Avoidance of 5th force versus vacuum energy from matter in Modified Measure Theory

Eduardo Guendelman, Ben Gurion University, Israel

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The use of the Riemannian integration measure leads to a Gravitating vacuum energy

We are motivated by the cosmological constant problem. On the mathematical level this boils down to an asymmetry between the matter sector and the gravitational sectors of the theory concerning the role of an origin for the energy density. As it is well known, in non-gravitational physics, like in particle mechanics, for example, the origin from which we measure energy is not important. In mathematical terms that means that the equations of motion are invariant under addition of a constant to the matter Lagrangian L_m

$$L_m \longrightarrow L_m + C\,.$$

In general coordinate invariant theories we woork with a lagrangian density however, which we will denote as $\mathcal{L}_m$. However, when the lagrangian density Lagrangian $\mathcal{L}_m$ is integrated with the standard Riemannian measure of integration, the square root of the determinant of the metric $\sqrt{-g}$, as in,

$$\int d^4x\,\sqrt{-g}\mathcal{L}_m\,,$$

The shift of the lagrangian density,

$$\mathcal{L}_m \longrightarrow \mathcal{L}_m + C\,.$$

when the lagrangian density Lagrangian $\mathcal{L}_m$ is integrated with the standard Riemannian measure of integration, the square root of the determinant of the metric $\sqrt{-g}$, is not a symmetry of the theory now, indeed, the gravitational equations of motion derived from the variation of the metric get an extra contribution of the form $Cg_{\mu\nu}$ after the above shift.

This motivates us to search for alternative measures of integrations, where a shift of the Lagrangian is a symmetry.

This could be achieved if the measure is a total derivative, then the shift above would not change the equations of motion of the theory and we could talk then, at least in some sense of a non gravitating vacuum energy model, as we will see, this possibility is present when we go on and discuss the Non-Riemannian Volume-Form Formalism . Invariance under a shift of the lagrangian density we call Non Gravitating vacuum energy.

A. a proposal for a measure of this sort using 4 scalars

and the *density* Φ is defined by

$$\Phi = \epsilon^{\mu\nu\rho\sigma}\epsilon_{ijkl}\partial_\mu\varphi^i\partial_\nu\varphi^j\partial_\rho\varphi^k\partial_\sigma\varphi^l,$$

in which φ^i, $i = 1, 2, 3, 4$ are 4-scalar fields.

B. Measure Φ built of a totally antisymmetric three index potential

The measure Φ may be constructed also [17] from a totally antisymmetric three index potential $A_{\alpha\beta\gamma}$:

$$\Phi = \varepsilon^{\mu\nu\alpha\beta}\partial_\mu A_{\nu\alpha\beta}. \qquad (13)$$

Espacio para el texto

In both cases the measures are topological densities, i.e. total derivatives

7. Scale Invariance, avoidance of Fifth Force in Fermionic and Dust Matter Models and Flat Potentials

Continuing with the issue of Dark matter formulated in the context of modified measure theories but now with spontaneously broken scale invariance, we show now that

more explicitly, let us consider the action containing a non Riemannian measure Φ which is a total divergence and is invariant under the global scale transformations:

$$g_{\mu\nu} \to e^{\theta} g_{\mu\nu}, \quad \Gamma^{\mu}_{\alpha\beta} \to \Gamma^{\mu}_{\alpha\beta}, \quad \phi \to \phi - \frac{M_p}{\alpha}\theta, \quad \Phi \to e^{2\theta}\Phi \tag{20}$$

where $\theta = const$. It is convenient to represent the action in the following form:

Using the first order formulation, where the metric and the connection are idependent

$$
\begin{aligned}
S &= S_g + S_\phi + S_m \qquad (21)\\
S_g &= -\frac{1}{\kappa}\int (\Phi + b_g\sqrt{-g})R(\Gamma,g)e^{\alpha\phi/M_p}d^4x\,;\\
S_\phi &= \int e^{\alpha\phi/M_p}\left[(\Phi + b_\phi\sqrt{-g})\frac{1}{2}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - (\sqrt{-g}V_2 - \Phi V_1)e^{\alpha\phi/M_p}\right]d^4x\,;\\
S_m &= \int (\Phi + b_m\sqrt{-g})L_m d^4x\,,
\end{aligned}
$$

where the Lagrangian for the matter, as collection of particles, which provides the scale invariance of S_m reads

$$
L_m = -m\sum_i \int e^{\frac{1}{2}\alpha\phi/M_p}\sqrt{g_{\alpha\beta}\frac{dx_i^\alpha}{d\lambda}\frac{dx_i^\beta}{d\lambda}}\,\frac{\delta^4(x - x_i(\lambda))}{\sqrt{-g}}d\lambda \qquad (22)
$$

where λ is an arbitrary parameter. For simplicity we consider the collection of the particles with the same mass parameter m. We assume in addition that $x_i(\lambda)$ do not participate in the scale transformations (20). We will assume that $d\vec{x}_i/d\lambda \equiv 0$ for all particles. It is convenient to proceed in the frame where $g_{0l} = 0, \ \ l = 1,2,3$. Then the particle density is defined by

$$n(\vec{x}) = \sum_i \frac{1}{\sqrt{-g_{(3)}}} \delta^{(3)}(\vec{x} - \vec{x}_i(\lambda)) \tag{23}$$

where $g_{(3)} = \det(g_{kl})$ and

$$S_m = -m \int d^4x (\Phi + b_m \sqrt{-g})\, n(\vec{x})\, e^{\frac{1}{2}\alpha\phi/M_p} \tag{24}$$

