A Snyder-inspired Effective Schwarzschild Geometry

A fixed-parameter model at order ℓ^2

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Motivation and objective

Question

What geometric and geodesic properties follow from a Snyder-inspired Schwarzschild ansatz at fixed parameter q?

- Motivate the ansatz through a spatial Snyder realization.
- Obtain the effective potential for test-particle geodesics.
- Study curvature with Cartan's torsion-free formalism.

Scope: a preliminary effective model at order ℓ^2. The relation to full Snyder particle dynamics remains under investigation.

Spatial realization and the fixed parameter

$$x'^i = x^i + \ell^2(\mathbf{x}\cdot\mathbf{p})p^i, \qquad p'_i = p_i, \quad t' = t, \quad i = 1,2,3. \tag{1}$$

The spatial contraction motivates the parameter q:

$$\mathbf{x}\cdot\mathbf{p} = rp_r \quad \longmapsto \quad q \text{ (fixed parameter)}. \tag{2}$$

Effective-model prescription

q labels a family of spacetime metrics. It is held fixed in both the geometric and geodesic calculations; it is not identified with the evolving momentum of the test particle.

Fixing q is a modeling assumption, not a derived conservation law.
Units: $c = \hbar = 1$; Euclidean spatial contraction with δ_{ij}.
Snyder (1947); realization approach: Battisti–Meljanac (2010) [References].

A family of effective Schwarzschild metrics

$$F_q(r) = 1 - \frac{2GM}{r} + \frac{2GM\ell^2 q^2}{r^3}, \qquad B_q(r) = r^2 + 2\ell^2 q^2. \tag{3}$$

Effective ansatz

$$ds_q^2 = -F_q\, dt^2 + F_q^{-1} dr^2 + B_q\, d\Omega^2. \tag{4}$$

- The radial expression $\sqrt{r^2 + 2\ell^2 q^2}$ motivates the coefficients.
- The coordinate differentials remain unchanged by prescription.
- This is an ansatz, not a full coordinate transformation of Schwarzschild.

Keep terms through ℓ^2, with $\ell^2 q^2/r^2 \ll 1$; the inverse-metric expansion needs an additional condition near $2GM$ [B3].
Schwarzschild: $q = 0$ or $\ell \to 0$. Motivation: Leiva [References]; the fixed-q and angular-sector prescriptions here are model assumptions.

Geodesics and conserved quantities at fixed q

For each metric in the family, take $\theta = \pi/2$ and $\dot{\theta} = 0$.

$$\mathcal{L} = \frac{1}{2}\left(-F_q \dot{t}^2 + F_q^{-1}\dot{r}^2 + B_q \dot{\phi}^2\right). \tag{5}$$

Time translation and axial symmetry give

$$E = F_q \dot{t}, \qquad L = B_q \dot{\phi}. \tag{6}$$

$$g^{(q)}_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = -k, \qquad k = \begin{cases} 0, & \text{null geodesics,} \\ 1, & \text{timelike geodesics.} \end{cases} \tag{7}$$

Dots denote an affine parameter (proper time for $k = 1$). For timelike motion, E, L are specific conserved quantities.
Geodesic method: Tong, Ch. 1 [References].

Effective potential at fixed q

Substituting $\dot{t} = E/F_q$ and $\dot{\phi} = L/B_q$ into the normalization gives

$$\begin{aligned} \dot{r}^2 &= E^2 - V_{\text{eff}}(r;q), \\ V_{\text{eff}}(r;q) &= F_q(r)\left(k + \frac{L^2}{B_q(r)}\right). \end{aligned} \tag{8}$$

Writing $f_0 = 1 - 2GM/r$, the consistent expansion is

$$\begin{aligned} V_{\text{eff}}(r;q) = f_0 &\left(k + \frac{L^2}{r^2}\right) \\ &+ \ell^2 q^2 \left[\frac{2GM}{r^3}\left(k + \frac{L^2}{r^2}\right) - \frac{2f_0 L^2}{r^4}\right] + \mathcal{O}(\ell^4). \end{aligned} \tag{9}$$

The undeformed limit is the Schwarzschild effective potential.

