

Gauge-invariant non-universal scalar couplings in teleparallel gravity

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Outline

1. Teleparallel gravity
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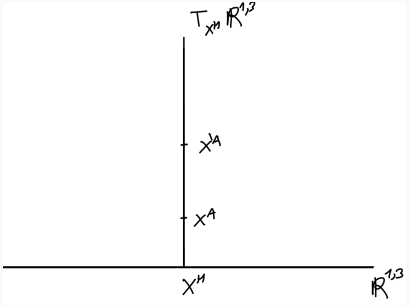
Teleparallel gravity

Geometrical setting: the tangent bundle

- Spacetime coordinates $\{x^\mu\}$; at each point, a tangent Minkowski space (the fiber) with coordinates $\{x^A\}$ and metric $\eta_{AB} = \mathrm{diag}(-1, 1, 1, 1)$.
- Bases $\{\partial_\mu\}$, $\{\partial_A\}$ and dual bases $\{dx^\mu\}$, $\{dx^A\}$, with $dx^\mu(\partial_\nu) = \delta^\mu_\nu$, $dx^A(\partial_B) = \delta^A_B$.
- Teleparallel gravity: a gauge theory for the translation group acting on the fibers,

$$x^A \;\rightarrow\; x'^A = x^A + \epsilon^A(x^\mu), \qquad (1)$$

a point-dependent translation.



Trivial and nontrivial tetrads

Trivial tetrads (frames of special relativity): $e_A = e_A{}^\mu \partial_\mu$, $e^A = e^A{}_\mu dx^\mu$, with $e^A{}_\mu e_A{}^\nu = \delta^\nu_\mu$, $e^A{}_\mu e_B{}^\mu = \delta^A_B$

$$[e_A, e_B] = f^C{}_{AB}\, e_C, \qquad f^C{}_{AB} = e_A{}^\mu e_B{}^\nu \left(\partial_\nu e^C{}_\mu - \partial_\mu e^C{}_\nu\right), \qquad \eta_{\mu\nu} = \eta_{AB}\, e^A{}_\mu e^B{}_\nu \tag{2}$$

- Inertial frames are holonomic: $f^C{}_{AB} = 0$.

Nontrivial tetrads $h^A{}_\mu$ (presence of gravitation):

$$[h_A, h_B] = f^C{}_{AB}\, h_C, \qquad g_{\mu\nu} = \eta_{AB}\, h^A{}_\mu h^B{}_\nu \tag{3}$$

- The anholonomy $f^C{}_{AB}$ now encodes both inertial effects and gravitation.
- The fundamental variable is the tetrad; the metric is a derived quantity.

Lorentz connections, curvature and torsion

Spin connection: a 1-form with values in the Lie algebra of the Lorentz group,

$$\omega_\mu = \tfrac{1}{2}\,\omega^{AB}{}_\mu S_{AB}, \qquad \mathcal{D}_\mu \phi^A = \partial_\mu \phi^A + \omega^A{}_{B\mu}\phi^B \tag{4}$$

Curvature (Lorentz algebra) and torsion (translation algebra, $P_A = \partial_A$):

$$\begin{aligned} R^A{}_{B\mu\nu} &= \partial_\mu \omega^A{}_{B\nu} - \partial_\nu \omega^A{}_{B\mu} + \omega^A{}_{C\mu}\omega^C{}_{B\nu} - \omega^A{}_{C\nu}\omega^C{}_{B\mu} \\ T^A{}_{\mu\nu} &= \mathcal{D}_\mu h^A{}_\nu - \mathcal{D}_\nu h^A{}_\mu \end{aligned} \tag{5}$$

Spacetime-indexed connection and torsion:

$$\Gamma^\rho{}_{\nu\mu} = h_A{}^\rho \left(\partial_\mu h^A{}_\nu + \omega^A{}_{B\mu} h^B{}_\nu\right), \qquad T^\rho{}_{\mu\nu} = \Gamma^\rho{}_{\nu\mu} - \Gamma^\rho{}_{\mu\nu} = h_A{}^\rho T^A{}_{\mu\nu} \tag{6}$$

Purely inertial connection and free particles

Inertial frames in general coordinates: ${e'^A}_\mu = \partial_\mu x'^A$. Under a local Lorentz transformation ${e^A}_\mu = {\Lambda^A}_B(x)\,{e'^B}_\mu$:

$$ {e^A}_\mu = \partial_\mu x^A + {\omega^A}_{B\mu} x^B \equiv \mathcal{D}_\mu x^A \qquad (7) $$

$$ {\omega^A}_{B\mu} = {\Lambda^A}_C(x)\,\partial_\mu {\Lambda_B}^C(x) \qquad (8) $$

- ${R^A}_{B\mu\nu}(\omega) = 0$: the connection represents only inertial effects.

