ROTATING NEUTRON STARS

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Spacetime Metric

For a stationary, axisymmetric perfect fluid:

$$ds^2 = -e^{2\nu}dt^2 + e^{2\psi}(d\phi - \omega dt)^2 + e^{2\mu}(d\varpi^2 + dz^2)$$

<u>Assumptions:</u>

1. Stationary : Killing vector t^α which is *timelike* at spatial infinity
2. Axisymmetric: Killing vector ϕ^α which is *spacelike* everywhere, vanishes on a *symmetry axis* and whose orbits are *closed curves*
3. Asymptotically flat : $t_\alpha t^\alpha \to -1$, $\phi_\alpha \phi^\alpha \to +1$, $t_\alpha \phi^\alpha \to 0$, at spatial inf.

From 1., 2., 3. => t^α and ϕ^α *commute*

=> we can *choose* coordinates t and ϕ such that t^α and ϕ^α are *coordinate vectors*:

$$t^\alpha = \frac{\partial}{\partial t} \quad \text{and} \quad \phi^\alpha = \frac{\partial}{\partial \phi}$$

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Assumption of Circularity

The flow is *circular*, if the 4-velocity u^α and the charge current density J^α are *toroidal*

$$u^{[\alpha}t^{\beta}\phi^{\gamma]} = 0 \Rightarrow u^\alpha = \{u^t, 0, 0, u^\phi\}$$

$$J^{[\alpha}t^{\beta}\phi^{\gamma]} = 0 \Rightarrow J^\alpha = \{J^t, 0, 0, J^\phi\}$$

The latter condition is satisfied by *poloidal* magnetic fields.

These conditions imply that $T_{\alpha\beta}$ satisfies:

$$t^\delta T_{\delta[\alpha}t_\beta\phi_{\gamma]} = 0 \Leftrightarrow T^\alpha{}_t = \text{linear combination of } t^\alpha, \phi^\alpha$$

$$\phi^\delta T_{\delta[\alpha}t_\beta\phi_{\gamma]} = 0 \Leftrightarrow T^\alpha{}_\phi = \text{linear combination of } t^\alpha, \phi^\alpha$$

which leads to the following identities for the Killing vectors:

$$t_{[\alpha;\beta}t_\gamma\phi_{\delta]} = 0$$
$$\phi_{[\alpha;\beta}t_\gamma\phi_{\delta]} = 0$$

One can then show that t^α and ϕ^α are everywhere *orthogonal* to the 2-surfaces formed by the integral curves of the remaining two coordinates x^1 and x^2

$$\Rightarrow g_{t1} = g_{t2} = g_{\phi 1} = g_{\phi 2} = 0$$

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Quasi-Isotropic Coordinates

The nonzero components of the metric involving t and ϕ are written as *invariant combinations* of the Killing vectors:

$$\begin{aligned} g_{tt} &= t^\alpha t_\alpha = -e^{2\nu} + \omega^2 e^{2\psi} \\ g_{t\phi} &= t^\alpha \phi_\alpha = -\omega e^{2\psi} \\ g_{\phi\phi} &= \phi^\alpha \phi_\alpha = e^{2\psi} \end{aligned}$$

Then

$$g_{\alpha\beta} = \begin{bmatrix} -e^{2\nu} + \omega^2 e^{2\psi} & 0 & 0 & -\omega e^{2\psi} \\ & g_{11} & g_{12} & 0 \\ \text{sym.} & & g_{22} & 0 \\ & & & e^{2\psi} \end{bmatrix}$$

If one chooses *orthogonal coordinates* x^1 and x^2, then $g_{12}=0$.
For the remaining components, one can choose an *isotropic gauge*, in which the (x^1, x^2) sub-space is *conformally flat*:

e.g. *cylindrical-like* coordinates: $e^{2\mu}(d\varpi^2 + dz^2)$

so that, finally:

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Rotation of the Fluid

The ratio of the two components of the 4-velocity

$$\Omega \equiv \frac{u^\phi}{u^t} = \frac{d\phi/ds}{dt/ds} = \frac{d\phi}{dt}$$

is the *angular velocity* as seen by a *nonrotating observer at infinity.*

The normalization of u^α specifies u^t :

$$u^\alpha u_\alpha = -1 \Rightarrow u^t = \frac{e^{-\nu}}{\sqrt{1-(\Omega-\omega)^2 e^{2(\psi-\nu)}}}$$

If we set

$$v = (\Omega - \omega)e^{\psi-\nu}$$

Then

$$u^\alpha = \frac{e^{-\nu}}{\sqrt{1-v^2}}\{1,0,0,\Omega\}$$

Since ϕ^α is a Killing vector, there is a conserved specific angular momentum for freely-falling observers:

$$u_\phi = (\Omega - \omega)e^{2\psi}u^t$$

Thus, *zero-angular momentum* observers (*ZAMO's*) with $u_\phi=0$, are rotating with an angular velocity $\Omega=\omega$ w.r.t. infinity (*dragging of inertial frames*)!

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3-Velocity

We can define an orthonormal tetrad frame attached to the ZAMO:

$$
\begin{aligned}
\omega^{(0)} &= e^{\nu} dt \\
\omega^{(1)} &= e^{\mu} d\varpi \\
\omega^{(2)} &= e^{\mu} dz \\
\omega^{(3)} &= e^{\psi}(d\phi - \omega dt)
\end{aligned}
$$

in which the metric becomes Minkowskian: $g_{\alpha\beta}$=diag{-1, 1,1,1} .

Transforming the 4-velocity from the coordinate frame to the *ZAMO* tetrad, one finds the components:

$$u^{(\alpha)} = \frac{1}{\sqrt{1-v^2}}\{1,0,0,v\}$$

Therefore, v is the *3-velocity* measured by the *ZAMO*.

