Polylogarithmic Matching at Spatial Infinity: *The massless Scalar Field*

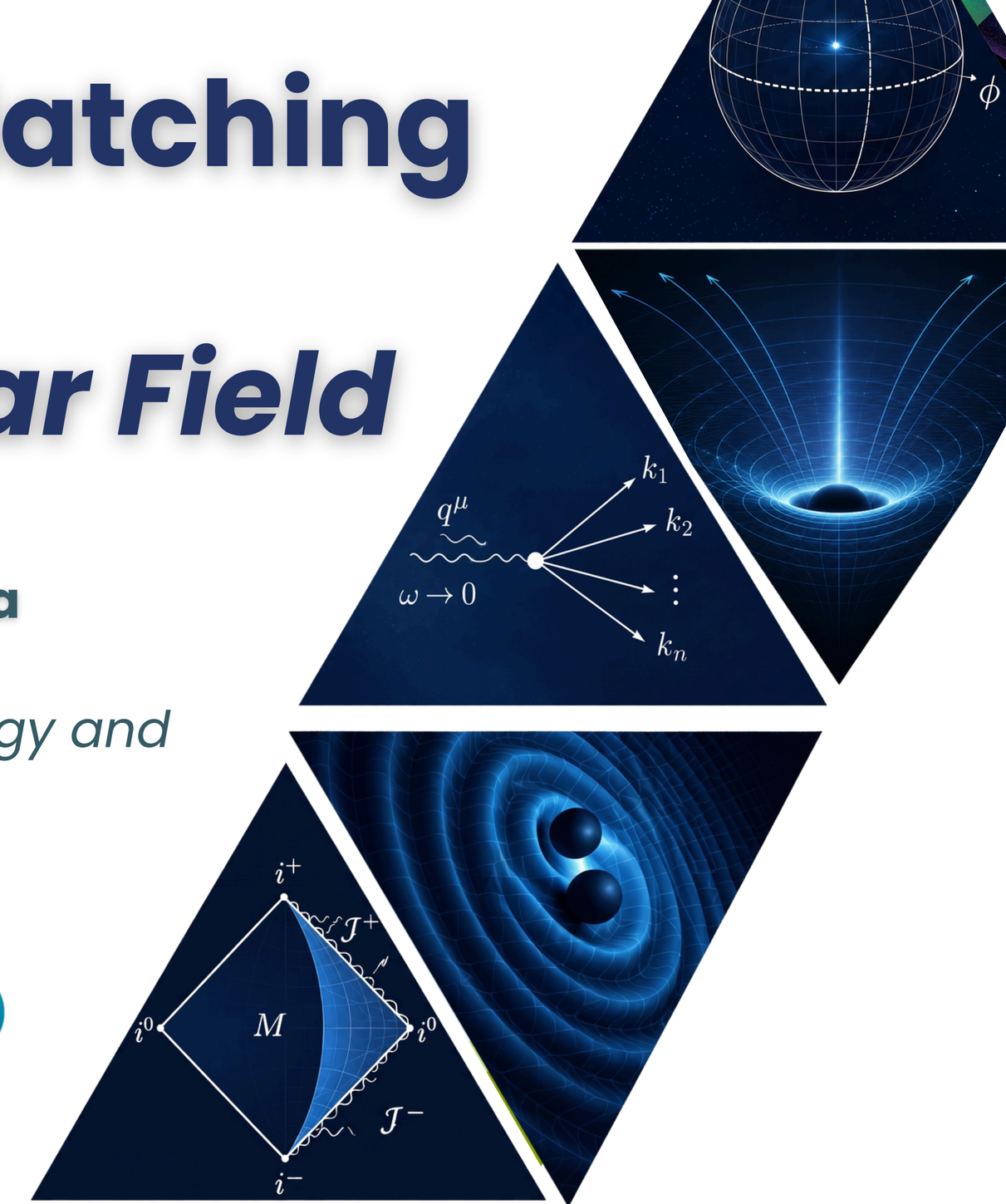
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UNIVERSIDAD ARTURO PRAT
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Overview