Free particle: $du'^A/d\sigma = 0$ in an inertial frame; in a general frame

$$ \frac{du^A}{d\sigma} + {\omega^A}_{B\mu}\, u^B u^\mu = 0 \qquad (9) $$

- The second term describes inertial effects.
- This purely inertial connection is the spin connection of teleparallel gravity; ${\Gamma^\rho}_{\nu\mu} = {h_A}^\rho \mathcal{D}_\mu {h^A}_\nu$ is the Weitzenböck connection.
- Weitzenböck gauge: ${\omega^A}_{B\mu} = 0$.

Translational coupling prescription

For a source field $\Psi(x^A(x^\mu))$ under a local translation:

$$\delta\Psi = \epsilon^A \partial_A \Psi, \qquad \delta(\partial_\mu \Psi) = \epsilon^A \partial_A(\partial_\mu \Psi) + (\partial_\mu \epsilon^A)\, \partial_A \Psi \tag{10}$$

Gauge potential $B_\mu = B^A{}_\mu P_A$ and translational covariant derivative:

$$h_\mu \Psi = \partial_\mu \Psi + B^A{}_\mu \partial_A \Psi = h^A{}_\mu\, \partial_A \Psi, \qquad \delta(h_\mu \Psi) = \epsilon^A \partial_A (h_\mu \Psi) \quad \text{if} \quad \delta B^A{}_\mu = -\mathcal{D}_\mu \epsilon^A \tag{11}$$

Coupling prescription

$$e^A{}_\mu = \mathcal{D}_\mu x^A \;\longrightarrow\; h^A{}_\mu = \mathcal{D}_\mu x^A + B^A{}_\mu, \qquad \eta_{\mu\nu} \;\longrightarrow\; g_{\mu\nu} = \eta_{AB} h^A{}_\mu h^B{}_\nu \tag{12}$$

- The tetrad is gauge invariant: $\delta_\epsilon h^A{}_\mu = \mathcal{D}_\mu \epsilon^A - \mathcal{D}_\mu \epsilon^A = 0$.

Field strength and the universality of the translational coupling

Translational field strength (using $[\mathcal{D}_\mu, \mathcal{D}_\nu]x^A = 0$):

$$T^A{}_{\mu\nu} = \mathcal{D}_\mu B^A{}_\nu - \mathcal{D}_\nu B^A{}_\mu \tag{13}$$

Gravitation is represented by torsion, not curvature. The gravitational coupling has two parts:

- **Translational** coupling prescription: translational gauge invariance; universal, the same $B^A{}_\mu$ for all fields.
- **Lorentz** coupling prescription: local Lorentz invariance (general covariance principle); depends on the spin of the field.

In this talk

The gravitational field $B^A{}_\mu$ stays universal; the universality of the *translational coupling* is relaxed for a scalar sector.

Teleparallel equivalent of General Relativity

$$S_{\rm TG} = -\frac{1}{2\kappa^2}\int d^4x\, h\, T, \qquad h = \det(h^A{}_\mu) = \sqrt{-g}, \qquad \kappa^2 = 8\pi G \tag{14}$$

$$T = S_\rho{}^{\mu\nu} T^\rho{}_{\mu\nu} = \tfrac{1}{4} T^\rho{}_{\mu\nu} T_\rho{}^{\mu\nu} + \tfrac{1}{2} T^\rho{}_{\mu\nu} T^{\nu\mu}{}_\rho - T^\rho{}_{\mu\rho} T^{\nu\mu}{}_\nu \tag{15}$$

$$\mathring{R} = -T + \frac{2}{h}\,\partial_\mu (h T^\nu{}_\nu{}^\mu) \tag{16}$$

- TEGR and Einstein–Hilbert actions differ by a boundary term: equivalent field equations for universal couplings.
- The equivalence does not extend in general to modified theories, where the boundary term affects the dynamics.