Schwarzschild reference and orbital analysis

Schwarzschild reference ($q = 0$)

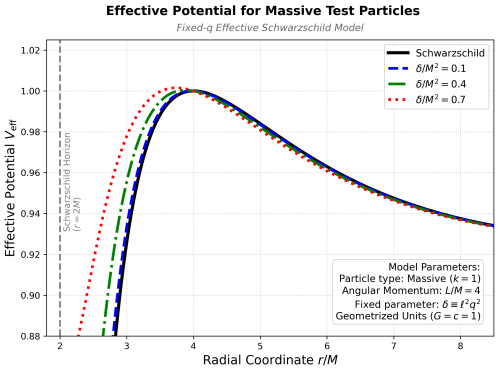
$$r_{\rm ph} = 3GM, \qquad r_{\rm ISCO} = 6GM. \tag{10}$$

For $q \neq 0$, circular geodesics require

$$E^2 = V_{\rm eff}, \qquad V'_{\rm eff} = 0. \tag{11}$$

with derivatives at fixed q, L. A minimum has $V''_{\rm eff} > 0$; physical conclusions also require perturbative validity.

Circular motion does not imply $q = 0$.



$k = 1$, $L/M = 4$, $G = c = 1$; $\delta = \ell^2 q^2$. Parameter values are illustrative and given in the legend.

Near-horizon behavior requires a separate validity analysis [B3].

Cartan formulation at fixed q

In the static region $F_q > 0$, choose

$$e^{\hat{t}} = \sqrt{F_q}\, dt, \quad e^{\hat{r}} = F_q^{-1/2} dr, \\ e^{\hat{\theta}} = \sqrt{B_q}\, d\theta, \quad e^{\hat{\phi}} = \sqrt{B_q} \sin\theta\, d\phi. \tag{12}$$

$$T^a = de^a + \omega^a{}_b \wedge e^b = 0, \\ \mathcal{R}^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b. \tag{13}$$

Differentiation prescription

All spacetime derivatives are taken at fixed q ($dq = 0$). Signature $\eta_{ab} = \mathrm{diag}(-1, 1, 1, 1)$; $\omega_{ab} = -\omega_{ba}$, hence $\omega^{\hat{t}}{}_{\hat{i}} = \omega^{\hat{i}}{}_{\hat{t}}$.

Connection and curvature forms: Tong, Ch. 3 [References].

Curvature at first order

At fixed q, with $R_{bd} = R^{a}{}_{bad}$ and signature $(-+++)$,

$$R = -\frac{4\ell^2 q^2}{r^4}\left(1 - \frac{3GM}{r}\right) + \mathcal{O}(\ell^4), \tag{14}$$

$$G_{\hat{t}\hat{t}} = -\frac{2\ell^2 q^2}{r^4}\left(1 - \frac{4GM}{r}\right) + \mathcal{O}(\ell^4). \tag{15}$$

Geometric interpretation

The Schwarzschild limit has $R_{ab} = 0$. For nonzero $\ell^2 q^2$, the effective family is generally nonvacuum in Einstein gravity.

Full Ricci and Einstein tensors are given in B9 and B8. The source interpretation remains open.

Formal effective source

With $\Lambda = 0$, define the source required by the chosen metric:

$$T^{\text{eff}}_{\hat{a}\hat{b}} := \frac{G_{\hat{a}\hat{b}}}{8\pi G}, \qquad \delta = \ell^2 q^2. \tag{16}$$

For static orthonormal observers ($F_q > 0$), the nonzero components are

$$\begin{aligned} \rho_{\text{eff}} &= T^{\text{eff}}_{\hat{t}\hat{t}} = -\frac{\delta}{4\pi G r^4}\left(1 - \frac{4GM}{r}\right), \\ p_r^{\text{eff}} &= T^{\text{eff}}_{\hat{r}\hat{r}} = -\frac{\delta}{4\pi G r^4}, \\ p_\perp^{\text{eff}} &= T^{\text{eff}}_{\hat{\theta}\hat{\theta}} = T^{\text{eff}}_{\hat{\phi}\hat{\phi}} = \frac{\delta}{4\pi G r^4}\left(1 - \frac{GM}{r}\right). \end{aligned} \tag{17}$$

Terms of order ℓ^4 are omitted; all off-diagonal components vanish. Here p_r^{eff} denotes radial *pressure*, not particle momentum.

Interpretation

An effective anisotropic source: generally $p_r^{\text{eff}} \neq p_\perp^{\text{eff}}$. This identifies the required stress tensor; a microscopic matter model remains unspecified.

Scope and next steps

Present scope

- A fixed-q effective ansatz and its geodesic potential.
- A torsion-free Cartan calculation through order ℓ^2.

Work in progress

- Specify an action and symplectic structure for dynamical $\mathbf{x} \cdot \mathbf{p} = r p_r$.
- Study orbital stability and precession within the validity range.