01 Motivation to study the infinity

02 The Asymptotic Structure of the Minkowski Spacetime

03 New Exploration with Asymptotic Conditions

04 Preliminary Results and Next Steps

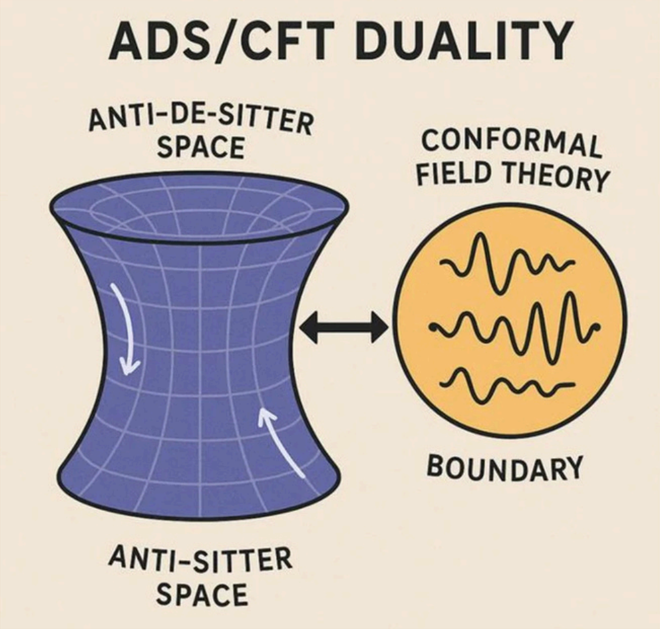
Boundaries hold more symmetries than expected

Why do boundaries matter?:

Beyond gauge redundancy: Boundary conditions can turn gauge transformations into nontrivial physical symmetries.

Boundary observables: Associated surface charges encode physical information and conservation laws.

Holography: Boundary symmetries provide powerful constraints on bulk gravitational physics.



Two landmark examples

Asymptotic symmetries of $\mathbf{AdS}_3 \rightarrow$ Conformal Symmetry in 2D

$$\mathfrak{so}(2,2) \quad \longrightarrow \quad \mathfrak{vir}_L \oplus \mathfrak{vir}_R$$

$$[L^{\pm}{}_m, L^{\pm}{}_n] = (m-n)L^{\pm}{}_{m+n} + \frac{c}{12}\, m(m^2-1)\,\delta_{m+n,0} \quad , \qquad c = \frac{3\ell}{2G}$$

Asymptotic symmetries at null infinity $\rightarrow$ Infinite dimensional $\mathbf{BMS}_4$

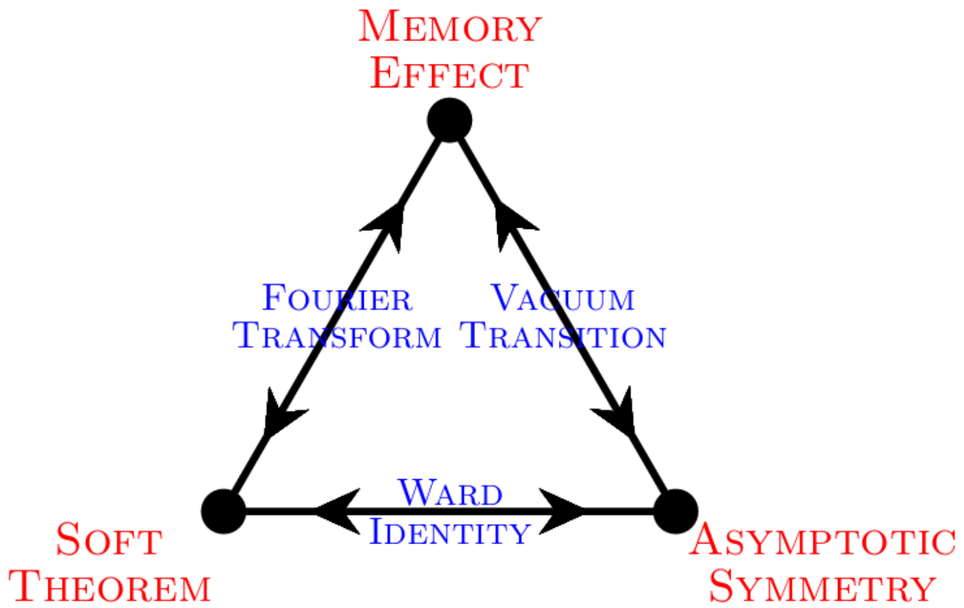
$$\mathfrak{so}(3,1) \ltimes \mathbb{R}^4 \quad \longrightarrow \quad \mathfrak{so}(3,1) \ltimes C^{\infty}(S^2)$$

$$[L^{\pm}_m, L^{\pm}_n] = (m-n)\, L^{\pm}_{m+n}$$

$$[L^{+}_m, T_{r,s}] = \left(\tfrac{m+1}{2} - r\right) T_{m+r,\,s}, \qquad [L^{-}_m, T_{r,s}] = \left(\tfrac{m+1}{2} - s\right) T_{r,\,m+s}$$

