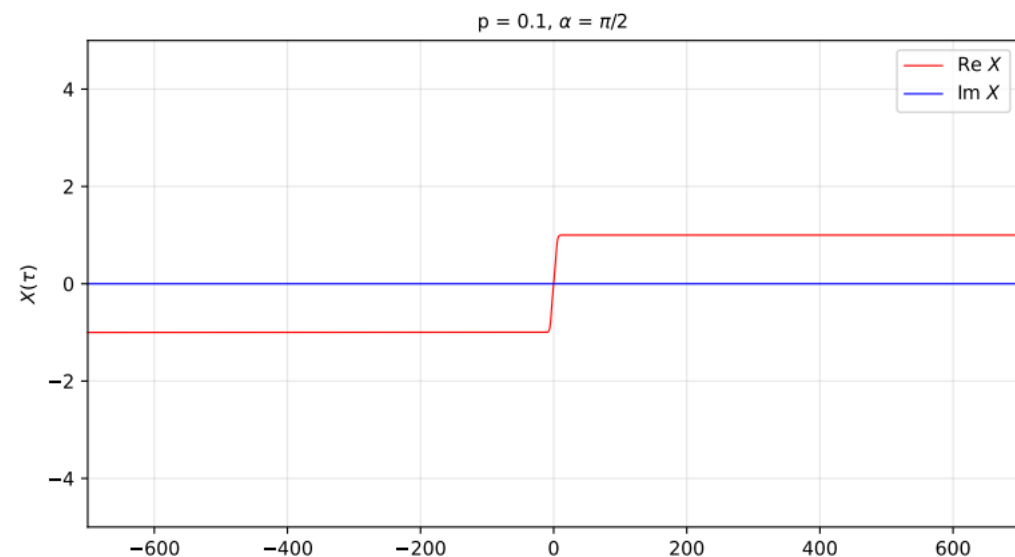
Contact: vojtech.loubal922@student.cuni.cz