Dynamical comparison: Mignemi–Štrajn (2014) [References].

Conclusions

1. The proposed family contains corrections controlled by $\ell^2 q^2$ and recovers Schwarzschild in the undeformed limit.
2. At fixed q, ordinary geodesic dynamics yields a well-defined effective potential within the perturbative regime.
3. Cartan's formalism gives a generally nonzero Ricci tensor at order ℓ^2. The physical source interpretation and the connection to dynamical Snyder particles remain open.

Thank you for your attention

Questions

Selected references

1. H. S. Snyder, "Quantized Space-Time," *Phys. Rev.* **71**, 38–41 (1947). doi:10.1103/PhysRev.71.38.
2. M. V. Battisti and S. Meljanac, "Scalar Field Theory on Non-commutative Snyder Space-Time," *Phys. Rev. D* **82**, 024028 (2010). arXiv:1003.2108.
3. C. Leiva, "Some features of a Schwarzchild Black Hole from a Snyder perspective," arXiv:1607.05972v2 (2017).
4. S. Mignemi and R. Štrajn, "Snyder dynamics in a Schwarzschild spacetime," *Phys. Rev. D* **90**, 044019 (2014). arXiv:1404.6396.
5. D. Tong, *General Relativity*, lecture notes, Chs. 1 and 3 (geodesics; connection and curvature forms). Author's lecture notes.

These references motivate the framework and methods; they do not uniquely derive the fixed-q ansatz used here.

B1. Spatial Snyder realization

Start with canonical auxiliary variables:

$$\{x^i, p_j\} = \delta^i{}_j, \qquad \{x^i, x^j\} = \{p_i, p_j\} = 0. \tag{B1}$$

$$x'^i = x^i + \ell^2(\mathbf{x}\cdot\mathbf{p})p^i, \qquad p'_i = p_i. \tag{B2}$$

The resulting spatial brackets are

$$\{x'^i, p'_j\} = \delta^i{}_j + \ell^2 p^i p_j, \qquad \{x'^i, x'^j\} = \ell^2(x^i p^j - x^j p^i). \tag{B3}$$

Euclidean contractions use δ_{ij}. This realization motivates, but does not uniquely determine, the effective ansatz.

B2. Fixed parameter versus dynamical contraction

In the original auxiliary phase space,

$$r = |\mathbf{x}|, \qquad p_r = \frac{\mathbf{x}\cdot\mathbf{p}}{r}, \qquad \mathbf{x}\cdot\mathbf{p} = r p_r. \tag{B4}$$

Along a trajectory,

$$\frac{d}{d\tau}(r p_r) = \dot{r}\, p_r + r\dot{p}_r. \tag{B5}$$

- Holding q fixed labels a metric in the effective family.
- It does not prove that $r p_r$ is conserved for a Snyder particle.
- Holding q fixed differs from holding p_r fixed.

B3. Perturbative regime and a formal root

$$f_0 = 1 - \frac{2GM}{r}, \qquad \frac{\ell^2 q^2}{r^2} \ll 1, \qquad \left| \frac{2GM\ell^2 q^2}{r^3 f_0} \right| \ll 1. \tag{B6}$$

The last condition controls the expansion of F_q^{-1} away from $f_0 = 0$. The formal root near $2GM$ of the truncated equation $F_q(r) = 0$ is

$$r_* = 2GM - \frac{\ell^2 q^2}{2GM} + \mathcal{O}(\ell^4). \tag{B7}$$

Interpretation

This algebraic root alone does not establish an event horizon. The expansion is nonuniform near $f_0 = 0$; a separate causal and validity analysis is required.

B4. Radial equation at fixed q

$$\mathcal{L} = \frac{1}{2}\left(-F_q \dot{t}^2 + F_q^{-1}\dot{r}^2 + B_q \dot{\phi}^2\right), \tag{B8}$$

$$E = F_q \dot{t}, \qquad L = B_q \dot{\phi}.$$

$$-\frac{E^2}{F_q} + \frac{\dot{r}^2}{F_q} + \frac{L^2}{B_q} = -k. \tag{B9}$$

Multiplying by F_q yields

$$\boxed{\dot{r}^2 = E^2 - F_q\left(k + \frac{L^2}{B_q}\right).} \tag{B10}$$

q is an external parameter of this geodesic calculation.