The infrared triangle

Modern results in infrared physics have taught us that **soft theorems**, **memory effects** and **asymptotic symmetries** are not three separate phenomena, but ***three manifestations of a single one.***



Memmory effect

Gravitational radiation produces a **permanent relative displacement between freely falling observers**

$$\Delta C_{AB} = \int_{u_i}^{u_f} du\, N_{AB} \qquad \Delta s^A = \frac{1}{2r} \Delta C^A{}_B\, s^B$$

Soft theorems

Universal factorization of scattering amplitudes when an emitted **massless particle becomes soft**.

Leading soft theorem

$$\lim_{\omega\to 0} \mathcal{A}_{n+1} = \left[\omega^{-1} S^{(0)}{}_n + \omega^0 S^{(1)}{}_n\right] \mathcal{A}_n + \mathcal{O}(\omega).$$

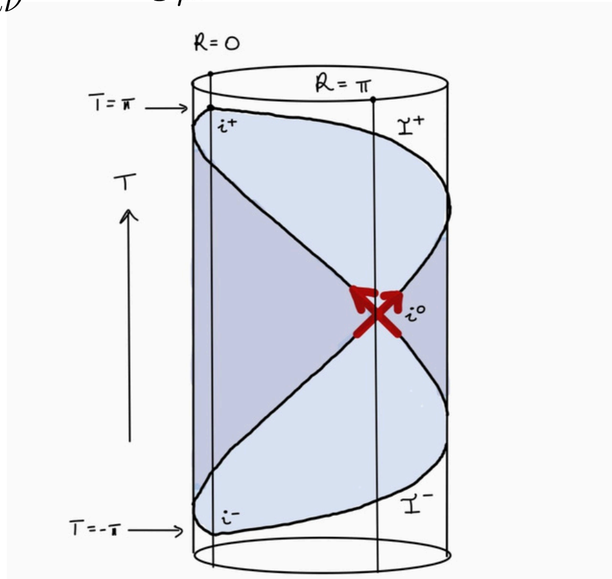
Subleading soft theorem

$$\lim_{\omega\to 0} \mathcal{A}_{n+1} = \left[\omega^{-1} S^{(0)}{}_n - \frac{k^2}{4} \ln\omega\, S^{(\ln)}{}_n\right] \mathcal{A}_n + \mathcal{O}(\omega^0).$$

The structure of the Infinity

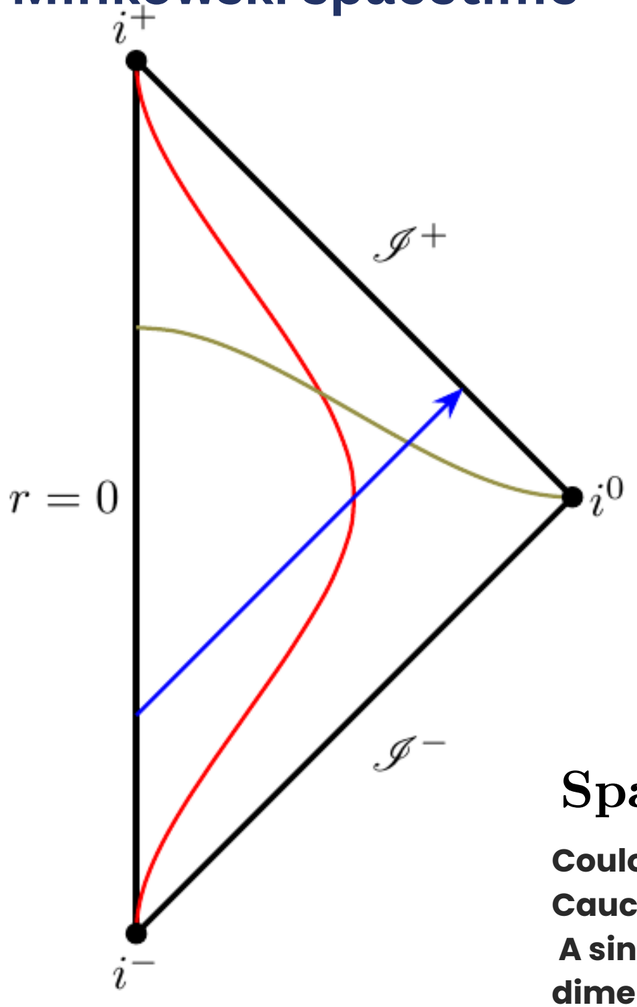
Penrose conformal compactification :