Proper Circumference

The *proper circumference* of a spatial circle at ϖ, z and t=const. is

$$\mathcal{C} = 2\pi\sqrt{g_{\phi\phi}} = 2\pi e^{\psi}$$

Thus, the *circumferential radius* is defined as

$$R := \mathcal{C}/(2\pi) = e^{\psi}$$

Since $R=\varpi$ at infinity, we can redefine:

$$e^{\psi} = \varpi B e^{-\nu}$$

Notice that *in vacuum*, one can *absorb* B in the definition of ϖ and z (only *three metric functions* are required).

In the *spherical limit*: $B = 1, \;\; \mu = -1$

$$\Rightarrow ds^2 = -e^{2\nu}dt^2 + e^{-2\nu}(d\varpi^2 + dz^2 + \varpi^2 d\phi^2)$$

which is the metric of a spherical star in *isotropic coordinates*.

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Time Dilation and Redshifts

At a given location ϖ, z, the proper time for the *ZAMO* ($d\phi = \omega\, dt$) is:

$$d\tau = e^{\nu} dt$$

Thus, $e^{-\nu}$ is the *time dilation* factor between the proper time for the *ZAMO* and coordinate time at infinity.

The *redshift* of emmited photons can be found from

$$z = \frac{\omega_E}{\omega_\infty} - 1$$

where

$$\frac{\omega_E}{\omega_\infty} = \frac{-p_\alpha u^\alpha}{-p_\alpha t^\alpha}$$

For photons emitted in the *forward* (+) or *backward* (-) direction at the equator:

$$p^\alpha = const. \times [t^\alpha + (e^{\nu-\psi} \pm \omega)\phi^\alpha]$$

so that

$$\{z_F, z_B\} = \left(\frac{1 \mp v}{1 \pm v}\right)^{1/2} \frac{e^{-\nu}}{1 \pm \omega e^{\psi-\nu}} - 1$$

while, for photons emitted at the *pole*, along the symmetry axis:

$$z_P = e^{-\nu} - 1$$

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Summary, so far

The metric of a stationary, axisymmetric star with purely circular flow is:

$$ds^2 = -e^{2\nu}dt^2 + e^{2\psi}(d\phi - \omega dt)^2 + e^{2\mu}(d\varpi^2 + dz^2)$$

where three metric functions have an invariant meaning:

e^{ν} (*time dilation)*

e^{ψ} *(circumferential radius)*

ω *(dragging of intertial frames)*

while $e^{2\mu}$ is a *conformal factor* for the geometry of the (ϖ,*z*) 2-planes.

The *angular velocity* is measured by an *observer at infinity*

$$\Omega \equiv \frac{u^\phi}{u^t} = \frac{d\phi}{dt}$$

while the *3-velocity v* is measured by the *zero-angular-momentum observer* (*ZAMO).*

Description of the Fluid

For a perfect fluid, we define the following *intensive properties*, all measured by an observer *comoving* with the fluid:

n	*baryon number density*
ρ	*baryon mass density*
e	*total energy density*
p	*isotropic pressure*
ε	*specific internal energy*
h	*specific enthalpy*
s	*specific entropy*

where

$$e = \rho(1 + \varepsilon)$$

$$h = \frac{e + p}{\rho}$$

The *equation of state* can be considered to be of the form

$p=p(\rho)$,	$e=e(\rho)$	*barotropic*	(cold stars)
$p=p(\rho, s)$,	$e=e(\rho,s)$	*general*	(hot stars)

If we assume that the fluid is *nonmagnetized*, then the stress-energy tensor is

$$T_{\alpha\beta} = (e + p)u_\alpha u_\beta + p g_{\alpha\beta}$$

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The Magnetized Fluid (I)

In relativistic MHD, one defines "electric" and "magnetic" 4-vectors

$$
\begin{aligned}
E^\alpha &= F^{\alpha\beta} u_\beta \\
B^\alpha &= {}^*F^{\alpha\beta} u_\beta
\end{aligned}
$$

where $F^{\alpha\beta}$ is the Faraday tensor and ${}^*F^{\alpha\beta}$ its dual. The two vectors have only three non-vanishing components normal to the 4-velocity:

$$E_\alpha u^\alpha = B_\alpha u^\alpha = 0$$

Ohm's law relates the "electric" 4-vector to the current density J^α

$$\sigma E^\alpha = J^\alpha + u_\beta J^\beta u^\alpha$$

In the *ideal MHD* approximation, the *conductivity* σ is assumed to be *infinite*. In order for J^α to be finite:

$$E^\alpha = 0$$

so that only B^α is needed to describe ideal MHD.

The usual electric field 3-vector E^i coincides with the normal components of E^α in a comoving frame. Thus, in ideal MHD, *there is no electric field in the comoving frame.*

The Magnetized Fluid (II)

The Faraday tensor becomes

$$\begin{aligned} F^{\alpha\beta} &= \epsilon^{\alpha\beta\gamma\delta} u_\gamma B_\delta \\ {}^*F_{\alpha\beta} &= u^\alpha B^\beta - u^\beta B^\alpha \end{aligned}$$

and the stress-energy tensor of the E/M field is

$$\begin{aligned} T^{\alpha\beta} &= \frac{1}{4\pi}(F^{\alpha\sigma} F^\beta{}_\sigma - \frac{1}{4} g^{\alpha\beta} F_{\lambda\mu} F^{\lambda\mu}) \\ &= \frac{1}{4\pi}(B^2 u^\alpha u^\beta + \frac{1}{2} B^2 g^{\alpha\beta} - B^\alpha B^\beta) \end{aligned}$$

To a first approximation, we can assume that the total stress-energy tensor is

$$\begin{aligned} T^{\alpha\beta} &= (e+p) u^\alpha u^\beta + p g^{\alpha\beta} \\ &\quad + \frac{1}{4\pi}(B^2 u^\alpha u^\beta + \frac{1}{2} B^2 g^{\alpha\beta} - B^\alpha B^\beta) \end{aligned}$$