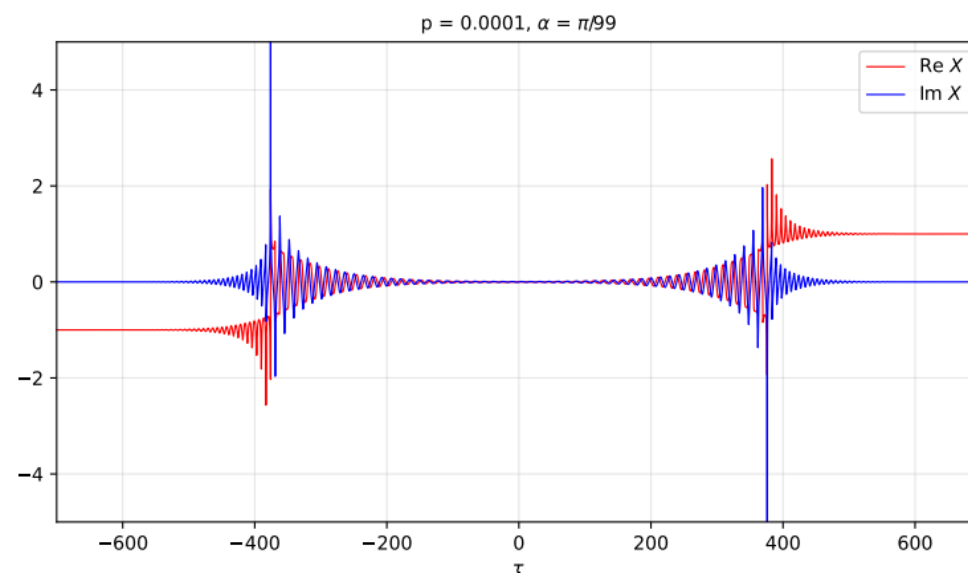
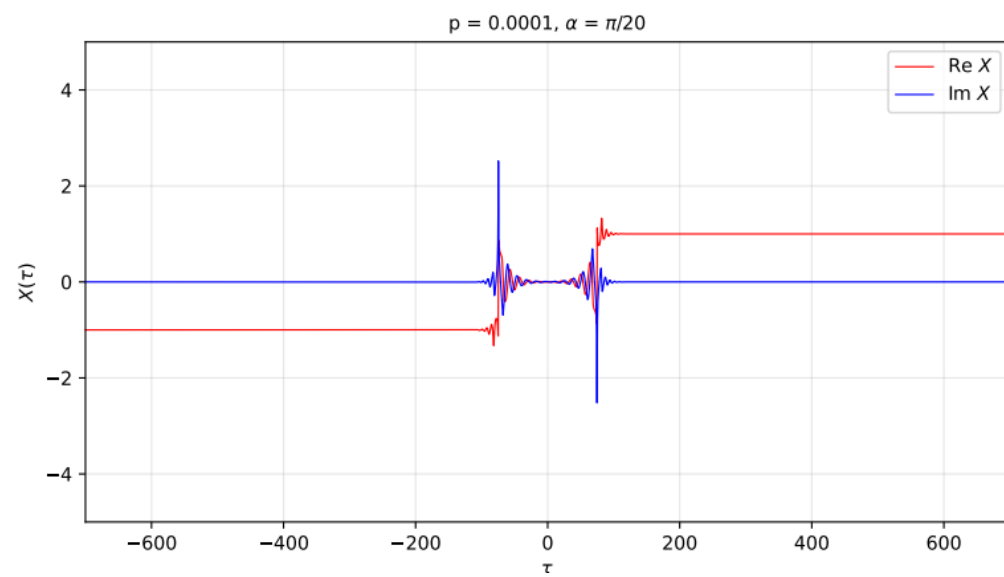
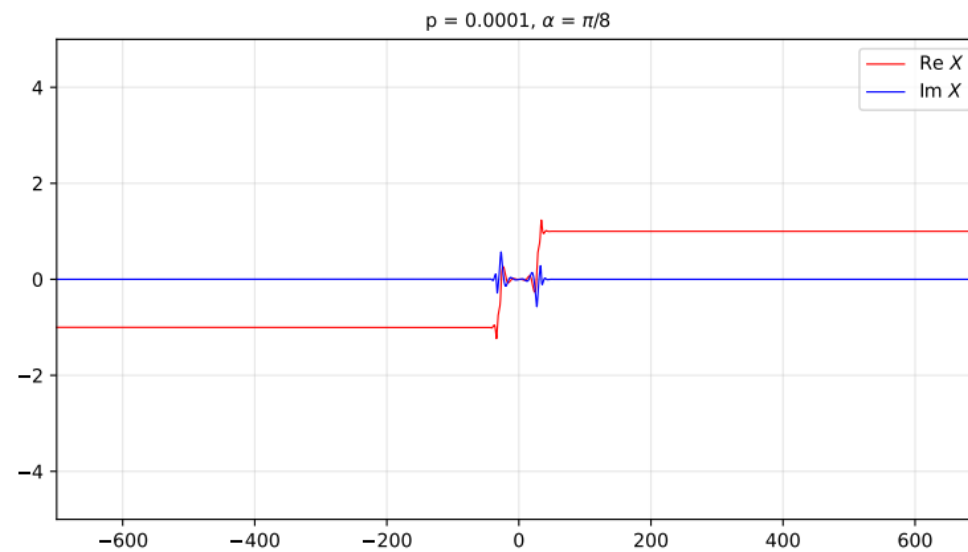
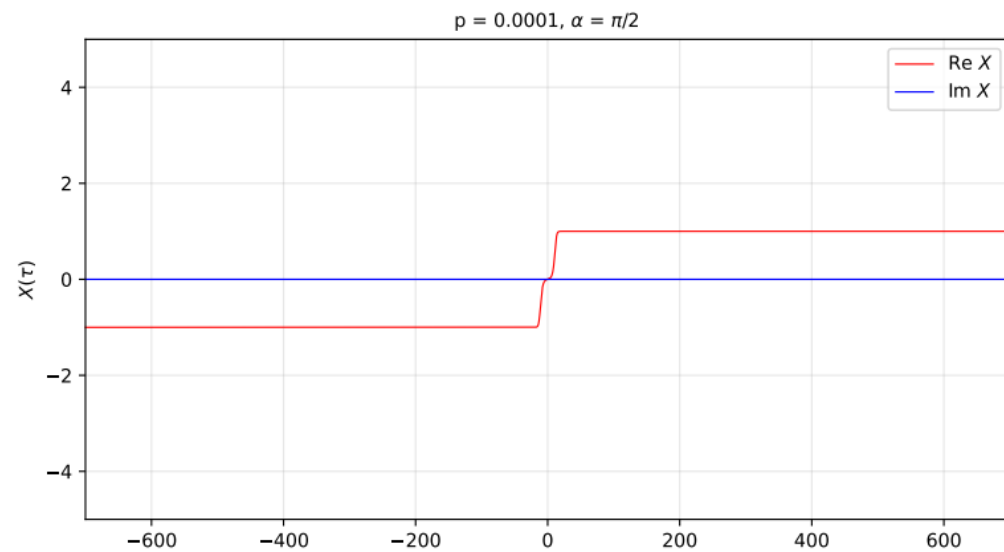
BackUp Slides