Rescale the physical metric by a conformal factor, preserving the causal structure while **bringing infinity to a finite boundary** $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$



- Null vectors are shared $g_{\mu\nu}k^\mu k^\nu = 0 \iff \tilde{g}_{\mu\nu}k^\mu k^\nu = 0$
- Light cones, hence causal relations between events, are unchanged
- Angles are preserved.

Penrose diagram of Minkowski spacetime



- ***Timelike geodesic***
- ***Nulll geodesic***
- ***Spacelike geodesic***

Timelike infinity $i^\pm$

Points. Blown up, hyperbolic space $\mathbb{H}_3$
Relevant only for massive particles

Null infinity $\mathcal{I}^\pm$

Where radiation escapes.
Every point of a line is a 2-sphere $\mathbb{R} \times S^2$

Spatial infinity i^0

Coulombic data, ADM-type charges.
Cauchy data end here.
A single point — blown up, a three-dimensional hyperboloid

The asymptotic structure of the scalar field

Why this theory ? Simplest arena to explore infrared structure: no gauge redundancy, no constraints, and yet a surprisingly rich asymptotic behaviour --- an unexpectedly instructive laboratory.

$$S = -\frac{1}{2}\int d^4x\, \partial_\mu\phi\, \partial^\mu\phi, \qquad \Box\phi = 0 \qquad \longleftrightarrow S_H[\phi,\pi] = \int dt\, d^3x \left[\pi\dot{\phi} - \frac{1}{2}\pi^2 - \frac{1}{2}\partial_i\phi\, \partial^i\phi\right]$$

Standard fall-off conditions

$$\phi = \frac{\bar{\phi}(t,x^A)}{r} + \frac{\phi^{(1)}(t,x^A)}{r^2} + \mathcal{O}(r^{-3}), \qquad \pi = \bar{\pi}(t,x^A) + \frac{\pi^{(1)}(t,x^A)}{r} + \mathcal{O}(r^{-2}).$$

What this reveals at the boundary?

Our proposal for asymptotic behaviour

We propose a relaxed fall-off conditions on the fields

$$\phi = \frac{\log r}{r}\,\phi_{\log}(t, x^A) + \frac{\bar{\phi}}{r} + \frac{\phi^{(1)}}{r^2} + \mathrm{o}(r^{-2})$$

$$\pi = \log r\;\pi_{\log}(t, x^A) + \bar{\pi} + \frac{\pi^{(1)}}{r} + \mathrm{o}(r^{-1})$$

Contains the standard treatment as the particular case $\phi_{\log} = \pi_{\log} = 0$
Higher powers define the general family of relaxations under study $\log^n r$

Whether they are physically admisible?

Fall-off invariance under the Poincaré group: $\delta_\xi\phi\,,\delta_\xi\pi$ do not leave the ansatz.

Finiteness of the symplectic form Ω

Poincaré invariance of the symplectic form $\mathcal{L}_\xi\Omega = 0$

Finite canonical generator: $i_\xi\Omega = -d_V Q_\xi$ with Q_ξ finite (and integrable)

This conditions pass the test?

Fall-off invariance

we act with the **Poincaré killing vectors** on the fields using the canonical transformations

$$\xi^t = br + T, \qquad \xi^r = W, \qquad \xi^A = Y^A + \frac{1}{r} D^A W.$$

with b boosts, Y^A rotations and T, W time and space translations. **Acting canonically on spin-0 fields**

$$\delta_\xi \phi = \frac{\xi^\perp \pi}{\sqrt{\bar{\gamma}}} + \xi^i \partial_i \phi, \qquad \delta_\xi \pi = \partial_i \left(\sqrt{\gamma}\, \xi^\perp \gamma^{ij} \partial_j \phi \right) + \partial_i (\xi^i \pi)$$

The logarithmic anzats is mapped into itself!