In addition, if the flow is circular, the Faraday tensor is given by a *vector potential*

$$F_{\alpha\beta} = A_{\beta,\alpha} - A_{\alpha,\beta}$$

with components:

$$A_\alpha = \{A_t, 0, 0, A_\phi\}$$

The Equations of Structure

To obtain a self-consistent equilibrium solution, we need:

4 components of the *Einstein field equations*, for ν, ψ, ω, and μ

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta}$$

2 components of the *fluid conservation equations* for the specific enthalpy h and the angular velocity Ω

$$\begin{aligned} T^{\alpha\beta}{}_{;\alpha} &= 0 \\ (\rho u^{\alpha})_{;\alpha} &= 0 \end{aligned}$$

2 components of *Maxwell's equations* for A_t, A_ϕ

$$\begin{aligned} {}^*F^{\alpha\beta}{}_{;\beta} &= 0 \\ F^{\alpha\beta}{}_{;\beta} &= 4\pi J^{\alpha} \end{aligned}$$

An nonmagnetized equilibrium solution is *uniquely determined*, once we *specify*:

$p=p(\rho, s)$, $e=e(\rho, s)$	*equation of state*
$\Omega=\Omega(\varpi, z, A)$	*rotation law*
ρ_c	*central density*
r_p/r_e	*axes ratio*

The Einstein Field Equations

The 4 components of the Einstein field equations can be derived, e.g. in the *ZAMO* tetrad. The $R_{(t)(t)}$, $R_{(t)(\phi)}$ and $R_{(t)}^{(t)}+R_{(\phi)}^{(\phi)}$ components yield 3 *elliptic* equations:

$$\nabla\cdot(B\nabla\nu) = \frac{1}{2}\varpi^2 B^3 e^{-4\nu}\nabla\omega\cdot\nabla\omega + 4\pi B e^{2\mu}\left[\frac{(e+p)(1+v^2)}{1-v^2}+2p\right]$$

$$\nabla\cdot(\varpi^2 B^3 e^{-4\nu}\nabla\omega) = -16\pi\varpi B^2 e^{2(\mu-\nu)}\frac{(e+p)v}{1-v^2}$$

$$\nabla\cdot(\varpi\nabla B) = 16\pi\varpi B e^{2\mu} p$$

where ∇ is the 2-dimensional *flat-space* Laplacian operator.

For *nonmagnetized stars*, one can use the *Ricci identities* $R_{(\varpi)(\varpi)}=R_{(z)(z)}$ and $R_{(\varpi)(z)}=0$, to construct a *first-order differential equation* for the metric function μ.

For *magnetized stars*, one can still construct an *elliptic equation* for μ from the remaining components of the Einstein equations.

The elliptic equations can be written in *integral form*, using appropriate *Green's functions.*

RNS code

Metric:

Stergioulas & Friedman, 1995

$$2\nu \to \gamma + \rho$$

$$2\psi \to \gamma - \rho$$

$$\mu \to \alpha$$

Then:

$$ds^2 = -e^{\gamma+\rho} dt^2 + e^{\gamma-\rho} r^2 \sin^2\theta (d\phi - \omega dt)^2 + e^{2\alpha}(dr^2 + r^2 d\theta^2)$$

Set: $\cos\theta \to \mu$

Convert field equations using *Green's functions*

Differential form:

$$\Delta[\rho e^{\gamma/2}] = S_\rho(r,\mu),$$

$$\left(\Delta + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\mu\frac{\partial}{\partial\mu}\right)\gamma e^{\gamma/2} = S_\gamma(r,\mu),$$

$$\left(\Delta + \frac{2}{r}\frac{\partial}{\partial r} - \frac{2}{r^2}\mu\frac{\partial}{\partial\mu}\right)\omega e^{(\gamma-2\rho)/2} = S_\omega(r,\mu),$$

$\Longrightarrow$

Integral form:

$$\rho = -\frac{1}{4\pi} e^{-\gamma/2}\int_0^\infty dr' \int_{-1}^1 d\mu' \int_0^{2\pi} d\phi' \, r'^2 S_\rho(r',\mu') \frac{1}{|\mathbf{r}-\mathbf{r}'|}$$

$$r\sin\theta\,\gamma = \frac{1}{2\pi} e^{-\gamma/2}\int_0^\infty dr' \int_0^{2\pi} d\theta' \, r'^2 \sin\theta' S_\gamma(r',\theta') \log|\mathbf{r}-\mathbf{r}'|,$$

$$r\sin\theta\cos\phi\,\omega = -\frac{1}{4\pi} e^{(2\rho-\gamma)/2}\int_0^\infty dr' \int_0^{\pi} d\theta' \int_0^{2\pi} d\phi' \, r'^3 \sin^2\theta' \cos\phi' S_\omega(r',\theta') \frac{1}{|\mathbf{r}-\mathbf{r}'|}$$

RNS code

Stergioulas & Friedman, 1995

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RNS code

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$$\mu \rightarrow \alpha$$

Stergioulas & Friedman, 1995

Komatsu, Eriguchi & Hachisu (1989) method

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RNS code

Equation for α is an ordinary ODE:

$$
\begin{aligned}
\alpha_{,\mu} = &-\nu_{,\mu} - \{(1-\mu^2)(1+rB^{-1}B_{,r})^2 + [\mu - (1-\mu^2)B^{-1}B_{,\mu}]^2\}^{-1} \\
&[\![\tfrac{1}{2}B^{-1}\{r^2 B_{,rr} - [(1-\mu^2)B_{,\mu}]_{,\mu} - 2\mu B_{,\mu}\}[-\mu + (1-\mu^2)B^{-1}B_{,\mu}] \\
&+ rB^{-1}B_{,r}[\tfrac{1}{2}\mu + \mu r B^{-1}B_{,r} + \tfrac{1}{2}(1-\mu^2)B^{-1}B_{,\mu}] \\
&+ \tfrac{3}{2}B^{-1}B_{,\mu}[-\mu^2 + \mu(1-\mu^2)B^{-1}B_{,\mu}] - (1-\mu^2)rB^{-1}B_{,\mu r}(1+rB^{-1}B_{,r}) \\
&- \mu r^2 \nu_{,r}^2 - 2(1-\mu^2)r\nu_{,\mu}\nu_{,r} + \mu(1-\mu^2)\nu_{,\mu}^2 - 2(1-\mu^2)r^2B^{-1}B_{,r} \\
&\times \nu_{,\mu}\nu_{,r} + (1-\mu^2)B^{-1}B_{,\mu}[r^2\nu_{,r}^2 - (1-\mu^2)\nu_{,\mu}^2] + (1-\mu^2)B^2 e^{-4\nu} \\
&\times [\tfrac{1}{4}\mu r^4 \omega_{,r}^2 + \tfrac{1}{2}(1-\mu^2)r^3\omega_{,\mu}\omega_{,r} - \tfrac{1}{4}\mu(1-\mu^2)r^2\omega_{,\mu}^2 + \tfrac{1}{2}(1-\mu^2) \\
&\times r^4 B^{-1}B_{,r}\omega_{,\mu}\omega_{,r} - \tfrac{1}{4}(1-\mu^2)r^2B^{-1}B_{,\mu}[r^2\omega_{,r}^2 - (1-\mu^2)\omega_{,\mu}^2]\}]\!],
\end{aligned}
$$

Cook, Shapiro & Teukolsky (1994)

compactified radial coordinate s:

$$r = r_e \frac{s}{1-s}$$

$$r = 0 \rightarrow s = 0$$

$$r = r_e \rightarrow s = 0.5$$

$$r = \infty \rightarrow s = 1$$

Hydrostationary Equilibrium

The conservation of the stress-energy tensor leads to the equations of motion

$$\frac{1}{e+p}\nabla p + u_\phi u^t \nabla\Omega - \frac{1}{e+p}(J^\phi - \Omega J^t)\nabla A_\phi = \nabla(\ln u^t)$$

For stationary configurations, this equation is integrable in special cases, e.g.

(i) Cold, nonmagnetized models:

if $u_\phi u^t = F(\Omega) \;\Rightarrow\; \ln\left(\frac{h}{u^t}\right) + \int F(\Omega)d\Omega = const.$

(ii) Cold, magnetized models:

if Ω=*const.* and $\frac{J^\phi - \Omega J^t}{e+p} = f(A_\phi)$

$$\Rightarrow\; \ln\left(\frac{h}{u^t}\right) - \int f(A_\phi)dA_\phi = const.$$

(iii) Hot, nonmagnetized models:

if $T = T(n) \;\Rightarrow\; \int \frac{dp}{e+p} - \ln u^t + \int F(\Omega)d\Omega = const.$

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Maxwell's Equations

From

$$F^{\alpha\beta}{}_{;\beta} = 4\pi J^{\alpha}$$

one finds two *elliptic equations* for A_t and A_ϕ. These can be written in *integral form.*

Conserved Quantities Along the Flow

For the stationary, axisymmetric and circular flow:

$\mathcal{L}_u(hu_t) = 0$ Bernoulli's law

$\mathcal{L}_u(hu_\phi) = 0$ cons. of specific angular momentum

$\mathcal{L}_u \omega_{\alpha\beta} = 0$ cons. of circulation

where the *relativistic vorticity* $\omega_{\alpha\beta}$ is defined as

$$\omega_{\alpha\beta} = (hu_\beta)_{;\alpha} - (hu_\alpha)_{;\beta}$$

The Virial Theorems

The well-known Newtonian Virial theorem for equilibrium configurations

$$2T + (3\gamma - 1)U + W = 0$$

has been generalized in GR by Bonazzola & Gourgoulhon

$$\begin{aligned}\lambda_3 \equiv\ & 4\pi \int_0^{+\infty} \int_0^{\pi} \left[3p + (\varepsilon + p)\frac{v^2}{1-v^2}\right] \mathrm{e}^{2\mu+\psi}\, r\, \mathrm{d}r\, \mathrm{d}\theta \\ & \times \left\{ \int_0^{+\infty} \int_0^{\pi} \left[\partial\nu\partial\nu - \frac{1}{2}\partial\mu\partial\psi \right.\right. \\ & + \frac{\mathrm{e}^{2\mu-2\psi}}{2} r \sin^2\theta \left(\frac{\partial\mu}{\partial r} + \frac{1}{r\tan\theta}\frac{\partial\mu}{\partial\theta}\right) \\ & + \frac{1}{4r}\left(1 - \mathrm{e}^{2\mu-2\psi} r^2 \sin^2\theta\right)\left(\frac{\partial\psi}{\partial r} + \frac{1}{r\tan\theta}\frac{\partial\psi}{\partial\theta}\right. \\ & \left.\left.\left. - \frac{1}{r\sin^2\theta}\right) - \frac{3}{8}\mathrm{e}^{2\psi-2\nu}\partial\omega\partial\omega\right] \mathrm{e}^{\psi}\, r\, \mathrm{d}r\, \mathrm{d}\theta \right\}^{-1},\end{aligned}$$

A 2-D Virial theorem has also been found:

$$\lambda_2 \equiv \frac{8\pi \int_0^{+\infty} \int_0^{\pi} \left[p + (\varepsilon + p)\frac{v^2}{1-v^2}\right] \mathrm{e}^{2\mu}\, r\, \mathrm{d}r\, \mathrm{d}\theta}{\int_0^{+\infty} \int_0^{\pi} \left[\partial\nu\partial\nu - \frac{3}{4}\mathrm{e}^{2\psi-2\nu}\partial\omega\partial\omega\right] r\, \mathrm{d}r\, \mathrm{d}\theta},$$

Equilibrium Quantities

Using the unit normal $\hat{n}^\alpha$ to the *t*=const. spacelike surfaces and the proper volume

$$dV = \sqrt{|{}^3g|}d^3x$$

one can define various *extensive* equilibrium quantities

$$M = \int \left(T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}T\right) t^\alpha \hat{n}^\beta dV \quad \textit{gravitational mass}$$

$$M_0 = \int \rho u_\alpha \hat{n}^\alpha dV \quad \textit{baryon mass}$$

$$U = \int \rho\epsilon u_\alpha \hat{n}^\alpha dV \quad \textit{internal energy}$$