Non-universal scalar coupling

Non-universal particle in teleparallel gravity

Aldrovandi, Pereira & Vu (2004): spinless particle with inertial mass m_i and gravitational mass m_g.

$$S_p = -m_i c \int d\sigma - m_g c \int B^A{}_\mu u_A \, dx^\mu \tag{17}$$

$$\left(\partial_\mu x^A + \frac{m_g}{m_i} B^A{}_\mu\right) \frac{du_A}{ds} = \frac{m_g}{m_i} F^A{}_{\mu\nu} u_A u^\nu, \qquad q_g \equiv \frac{m_g}{m_i} \tag{18}$$

- $B^A{}_\mu$ is independent of q_g: the gravitational field stays universal.
- Geometrizing the motion would need a particle-dependent tetrad $h^A{}_\mu(q_g) = \mathcal{D}_\mu x^A + q_g B^A{}_\mu$: no single metric.
- The teleparallel force equation remains well defined: non-universality resides in the coupling.

This work: a field-dependent gravitational charge

$$q_g \longrightarrow q_\phi(\phi) = 1 + \alpha(\phi) \tag{19}$$

- Gravitational sector: TEGR, universal tetrad $h^A{}_\mu$, constant coupling.
- Scalar sector: couples to the same $B^A{}_\mu$ with strength $1 + \alpha(\phi)$; universal limit $\alpha \to 0$.

Framework	What is modified
$F(\phi)T$, $f(T,\phi)$	gravitational Lagrangian (scalar–torsion coupling)
Bigravity	second dynamical tetrad (massive spin-2)
Conformal/disformal, rainbow	transformation or probe-dependent geometry
This work	coupling of the scalar sector to $B^A{}_\mu$

Gauge-invariant construction

$h^A{}_\mu + \alpha B^A{}_\mu$ is not invariant $\Rightarrow$ Stückelberg fields π^A:

$$\delta_\epsilon \pi^A = \epsilon^A, \qquad \mathcal{B}^A{}_\mu \equiv B^A{}_\mu + \mathcal{D}_\mu \pi^A, \qquad \delta_\epsilon \mathcal{B}^A{}_\mu = 0 \tag{20}$$

Scalar-sector tetrad

$$h^A_{\phi\,\mu} = h^A{}_\mu + \alpha(\phi)\, \mathcal{B}^A{}_\mu = h^A{}_\nu M^\nu{}_\mu, \qquad M^\mu{}_\nu = \delta^\mu_\nu + \alpha(\phi)\, h_A{}^\mu \mathcal{B}^A{}_\nu \tag{21}$$

$$g^{(\phi)}_{\mu\nu} = g_{\rho\sigma} M^\rho{}_\mu M^\sigma{}_\nu, \qquad h_\phi = h \det M > 0 \tag{22}$$

$$S = S_{\mathrm{TG}}[h] + \int d^4x\, h_\phi \Big[-\tfrac{1}{2} g^{\mu\nu}_{(\phi)} \partial_\mu\phi \partial_\nu\phi - V(\phi) \Big] \tag{23}$$

One fundamental tetrad; h_ϕ is derived (no second spin-2 field). Unitary gauge $\pi^A = 0$: $h^A_{\phi\,\mu} = h^A{}_\mu + \alpha B^A{}_\mu$.

Field equations

Gravitational and scalar field equations

With $\Theta_A^{(\phi)\mu} \equiv h_{\phi A}{}^{\nu}\, T^{(\phi)\mu}{}_{\nu}$:

$$\mathring{G}_{\lambda}{}^{\mu} = \kappa^2\, T^{\mathrm{eff}\,\mu}{}_{\lambda}, \qquad T^{\mathrm{eff}\,\mu}{}_{\lambda} = [1+\alpha]\det M\,(M^{-1})^{\nu}{}_{\lambda}\, T^{(\phi)\mu}{}_{\nu} \tag{24}$$

$$\mathring{\nabla}_{\mu} T^{\mu}_{\mathrm{eff}\,\nu} = 0, \qquad g_{\rho[\mu}(M^{-1})^{\nu}{}_{\lambda]}\, T^{(\phi)\rho}{}_{\nu} = 0 \tag{25}$$

Scalar field:

$$\Box_{(\phi)}\phi - V_{,\phi} + \alpha_{,\phi}\, \mathcal{B}^{A}{}_{\mu}\, \Theta_A^{(\phi)\mu} = 0, \qquad \nabla_{\mu}^{(\phi)} T^{(\phi)\mu}{}_{\nu} = -\alpha_{,\phi}\, \mathcal{B}^{A}{}_{\mu} \Theta_A^{(\phi)\mu}\, \partial_{\nu}\phi \tag{26}$$

- The effective source is conserved in the universal geometry; the scalar energy–momentum tensor is not separately conserved.
- The final term in the scalar equation arises from the field dependence of the gravitational charge.