Poincaré invariance of the symplectic form

The radial integral produces **potentially divergent terms**.

$$\Omega = \int d^3x \;\; d_V \left(\log r\, \pi_{\log} + \bar{\pi} + \cdots \right) \wedge d_V \left(\frac{\log r}{r}\, \phi_{\log} + \frac{\bar{\phi}}{r} + \cdots \right)$$

The divergences cancel by assigning definite parities under the antipodal map of the sphere

$$\bar{\phi},\ \phi_{\log} \longrightarrow \text{even}, \qquad \bar{\pi},\ \pi_{\log} \longrightarrow \text{odd}$$

Finite symplectic form

Using **the Cartán formula** for differential forms Lie derivatives

$$\mathcal{L}_\xi \Omega = \left[d_V, i_\xi \right] \Omega$$

We obtain a boundary term directly

$$\mathcal{L}_\xi \Omega = \oint d^2 s_i \; r \left(\sqrt{\gamma}\, \xi^\perp \gamma^{ij} \partial_j d_V \phi \right) \wedge d_V \phi + \oint d^2 s_i \; \xi^i \, d_V \pi \wedge d_V \phi$$

Both surface integrals vanish once the relaxed fall-offs and the parity conditions are imposed.

Finite canonical generator

$$i_\xi \Omega = -d_V \int d^3x \;\; \left[\xi^\perp \left(\frac{\pi^2}{2\sqrt{\gamma}} + \frac{1}{2} \sqrt{\gamma}\, \gamma^{ij} \partial_i \phi\, \partial_j \phi \right) + \xi^i \pi\, \partial_i \phi \right]$$

Regge-Teitelboim structure

$$Q_{\xi,\xi^i} = \int d^3x \left(\xi\, \mathcal{H}_\perp + \xi^i \mathcal{H}_i \right) + B_{\xi,\xi^i}$$

The surface term carries the physical charge; it is finite for the logarithmic ansatz!

How to compute matching conditions

To connect with radiation we must evolve the initial data: solve $\Box\phi = 0$ towards $\mathcal{I}$.
In hyperbolic coordinates $ds^2 = d\eta^2 + \eta^2 h_{ab} dx^a dx^b$ the Klein Gordon equation becomes

$$D^a D_a \phi + \eta^{-1} \partial_\eta \left(\eta^3 \partial_\eta \phi \right) = 0$$

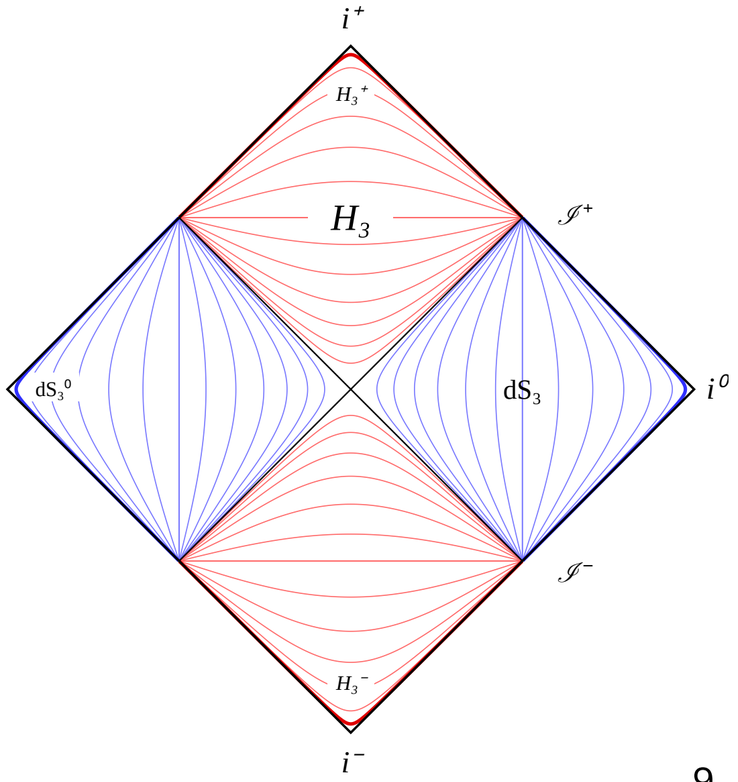
Why this coordinates?