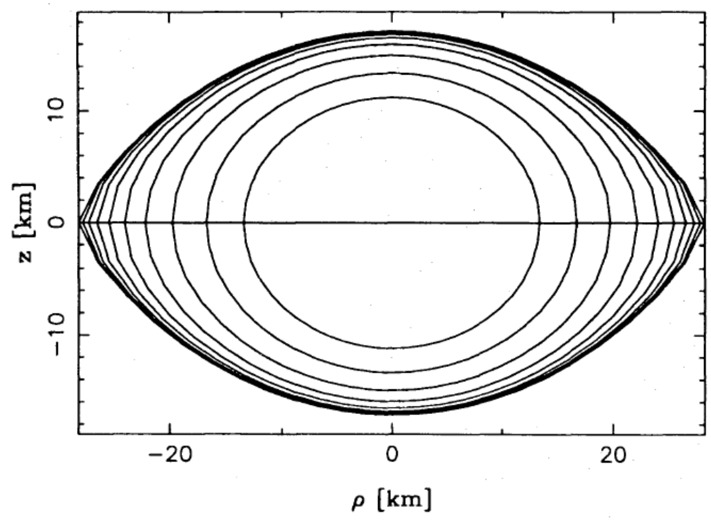
$$J = \int T_{\alpha\beta}\phi^\alpha n^\beta dV \quad \textit{angular momentum}$$

$$T = \frac{1}{2}\int \Omega dJ \quad \textit{rotational energy}$$

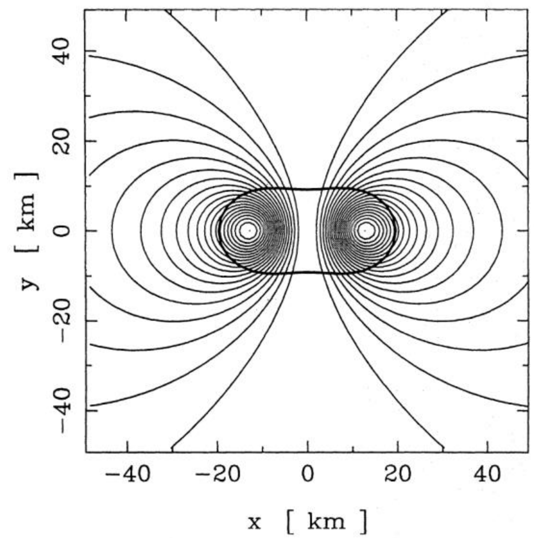
$$W = M - (M_0 + U + T) \quad \textit{gravitational binding energy}$$