Stückelberg equations and the Noether identity

$$\mathcal{D}_\mu \left[h\, \alpha(\phi) \det M\, \Theta_A^{(\phi)\mu} \right] = 0 \tag{27}$$

The conservation identity for $\overset{\circ}{\nabla}_\mu T^\mu_{\text{eff}\,\nu}$, the Bianchi identity and the scalar equation give, with $D^\rho{}_{\mu\nu} \equiv (M^{-1})^\rho{}_\sigma \overset{\circ}{\nabla}_\mu M^\sigma{}_\nu + C^\rho{}_{\mu\nu}$,

$$0 = \alpha_{,\phi} \mathcal{B}^A{}_\mu \Theta_A^{(\phi)\mu} \partial_\nu \phi - \tfrac{\alpha_{,\phi}}{1+\alpha} T^{(\phi)\mu}{}_\nu \partial_\mu \phi + T^{(\phi)\mu}{}_\rho D^\rho{}_{\mu\nu} \tag{28}$$

- Equivalent to the Stückelberg equation through the Noether identity of local translations: not an independent constraint.
- The Stückelberg equations must be retained after imposing $\pi^A = 0$.
- Universal limit: $\overset{\circ}{\nabla}_\mu T^{(\phi)\mu}{}_\nu = 0$ is recovered.

Homogeneous and isotropic cosmology

FLRW: one gravitational field, two geometries

Step 1 – choose the flat-FLRW proper tetrad and its inertial counterpart

$$h^A{}_\mu = \mathrm{diag}(1,a,a,a), \qquad e^A{}_\mu = \mathrm{diag}(1,1,1,1),$$
$$B^A{}_\mu = h^A{}_\mu - e^A{}_\mu = \mathrm{diag}(0,a-1,a-1,a-1). \tag{29}$$

Step 2 – apply the non-universal coupling

$$\beta \equiv 1 - \frac{1}{a}, \qquad m \equiv 1 + \alpha(\phi)\beta, \qquad h^A_{\phi\,\mu} = \mathrm{diag}(1,am,am,am). \tag{30}$$

Scalar-sector expansion

$$a_\phi = am, \qquad H_\phi = H + \frac{\dot m}{m} = \frac{1+\alpha}{m}H + \frac{\beta\dot\alpha}{m}. \tag{31}$$

$M^0{}_0 = 1$: only spatial geometry is deformed ($m > 0$). With diagonal M and $T^{(\phi)\mu}{}_\nu$, the antisymmetric constraint is identically satisfied.

From the effective tetrad to the Friedmann equations

The deformation matrix has $\det M = m^3$ and $(M^{-1})^{\mu}{}_{\nu} = \mathrm{diag}(1, m^{-1}, m^{-1}, m^{-1})$.

$$\rho_\phi = \frac{\dot\phi^2}{2} + V, \qquad p_\phi = \frac{\dot\phi^2}{2} - V. \tag{32}$$

From $T^{\mu}{}_{\mathrm{eff}\,\nu} = (1+\alpha)m^3(M^{-1})^{\rho}{}_{\nu}T^{(\phi)\mu}{}_{\rho}$, the time and spatial components give

$$\underbrace{\rho_{\mathrm{eff}} = (1+\alpha)m^3\rho_\phi}_{\text{time component}}, \qquad \underbrace{p_{\mathrm{eff}} = (1+\alpha)m^2 p_\phi}_{\text{spatial components}}. \tag{33}$$

Gravitational field equations

$$\begin{aligned} 3H^2 &= \kappa^2\rho_{\mathrm{eff}}, \\ -2\dot H &= \kappa^2(\rho_{\mathrm{eff}} + p_{\mathrm{eff}}). \end{aligned} \tag{34}$$

Point out the different powers m^3 and m^2: that is the origin of the modified cosmological equation of state.

Acceleration: from the Friedmann equations

Universal gravitational equations, effective scalar source

$$3H^2 = \kappa^2 \rho_{\text{eff}}, \qquad -2\dot{H} = \kappa^2(\rho_{\text{eff}} + p_{\text{eff}}). \tag{35}$$

Non-universal coupling changes the effective pressure/density ratio

$$\rho_{\text{eff}} = (1+\alpha)m^3\rho_\phi, \qquad p_{\text{eff}} = (1+\alpha)m^2 p_\phi, \tag{36}$$

$$\frac{p_{\text{eff}}}{\rho_{\text{eff}}} = \frac{1}{m}\frac{p_\phi}{\rho_\phi} = \frac{w_\phi}{m}. \tag{37}$$

First Hubble parameter

$$\epsilon \equiv -\frac{\dot{H}}{H^2} = \frac{3}{2}\left(1 + \frac{w_\phi}{m}\right). \tag{38}$$

Physical meaning: $m = a_\phi/a$ modifies the effective equation of state.