Connections to collider physics

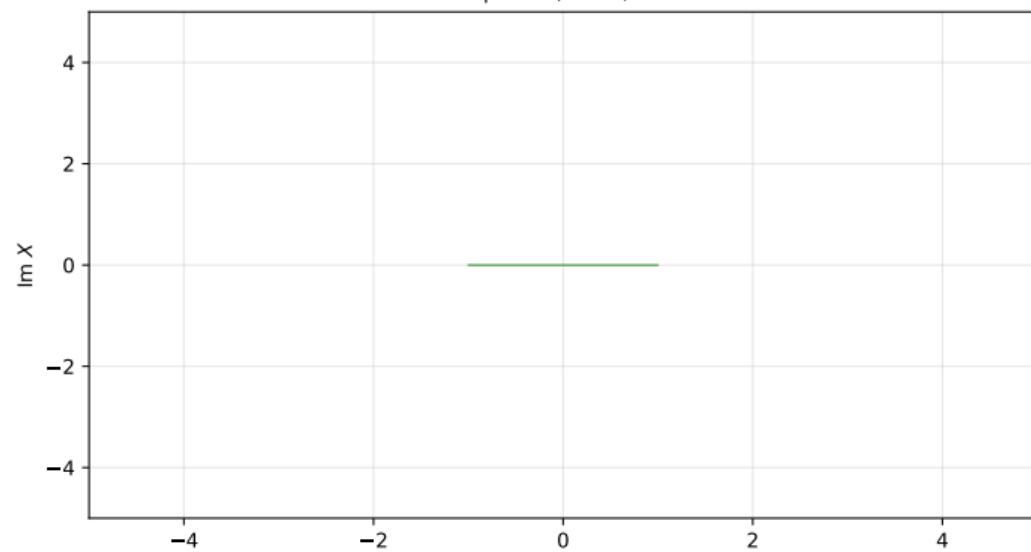
- The vacuum structure of QCD is non-trivial and consists of countably infinite degenerate minima (Callan et al. 1976)
- There exist instanton solutions in QCD connecting these nontrivial vacua (BPST 1975)
- Instantons are nonperturbative and modify the QCD Lagrangian through boundary terms
- Instanton coupling to gluons should be measurable (Khoze et al. 2021)
- Real-time instanton analysis in QFT through Picard-Lefschetz theory is still in development