$$I = J/\Omega \quad \textit{moment of inertia}$$

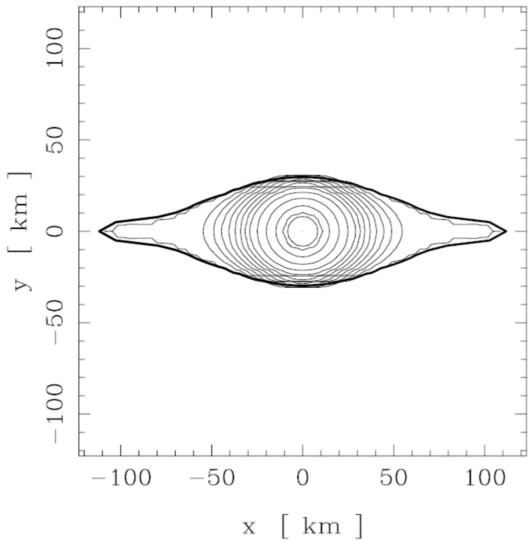
Examples of Equilibrium Models



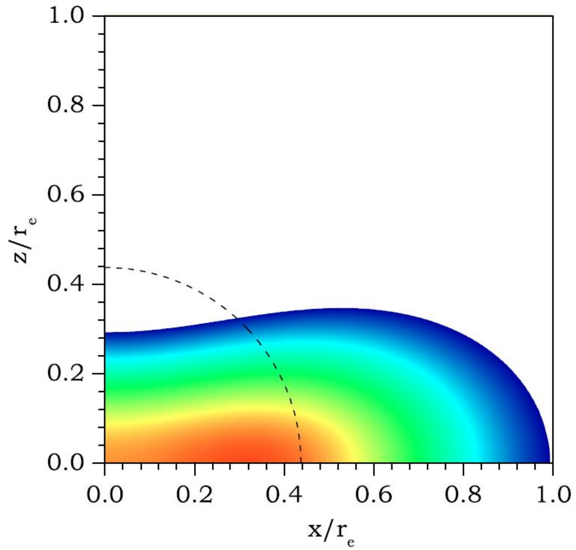
Bonazzola et al. 1993



Bocquet et al. 1995



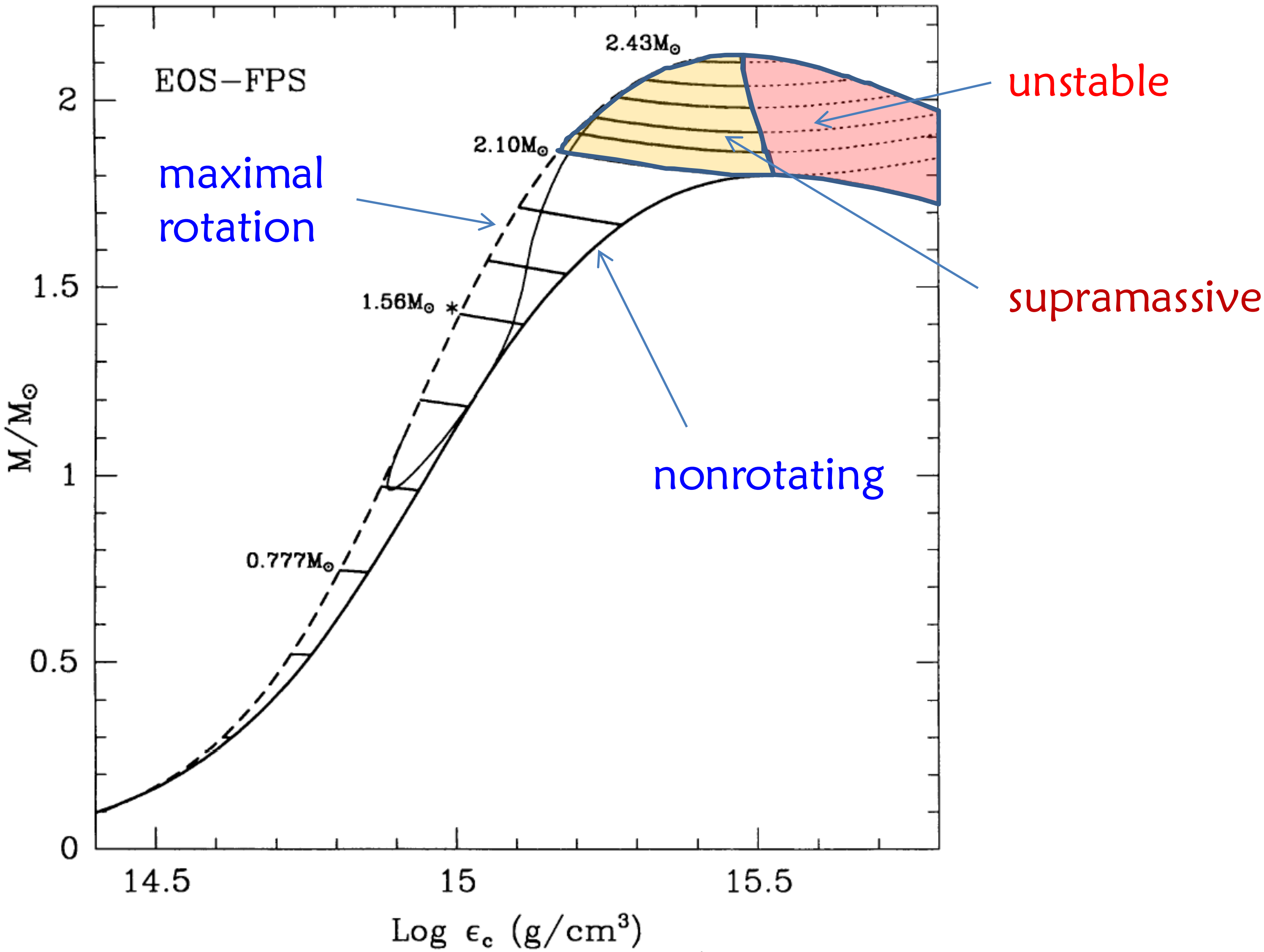
Goussard et al. 1998



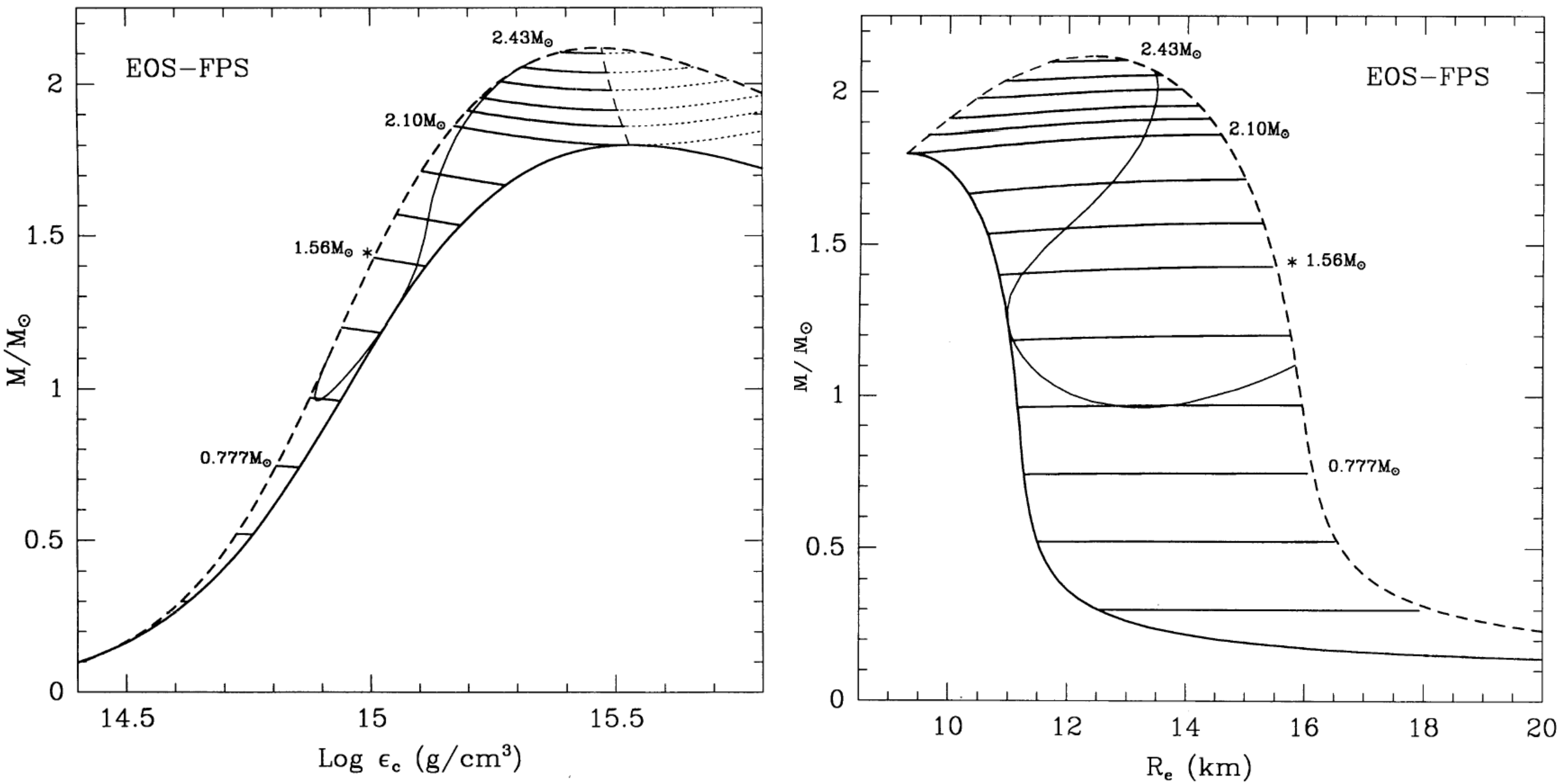
N. S., Apostolatos & Font, 2004