When does potential domination give acceleration?

Kinetic-to-potential ratio $V > 0,\ m > 0$

$$r_\phi \equiv \frac{\dot{\phi}^2}{2V}, \qquad w_\phi = \frac{r_\phi - 1}{r_\phi + 1}, \qquad w_{\text{eff}} = \frac{w_\phi}{m}. \tag{39}$$

Why potential domination is not enough for arbitrary m

$$r_\phi \ll 1 \quad \Longrightarrow \quad w_{\text{eff}} \simeq -\frac{1}{m}. \tag{40}$$

Acceleration requires $w_{\text{eff}} < -1/3$; therefore $m < 3$.

For nonzero kinetic energy, the exact condition is stronger:

$$\epsilon < 1 \quad \Longleftrightarrow \quad r_\phi < \frac{3-m}{3+m}, \qquad 0 < m < 3. \tag{41}$$

Universal limit: $m \to 1$
$r_\phi < \frac{1}{2}$.

Strong deformation: $m \to 3^-$
$r_\phi \to 0$ is required.

Stückelberg constraint: the central result

The temporal Stückelberg equation integrates to

$$\alpha\, a_\phi^3 \rho_\phi = \mathcal{C}. \tag{42}$$

Eliminate ρ_ϕ using the Friedmann equation:

$$\begin{aligned} 3H^2 = \kappa^2(1+\alpha)m^3\rho_\phi \quad &\Rightarrow \quad m^3\rho_\phi = \frac{3H^2}{\kappa^2(1+\alpha)}, \\ \mathcal{C} = \alpha a^3 m^3 \rho_\phi \quad &\Rightarrow \quad \frac{\alpha}{1+\alpha} a^3 H^2 = \frac{\kappa^2 \mathcal{C}}{3}. \end{aligned} \tag{43}$$

Physical implication

This identity links the non-universal coupling directly to the background expansion. It is exact on the chosen FLRW configuration.

One derivative determines the evolution of α

Take $N \equiv \ln a$ and differentiate the previous identity:

$$\frac{1}{\alpha(1+\alpha)}\frac{d\alpha}{dN} + 3 + 2\frac{d\ln H}{dN} = 0. \tag{44}$$

Since $d\ln H/dN = -\epsilon$ and $2\epsilon = 3(1 + w_\phi/m)$, we obtain

$$\frac{1}{\alpha(1+\alpha)}\frac{d\alpha}{dN} = -3 + 2\epsilon = \frac{3w_\phi}{m}. \tag{45}$$

Exact coupling evolution

$$\frac{d\alpha}{dN} = 3w_\phi \frac{\alpha(1+\alpha)}{m}. \tag{46}$$

Kination: $w_\phi \simeq +1$
$|\alpha|$ grows with expansion.

Potential domination: $w_\phi \simeq -1$
$|\alpha|$ decreases with expansion.

Kination: why $\alpha \sim a^3$ at early times

For $\dot{\phi}^2/2 \gg V$, we have $w_\phi \simeq 1$. The exact evolution equation reduces to

$$\alpha_N \simeq \frac{3\alpha(1+\alpha)}{m}, \qquad \epsilon \simeq \frac{3}{2}\left(1+\frac{1}{m}\right). \tag{47}$$

As $a \to 0$ on the weak-coupling branch, take $\alpha \ll 1$ and $m \to 1$:

$$\frac{d\ln|\alpha|}{d\ln a} \simeq 3 \quad \Longrightarrow \quad \alpha \simeq A a^3. \tag{48}$$

Then $m = 1 + \alpha(1 - a^{-1})$, giving

$$m \simeq 1 - A a^2, \qquad H \simeq H_* a^{-3}\left(1 - \frac{3}{4} A a^2\right). \tag{49}$$

Interpretation

The leading expansion is the universal stiff-fluid solution ($w_{\rm eff} \to 1$); the first non-universal correction is $\propto a^2$.