The hyperbolic foliation of the Minkowski background allows us both **making a direct contact with the spacelike Cauchy hypersurface** $t = 0$ and **"touching" the so-called critical sets** at spatial infinity.

The procedure in simple steps (after the spatial infinity)

- Pass to **Friedrich coordinates**, which interpolate between the hyperboloid and $\mathcal{I}$:
- Express ϕ (and its log branches) there
- Construct the conserved quantities at null infinity
- Derive the **matching conditions**, relating the charges at i^0 with those at $\mathcal{I}^{\pm}$

Penrose diagram of hyperbolic slicing of Minkowski space. The slices correspond to surfaces of constant $t^2 - r^2$. The slices in the past and future lightcones of the origin have the geometry of $\mathbb{H}_3$, and the slices with spacelike separation from the origin have the geometry of dS_3



Now we can start computing

Ultraspherical equations

We can express also our logarithmic anzats in hyperbolic coordinates

$$\phi(\eta, x^a) = \sum_{k=0}^{\infty} \frac{1}{\eta^{1+k}} \left[\varphi^{(k)}(x^a) + \log\eta\, \varphi^{(k)}{}_{\log}(x^a)\right]$$

Then, the Klein-Gordon equation will take the form

$$(1-s^2)\partial^2{}_s\varphi^{(k)}_{\log} - \overline{D}^A\overline{D}_A\varphi^{(k)}_{\log} - \frac{(k^2-1)}{1-s^2}\varphi^{(k)}_{\log} = 0\,,$$

$$(1-s^2)\partial^2{}_s\varphi^{(k)} - \overline{D}^A\overline{D}_A\varphi^{(k)} - \frac{(k^2-1)}{1-s^2}\varphi^{(k)} = -\frac{2k}{1-s^2}\varphi^{(k)}_{\log}$$

From the homogeneous equation, **k=0** reduces to Legendere equation! **The general case** for any value of **k** is given by

$$\varphi^{(k)}{}_{\log}(s, \bar{x}^A) = \varphi^{P(k)}{}_{\log}(s, \bar{x}^A) + \varphi^{Q(k)}{}_{\log}(s, \bar{x}^A)$$

Where each term is associated to a **Gegenbauer's polynomial** or **functions of the second kind**

$$\varphi^{P(k)}{}_{\log}(s, \bar{x}^A) = (1-s^2)^{\frac{1-k}{2}} \sum_{l,m} \Phi^{P(k)}{}_{lm} \tilde{P}^{k+\frac{1}{2}}{}_{l-k}(s) Y_{lm}(\bar{x}^A)\,,$$

$$\varphi^{Q(k)}{}_{\log}(s, \bar{x}^A) = (1-s^2)^{\frac{1-k}{2}} \sum_{l,m} \Phi^{Q(k)}{}_{lm} \tilde{Q}^{k+\frac{1}{2}}{}_{l-k}(s) Y_{lm}(\bar{x}^A)$$

The homogeneous case matches the standard fall-off structure

To see the behaviour at null infinity (near $\mathcal{I}^+$) we use Bondi coordinates also $s = 1 + \frac{u}{r}$. For the P-branch

$$\phi_P = \frac{\Phi(u, x^A)}{r} + \mathcal{O}\left(\frac{1}{r^2}\right), \qquad \lim_{u\to-\infty} \Phi(u, x^A) = \sum_{l,m} \Theta^{P(0)}{}_{lm} Y_{lm}(x^A)$$

And for the Q-branch

$$\phi_Q = \frac{1}{2}\frac{\log r}{r}\Psi(x^A) + \mathcal{O}\left(\frac{1}{r}\right), \qquad \Psi(x^A) = \sum_{l,m} \Theta^{Q(0)}{}_{lm} Y_{lm}(x^A)$$

The dominant $\frac{\log r}{r}$ term originates exclusively from the **K=0** mode of the **Q-branch**, even though **the initial data contain no logarithm at all.**