Equilibria of Rotating Stars

Uniformly rotating equilibrium models for a realistic neutron star equation of state.

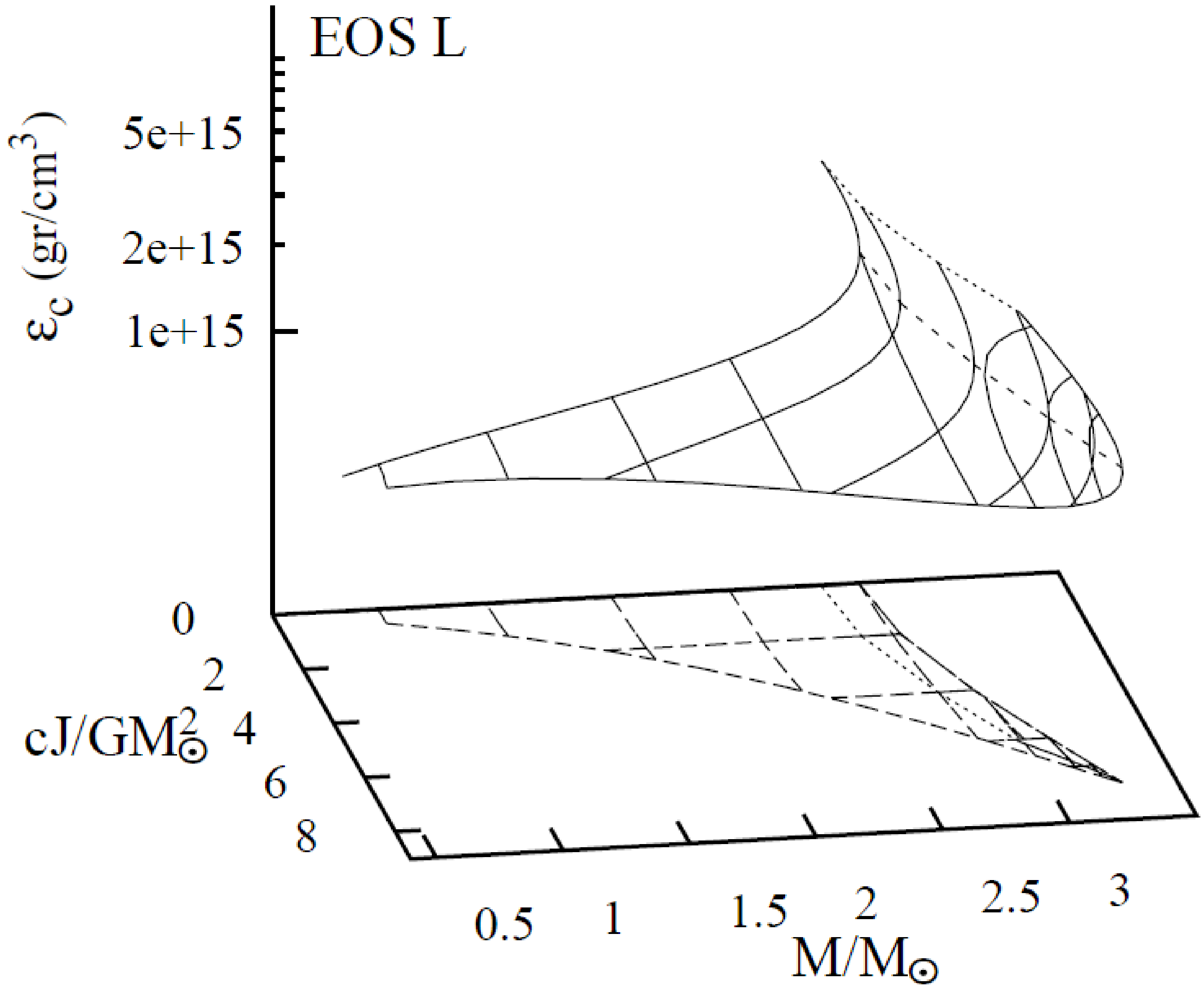


Equilibrium Sequences (I)



(Cook, Shapiro & Teukolsky 1994)

Equilibrium Sequences (II)



(N. S., Friedman 1995)

Comparison of Different Codes

N.S., Living Reviews in Relativity (2003)