Potential domination: checking $\dot{\phi}^2 \ll 2V$

Goal: determine $r_\phi \equiv \dot{\phi}^2/(2V)$, without assuming $\ddot{\phi} \simeq 0$. Define $N = \ln a$ and $L_\alpha \equiv \partial_\phi \ln|\alpha|$.

1. Stückelberg constraint: scalar-field velocity

$$\alpha_N = 3\,\frac{r_\phi - 1}{r_\phi + 1}\,\frac{\alpha(1+\alpha)}{m},$$

$$\alpha_N = \alpha L_\alpha \phi_N \quad \Longrightarrow \quad \phi_N = \frac{3(r_\phi - 1)(1+\alpha)}{m(r_\phi + 1)L_\alpha}. \tag{50}$$

2. Friedmann equation: same velocity

$$\phi_N^2 = \frac{\dot{\phi}^2}{H^2} = \frac{6r_\phi}{\kappa^2(1+\alpha)m^3(1+r_\phi)}. \tag{51}$$

Equate both velocities: exact scalar-only result

$$\frac{r_\phi(1+r_\phi)}{(1-r_\phi)^2} = \frac{3\kappa^2(1+\alpha)^3 m}{2L_\alpha^2}. \tag{52}$$

What guarantees potential domination?

Rewrite the exact result as

$$\frac{r_\phi(1+r_\phi)}{(1-r_\phi)^2} = S_\alpha, \qquad S_\alpha \equiv \frac{3\kappa^2(1+\alpha)^3 m}{2L_\alpha^2}. \tag{53}$$

On the low-kinetic branch $0 \le r_\phi < 1$:

$$\frac{r_\phi(1+r_\phi)}{(1-r_\phi)^2} = r_\phi + 3r_\phi^2 + \mathcal{O}(r_\phi^3). \tag{54}$$

Thus, if $S_\alpha \ll 1$, the solution is self-consistently

$$r_\phi = S_\alpha + \mathcal{O}(S_\alpha^2) \ll 1. \tag{55}$$

Condition on the non-universal coupling

$$L_\alpha^2 \gg \frac{3\kappa^2}{2}(1+\alpha)^3 m \quad \Longrightarrow \quad \frac{\dot\phi^2}{2V} \ll 1. \tag{56}$$

Physical meaning: large $|L_\alpha|$ allows coupling evolution with little scalar-field motion.

Exponential coupling: kinetic ratio and decay

Coupling and logarithmic slope

$$\alpha(\phi) = Ae^{\lambda\kappa\phi}, \qquad L_\alpha = \partial_\phi \ln|\alpha| = \lambda\kappa. \tag{57}$$

1. Kinetic ratio: from Eq. (52)

$$\frac{r_\phi(1+r_\phi)}{(1-r_\phi)^2} = \frac{3(1+\alpha)^3 m}{2\lambda^2} \xrightarrow[m\to 1]{\alpha\to 0} \frac{r_\infty(1+r_\infty)}{(1-r_\infty)^2} = \frac{3}{2\lambda^2}. \tag{58}$$

$$r_\infty \ll 1 \quad \Longrightarrow \quad \boxed{r_\infty \simeq \frac{3}{2\lambda^2}}, \qquad \lambda^2 \gg \frac{3}{2}. \tag{59}$$

2. Coupling evolution: from Eq. (46)

$$\frac{d\ln|\alpha|}{dN} = (1+\alpha)(-3+2\epsilon) = \lambda\kappa\frac{d\phi}{dN}, \qquad N = \ln a. \tag{60}$$

$$\epsilon \to \epsilon_\infty < 1, \quad |\alpha| \ll 1 \quad \Longrightarrow \quad \phi_N \simeq -\frac{3-2\epsilon_\infty}{\lambda\kappa}, \qquad \boxed{|\alpha| \propto a^{-3+2\epsilon_\infty} \to 0}. \tag{61}$$

$X_\infty \equiv \lim_{a\to\infty} X(a)$; exponential field dependence need not imply growth with cosmic time.