B5. Connection at fixed q

For the stated tetrads, a connection component is

$$\omega^{\hat{t}}{}_{\hat{r}} = \frac{1}{2}\frac{dF_q}{dr}\,dt = \frac{GM}{r^2}\left(1 - \frac{3\ell^2 q^2}{r^2}\right)dt. \tag{B11}$$

$$\left.\frac{dF_q}{dr}\right|_q = \frac{2GM}{r^2} - \frac{6GM\ell^2 q^2}{r^4}. \tag{B12}$$

$$\omega_{ab} = -\omega_{ba}, \qquad \omega^{\hat{t}}{}_{\hat{r}} = \omega^{\hat{r}}{}_{\hat{t}}. \tag{B13}$$

Lowering the temporal index introduces a minus sign. All derivatives have $dq = 0$.

B6. Radial curvature component

With $\mathcal{R}^{a}{}_{b} = d\omega^{a}{}_{b} + \omega^{a}{}_{c} \wedge \omega^{c}{}_{b}$,

$$\mathcal{R}^{\hat{t}}{}_{\hat{r}} = \frac{1}{2} F_q'' \, dr \wedge dt = -\frac{2GM}{r^3} \left(1 - \frac{6\ell^2 q^2}{r^2} \right) dr \wedge dt. \tag{B14}$$

The wedge ordering is $dr \wedge dt$; reversing it reverses the sign.

B7. Transverse curvature

Writing $f_0 = 1 - 2GM/r$, the coordinate two-form is

$$\mathcal{R}^{\hat{t}}{}_{\hat{\theta}} = -\frac{GM}{r^2}\left[\sqrt{f_0} - \frac{4\ell^2 q^2}{r^2\sqrt{f_0}}\left(1 - \frac{9GM}{4r}\right)\right] dt \wedge d\theta + \mathcal{O}(\ell^4). \tag{B15}$$

In the orthonormal coframe this simplifies to

$$\mathcal{R}^{\hat{t}}{}_{\hat{\theta}} = -\frac{GM}{r^3}\left(1 - \frac{5\ell^2 q^2}{r^2}\right) e^{\hat{t}} \wedge e^{\hat{\theta}} + \mathcal{O}(\ell^4). \tag{B16}$$

$$\mathcal{R}^{\hat{\theta}}{}_{\hat{t}} = \mathcal{R}^{\hat{t}}{}_{\hat{\theta}}. \tag{B17}$$

The $\hat{\phi}$ component has the same orthonormal coefficient. The undeformed coordinate prefactor is $-GM/r^2$.

B8. Einstein tensor at fixed q

With $\delta = \ell^2 q^2$ and both frame indices lowered,

$$G_{\hat{t}\hat{t}} = -\frac{2\delta}{r^4}\left(1 - \frac{4GM}{r}\right),$$
$$G_{\hat{r}\hat{r}} = -\frac{2\delta}{r^4}, \tag{B18}$$
$$G_{\hat{\theta}\hat{\theta}} = G_{\hat{\phi}\hat{\phi}} = \frac{2\delta}{r^4}\left(1 - \frac{GM}{r}\right).$$

All expressions omit $\mathcal{O}(\ell^4)$ at fixed q. The trace satisfies $\eta^{ab} G_{ab} = -R$.

The formal source is $T^{\rm eff}_{ab} = G_{ab}/(8\pi G)$. These components describe the chosen effective ansatz.

B9. Ricci tensor and sign conventions

Let $A = 2GM$, $\delta = \ell^2 q^2$ and $d\delta = 0$.

$$R_{\hat{t}\hat{t}} = \frac{A\delta}{r^5}, \qquad R_{\hat{r}\hat{r}} = -\frac{4\delta}{r^4} + \frac{3A\delta}{r^5},$$
$$R_{\hat{\theta}\hat{\theta}} = R_{\hat{\phi}\hat{\phi}} = \frac{2A\delta}{r^5}, \qquad R = -\frac{4\delta}{r^4} + \frac{6A\delta}{r^5}. \tag{B19}$$

$$\mathcal{R}^a{}_b = \frac{1}{2} R^a{}_{bcd}\, e^c \wedge e^d, \quad R_{bd} = R^a{}_{bad}, \quad G_{ab} = R_{ab} - \frac{1}{2}\eta_{ab} R. \tag{B20}$$

Lorentzian index convention

$\mathcal{R}_{ab} = -\mathcal{R}_{ba}$, but $\mathcal{R}^{\hat{i}}{}_{\hat{t}} = \mathcal{R}^{\hat{t}}{}_{\hat{i}}$. The antisymmetry applies to the two lowered indices.

Terms beyond order ℓ^2 are omitted.