The previous known matching conditions

$$\phi|_{\mathcal{I}^+} = \frac{\Phi(u, x^A)}{r} + \mathcal{O}\left(\frac{\log r}{r^2}\right), \qquad \phi|_{\mathcal{I}^-} = \frac{\Phi'(v, x^A)}{r} + \mathcal{O}\left(\frac{\log r}{r^2}\right)$$

$$\lim_{v\to\infty} \Phi'(v, x^A) = \sum_{l,m} \Theta^{P(0)}{}_{lm} P_l(-1) Y_{lm}(x^A) = \Phi(-x^A).$$

$$\boxed{\lim_{v\to\infty} \Phi'(v, x^A) = \lim_{u\to-\infty} \Phi(u, -x^A)}$$

The sourced equation

Our new asympotic conditions on the fields left us with the previous asymptotic physics... but also with something new

$$(1-s^2)\partial^2{}_s\varphi^{(k)} - \bar{D}^A\bar{D}_A\varphi^{(k)} - \frac{(k^2-1)}{1-s^2}\varphi^{(k)} = -\frac{2k}{1-s^2}\varphi^{(k)}_{\log}$$

The previous solution acts as a source of the new equation!

Well, the solution will depend on the value of **k**.
For **K=1**

$$\rho^{(1)}{}_{lm}(s) = \log\left(\frac{1+s}{1-s}\right)\sum_{n=0}^{l-1} f_{[n],lm}s^n + \sum_{n=0}^{l-2} g_{[n],lm}s^n + \sum_{n=0}^{l-1}\rho^{P(1)}{}_{[n],lm}s^n$$

This solution consider the P-branch and Q-branch as sources, so it is natural the presence of logarithmic terms here.

There is a general solution?

For **K>1**, the solution depends on the value of **(l-k).**

Why? we take a particular form as anazts, and rewrite the differential operator and source in terms of this new constants. With the anzats for the particular solution

$$\varphi^{(k)}{}_p(s,\bar{x}^A) = (1-s^2)^{\frac{1-k}{2}}\sum_{l,m}\rho^{(k)}{}_{lm}(s)Y_{lm}(\bar{x}^A)$$

The equation takes a new form (a little bit more easy for solving it)

$$(1-s^2)\partial^2{}_s\rho^{(k)}{}_{lm} + 2s(k-1)\partial_s\rho^{(k)}{}_{lm} + [l(l+1)-k(k-1)]\rho^{(k)}{}_{lm}$$
$$= -\frac{2k}{1-s^2}\left[\Phi^{P(k)}{}_{lm}\tilde{P}^{(k+\frac{1}{2})}{}_{l-k} + \Phi^{Q(k)}{}_{lm}\tilde{Q}^{(k+\frac{1}{2})}{}_{l-k}\right]$$

The solution when (l-k) <0

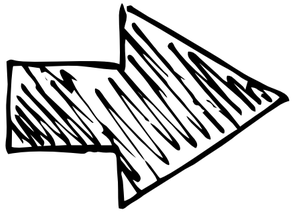
$$\rho^{P(k)}{}_{lm} = -\log\left(\frac{1+s}{1-s}\right)\frac{a_{(k+\frac{1}{2})}\Phi^{P(k)}{}_{lm}}{2}\tilde{Q}^{(k+\frac{1}{2})}{}_{l-k}(s) + \sum_{j=0}^{3k-l-4} g^{P(k)}{}_{[j],lm}s^j\,,$$

$$\rho^{Q(k)}{}_{lm} = -\log\left(\frac{1+s}{1-s}\right)\frac{\Phi^{Q(k)}{}_{lm}}{2a_{(k+\frac{1}{2})}}\tilde{P}^{(k+\frac{1}{2})}{}_{l-k}(s) + \sum_{j=0}^{3k-l-3} g^{Q(k)}{}_{[j],lm}s^j$$

Conclusions and Future Work

The remaining solution has something new!

When $(l-k) \geq 0$ there is precense of terms $\log^2(1-s)$ and polylogarithms!



This type of terms where knew only at null infinity before, but tracking the origin as initial data from the Cauchy surface is something powerfull

The work is more close with this solution, but still we need to compute:

- **Behaviour of the solutions near the critical sets.**
- **Compute the asymptotic charges at null infinity.**
- **Stablish the matching conditions with this new branches of the solution. We will connect with the tems that we have seen in null infinity before?**

As a paralell work:

The same idea in the **electromagnetic gauge theory** may work, but the gauge redundancy present there make the first tests more dificult to close.

Gravity?
Soft theorems?

Thanks!
Comments?