Exponential coupling: power-law inflation

From the Hubble slow-roll equation, Eq. (38)

$$\epsilon = \frac{3}{2}\left(1 + \frac{w_\phi}{m}\right), \qquad w_\phi = \frac{r_\phi - 1}{r_\phi + 1}. \tag{62}$$

For $m \to 1$ **and** $r_\phi \to r_\infty \ll 1$**:**

$$\epsilon_\infty = \frac{3r_\infty}{1 + r_\infty} \simeq 3r_\infty \quad \Longrightarrow \quad \boxed{\epsilon_\infty \simeq \frac{9}{2\lambda^2}}. \tag{63}$$

Integrate the definition of ϵ**:**

$$\epsilon = -\frac{d\ln H}{dN} \longrightarrow \epsilon_\infty \quad \Longrightarrow \quad H \propto a^{-\epsilon_\infty}. \tag{64}$$

Asymptotic expansion: $H = \dot{a}/a$

$$\boxed{a(t) \propto (t - t_0)^{1/\epsilon_\infty}}, \qquad 0 < \epsilon_\infty < 1. \tag{65}$$

Accelerated *power-law* expansion: $H \to 0$ as $t \to \infty$, although $\epsilon \to \epsilon_\infty > 0$.

Power-law coupling: field and kinetic suppression

Coupling and logarithmic slope

$$\alpha(\phi) = A(\phi - \phi_I)^n, \quad n > 0, \qquad \Delta\phi \equiv \phi - \phi_I, \quad L_\alpha = \frac{n}{\Delta\phi}. \tag{66}$$

1. From Eq. (52), for $r_\phi \ll 1$ and $|\alpha| \ll 1$:

$$\boxed{r_\phi \simeq \frac{3\kappa^2}{2n^2}(\Delta\phi)^2}. \tag{67}$$

2. From Eq. (46), for $\epsilon \ll 1$:

$$\frac{d\ln|\alpha|}{dN} \simeq -3, \qquad \ln|\alpha| = \ln|A| + n\ln|\Delta\phi|. \tag{68}$$

$$n\frac{d\ln|\Delta\phi|}{dN} \simeq -3 \quad \Longrightarrow \quad \boxed{|\Delta\phi| \propto a^{-3/n}}. \tag{69}$$

3. Substitute back into $\alpha(\phi)$ and r_ϕ:

$$\boxed{|\alpha| \propto a^{-3}}, \qquad \boxed{r_\phi \propto a^{-6/n}}. \tag{70}$$

As $\phi \to \phi_I$, both the coupling and the kinetic-to-potential ratio decay.

Power-law coupling: asymptotic de Sitter expansion

Previous slide: $|\alpha| \propto a^{-3}$ and $r_\phi \propto a^{-6/n}$ $(n > 0)$.

1. From Eq. (38), use $m = 1 + \alpha(1 - a^{-1})$:

$$\epsilon = \frac{3}{2}\left(1 - \frac{1}{m} + \frac{2r_\phi}{m(1 + r_\phi)}\right) \simeq \frac{3}{2}(m-1) + 3r_\phi. \tag{71}$$

$$m - 1 = \mathcal{O}(a^{-3}) \quad \Longrightarrow \quad \boxed{\epsilon = \mathcal{O}(a^{-3}) + \mathcal{O}(a^{-6/n}) \to 0}. \tag{72}$$

2. Integrate $\epsilon = -d\ln H/dN$, with $N = \ln a$:

$$\ln \frac{H(N)}{H(N_0)} = -\int_{N_0}^{N} \epsilon(\widetilde{N})\, d\widetilde{N}. \tag{73}$$

The integral converges as $N \to \infty$

$$\boxed{H \to H_\infty > 0} \quad \Longrightarrow \quad \boxed{a(t) \sim e^{H_\infty t}}. \tag{74}$$

For a regular expanding branch with $H(N_0) > 0$, the late-time limit is de Sitter.

Adding universally coupled matter

Add a fluid with $p_m = w\rho_m$ to the action; it is conserved in the *universal* geometry:

$$\rho_m = \rho_{m0} a^{-3(1+w)}, \qquad \frac{3H^2}{\kappa^2} = \rho_m + (1+\alpha) m^3 \rho_\phi. \tag{75}$$

The Stückelberg first integral remains unchanged. For $V(\phi) = V_0 > 0$ and $r_\phi = \dot\phi^2/(2V_0)$:

$$\alpha_N = 3\frac{r_\phi - 1}{r_\phi + 1}\frac{\alpha(1+\alpha)}{m}, \tag{76}$$

$$(r_\phi)_N + 6\frac{H_\phi}{H} r_\phi + 3\frac{\beta}{m}\alpha_N (1 - r_\phi) = 0. \tag{77}$$

The first equation comes from the Stückelberg constraint; the second follows by multiplying the scalar equation by $\dot\phi/(HV_0)$.

Late-time solution: where does the coefficient 7 come from?