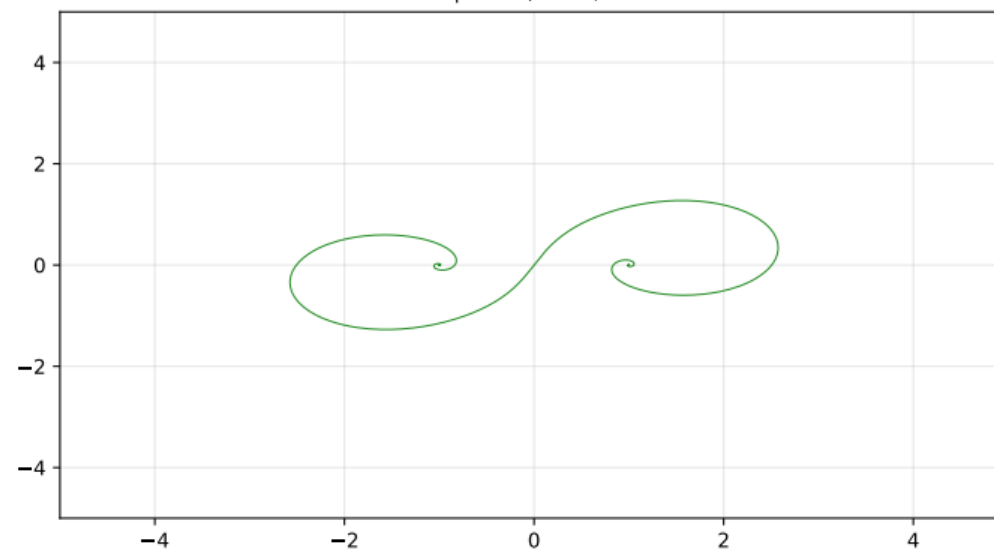




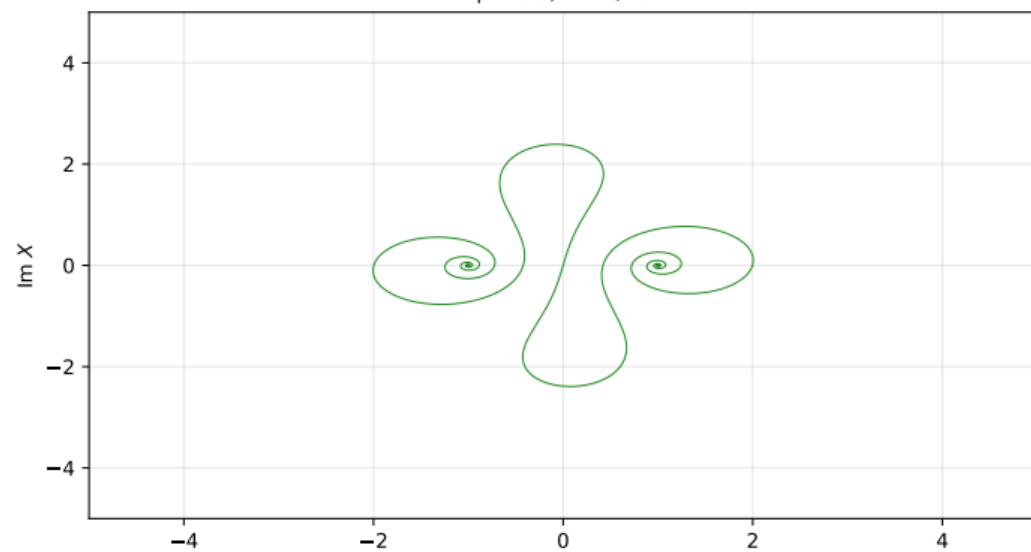
$\rho = 0.8, \alpha = \pi/2$



$\rho = 0.8, \alpha = \pi/8$



$\rho = 0.8, \alpha = \pi/20$



$\rho = 0.8, \alpha = \pi/99$