	AKM	Lorene/ rotstar	SF (260 × 400)	SF (70 × 200)	BGSM	KEH
$\bar{p}_\mathrm{c}$	1.0					
$r_\mathrm{p}/r_\mathrm{e}$	0.7				1×10^{-3}	
$\bar{\Omega}$	1.41170848318	9×10^{-6}	3×10^{-4}	3×10^{-3}	1×10^{-2}	1×10^{-2}
$\bar{M}$	0.135798178809	2×10^{-4}	2×10^{-5}	2×10^{-3}	9×10^{-3}	2×10^{-2}
$\bar{M}_0$	0.186338658186	2×10^{-4}	2×10^{-4}	3×10^{-3}	1×10^{-2}	2×10^{-3}
$\bar{R}_\mathrm{circ}$	0.345476187602	5×10^{-5}	3×10^{-5}	5×10^{-4}	3×10^{-3}	1×10^{-3}
$\bar{J}$	0.0140585992949	2×10^{-5}	4×10^{-4}	5×10^{-4}	2×10^{-2}	2×10^{-2}
Z_p	1.70735395213	1×10^{-5}	4×10^{-5}	1×10^{-4}	2×10^{-2}	6×10^{-2}
Z^f_eq	−0.162534082217	2×10^{-4}	2×10^{-3}	2×10^{-2}	4×10^{-2}	2×10^{-2}
Z^b_eq	11.3539142587	7×10^{-6}	7×10^{-5}	1×10^{-3}	8×10^{-2}	2×10^{-1}
|GRV3|	4×10^{-13}	3×10^{-6}	3×10^{-5}	1×10^{-3}	4×10^{-3}	1×10^{-1}

AKM: Ansorg et al.

Lorene/rotstar + BGSM: Meudon group

SF: RNS code

KEH: original KEH code (not compactified)

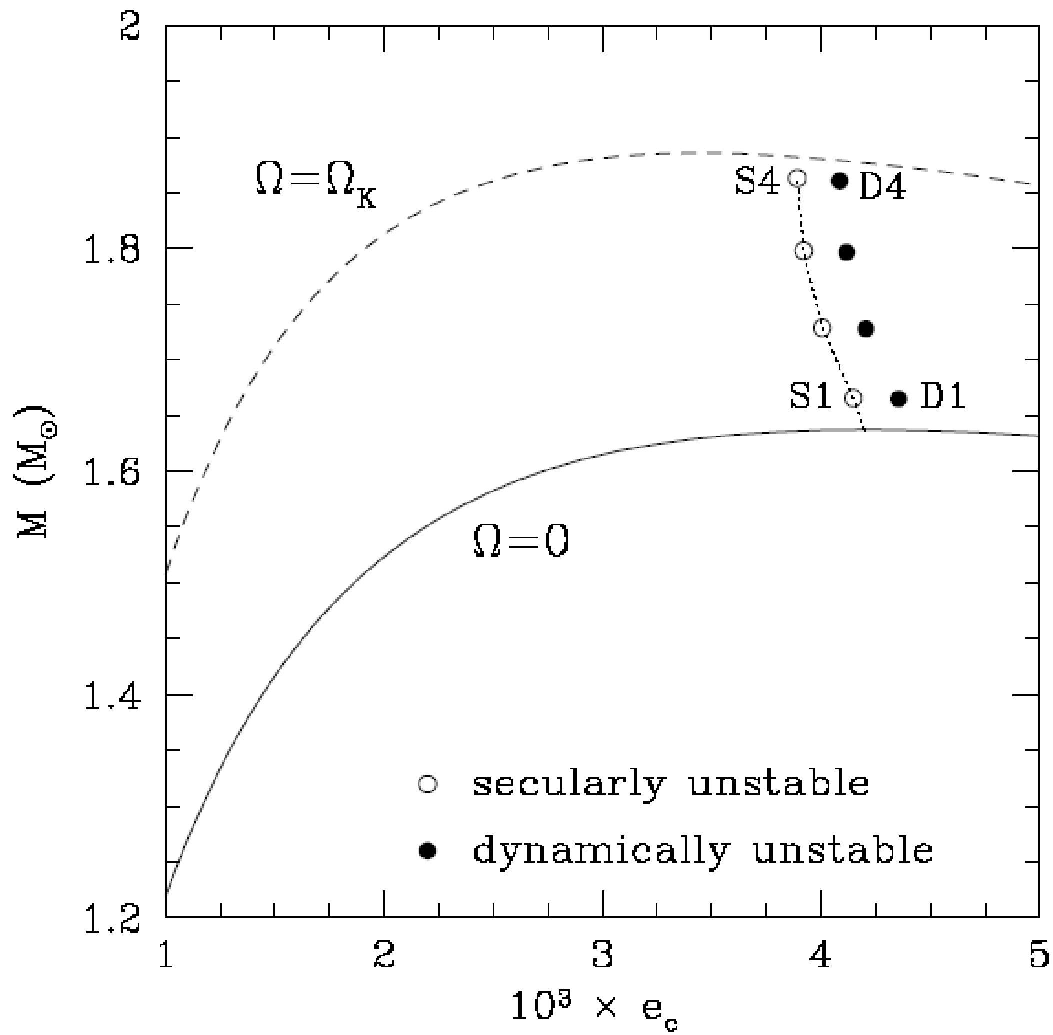
Axisymmetric Instability to Collapse

Rotating stars are subject to a ***secular*** axisymmetric instability, if:

$$\left(\frac{\partial M}{\partial \epsilon_c}\right)_J < 0$$

(Friedman, Ipser & Sorkin, 1988).

Dynamical instability soon after onset of secular instability.



DIFFERENTIAL ROTATION

Th specific angular momentum measured by proper time of matter is

$$j \equiv u^t u_\phi = j(\Omega)$$

Rotation Law:

$$\Omega = \Omega_C - \frac{(\Omega-\omega) r^2 \sin^2\theta \, e^{-2\rho}}{\mathrm{A}^2 \left[1-(\Omega-\omega)^2 r^2 \sin^2\theta \, e^{-2\rho}\right]}$$

Dimensionless constant:

$$\hat{\mathrm{A}} = \mathrm{A}/r_e$$

Limits:

$$\hat{\mathrm{A}}^{-1} \rightarrow \begin{cases} 0 & \text{uniform rotation} \\ \infty & j-\text{constant rotation law} \end{cases}$$

Specific angular momentum conserved during homologous collapse:

$$\tilde{j} \equiv u_\phi \left(\frac{\varepsilon + p}{\rho_0} \right)$$

Satisfies Rayleigh criterion for local dynamical stability to axisymmetric perturbations:

$$\frac{d\tilde{j}}{d\Omega} < 0$$

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