For $a \gg 1$, $|\alpha| \ll 1$ and $r_\phi \ll 1$, the preceding equations give

$$\alpha_N \simeq -3\alpha, \qquad (r_\phi)_N + 6r_\phi \simeq 9\alpha. \tag{78}$$

Therefore,

$$\alpha \simeq \frac{\mathcal{C}}{V_0 a^3}, \qquad r_\phi \simeq 3\alpha \quad \text{(up to a decaying } a^{-6} \text{ mode)}. \tag{79}$$

Expand all factors to first order in α:

$$\begin{aligned} \frac{\rho_{\rm eff}}{V_0} &= (1+\alpha) \underbrace{m^3}_{\simeq 1+3\alpha} \underbrace{(1+r_\phi)}_{\simeq 1+3\alpha} \\ &\simeq 1 + \underbrace{\alpha}_{\text{charge}} + \underbrace{3\alpha}_{\text{geometry}} + \underbrace{3\alpha}_{\text{kinetic}} = 1 + 7\alpha. \end{aligned} \tag{80}$$

Effective matter-like correction

$$\rho_{\rm eff} \simeq V_0 + \frac{7\mathcal{C}}{a^3}. \tag{81}$$

Late-time expansion: from matter to de Sitter

With pressureless ordinary matter ($w = 0$), define $D \equiv \rho_{m0} + 7\mathcal{C} > 0$. Then

$$\frac{3H^2}{\kappa^2} \simeq V_0 + \frac{D}{a^3}. \tag{82}$$

Integrating with $H = \dot{a}/a$ gives

$$a(t) \simeq \left(\frac{D}{V_0}\right)^{1/3} \sinh^{2/3}\left[\frac{\sqrt{3\kappa^2 V_0}}{2}(t - t_0)\right]. \tag{83}$$

Check acceleration directly:

$$\frac{\ddot{a}}{a} = \frac{\kappa^2}{3}\left(V_0 - \frac{D}{2a^3}\right), \qquad a_{\rm acc} = \left(\frac{D}{2V_0}\right)^{1/3}. \tag{84}$$

Formal limits: $a \propto (t - t_0)^{2/3}$ for small $t - t_0$; $a \propto e^{\sqrt{\kappa^2 V_0/3}\, t}$ at late times. Only the late-time regime is controlled.

Late-time reconstruction and validity of the solution

The coupling function is *reconstructed along the trajectory*. Using $\dot{\alpha} = \alpha_{,\phi}\dot{\phi}$ and $r_\phi \simeq 3\alpha$:

$$\alpha_{,\phi}^{2} \simeq \frac{3H^2\alpha}{2V_0}. \tag{85}$$

For $w = 0$, integrate with $H^2 = \frac{\kappa^2}{3}(V_0 + D/a^3)$:

$$\alpha(\phi) \simeq \frac{\mathcal{C}}{D}\sinh^2\left[\kappa\sqrt{\frac{D}{8\mathcal{C}}}\,(\phi - \phi_I)\right] \longrightarrow \frac{\kappa^2}{8}(\phi - \phi_I)^2. \tag{86}$$

Domain of validity – do not extrapolate to the early Universe

$a \gg 1, \quad |\alpha| \ll 1, \quad r_\phi \ll 1.$

At the formal onset of acceleration: $r_\phi(a_{\rm acc}) \simeq 6\mathcal{C}/D \ll 1$ requires $\mathcal{C} \ll D$.

The $7\mathcal{C}/a^3$ contribution is matter-like at background level; this approximation does not establish a dominant dark-matter component or its clustering.

Conclusions and outlook

- **Framework:** a gauge-invariant non-universal scalar coupling with a universal TEGR gravitational sector.
- **Background constraints:** $\alpha a_\phi^3 \rho_\phi = \mathcal{C}$ governs the coupling evolution; the antisymmetric constraint is identically satisfied in FLRW.
- **Two distinct requirements:** a sufficiently large $|L_\alpha|$ allows $r_\phi \ll 1$, whereas acceleration *also* requires
$$r_\phi < \frac{3-m}{3+m}, \qquad 0 < m < 3.$$
- **Inflationary limits:** on weak-coupling solutions $\alpha \to 0$ and $m \to 1$; exponential couplings yield power-law inflation, while power-law couplings can approach de Sitter.
- **Late-time cosmology:** with a universal fluid and $V = V_0$, $\rho_{\rm eff} \simeq V_0 + 7\mathcal{C}/a^3$ in the controlled weak-coupling regime.

Outlook: perturbations, stability and observational constraints.

Thank you!!!