It turns out that when working with the new metric (ϕ remains the same)

$$\tilde{g}_{\mu\nu} = e^{\alpha\phi/M_p}(\chi + b_g) g_{\mu\nu}, \tag{25}$$

here $\chi \equiv \Phi/\sqrt{-g}$, and we call this metric the Einstein frame metric, the connection becomes Riemannian. Notice that $\tilde{g}_{\mu\nu}$ is invariant under the scale transformations (20). The transformation (25) causes the transformation of the particle density

$$\tilde{n}(\vec{x}) = (\chi + b_g)^{-3/2}\, e^{-\frac{3}{2}\alpha\phi/M_p}\, n(\vec{x}) \tag{26}$$

The Lagrangian density coupled to the Measure Φ has to be a constant, as we have discussed in general. Let us call this constant M

After the change of variables to the Einstein frame (25) and some simple algebra, the gravitational equations take the standard GR form

$$G_{\mu\nu}(\tilde{g}_{\alpha\beta}) = \frac{\kappa}{2} T^{eff}_{\mu\nu} \tag{27}$$

where $G_{\mu\nu}(\tilde{g}_{\alpha\beta})$ is the Einstein tensor in the Riemannian space-time with the metric $\tilde{g}_{\mu\nu}$.
The components of the effective energy-momentum tensor are as follows:

$$T^{eff}_{00} = \frac{\zeta + b_\phi}{\zeta + b_g}\left(\dot{\phi}^2 - \tilde{g}_{00}X\right) + \tilde{g}_{00}\left[V_{eff}(\phi;\zeta,M) - \frac{\delta \cdot b_g}{\chi + b_g}X + \frac{3\chi + b_m + 2b_g}{2\sqrt{\zeta + b_g}}\, m\,\tilde{n}\right] \tag{28}$$

$$T^{eff}_{ij} = \frac{\chi + b_\phi}{\zeta + b_g}(\phi_{,k}\phi_{,l} - \tilde{g}_{kl}X) + \tilde{g}_{kl}\left[V_{eff}(\phi;\zeta,M) - \frac{\delta \cdot b_g}{\zeta + b_g}X + \frac{\zeta - b_m + 2b_g}{2\sqrt{\zeta + b_g}}\, m\,\tilde{n}\right] \tag{29}$$

Here the following notations have been used:

$$X \equiv \frac{1}{2}\tilde{g}^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} \qquad and \qquad \delta = \frac{b_g - b_\phi}{b_g} \tag{30}$$

and the function $V_{eff}(\phi;\zeta)$ is defined by

$$V_{eff}(\phi;\chi) = \frac{b_g\left[Me^{-2\alpha\phi/M_p} + V_1\right] + V_2}{(\chi + b_g)^2} \tag{31}$$

The dilaton ϕ field equation in the Einstein frame is as follows

$$\frac{1}{\sqrt{-\tilde{g}}}\partial_\mu\left[\frac{\chi + b_\phi}{\chi + b_g}\sqrt{-\tilde{g}}\tilde{g}^{\mu\nu}\partial_\nu\phi\right] - \frac{\alpha}{M_p}\frac{(\chi + b_g)Me^{-2\alpha\phi/M_p} - (\chi - b_g)V_1 + 2V_2 - \delta b_g(\chi + b_g)X}{(\chi + b_g)^2}$$
$$= \frac{\alpha}{M_p}\frac{\chi - b_m + 2b_g}{2\sqrt{\chi + b_g}}\, m\,\tilde{n} \tag{32}$$

In the above equations, the scalar field χ is determined as a function $\chi(\phi, X, \tilde{n})$ by means of the following constraint:

$$\frac{(b_g - \chi)\left(Me^{-2\alpha\phi/M_p} + V_1\right) + 2V_2}{(\chi + b_g)^2} - \frac{\delta \cdot b_g X}{\chi + b_g} = \frac{\chi - b_m + 2b_g}{2\sqrt{\chi + b_g}}\, m\,\tilde{n} \tag{33}$$

One should now pay attention to the interesting result that the explicit $\tilde{n}$ dependence involving **the same form of χ dependence**

$$\frac{\chi - b_m + 2b_g}{2\sqrt{\chi + b_g}}\, m\,\tilde{n} \tag{34}$$

appears simultaneously in the dust contribution to the pressure (through the last term in Eq. (39)), in the effective dilaton to dust coupling (in the r.h.s. of Eq. (32)) and in the r.h.s. of the constraint (44).

Let us analyze consequences of this wonderful coincidence in the case when the matter energy density (modeled by dust) is much larger than the dilaton contribution to the dark energy density in the space region occupied by this matter. Evidently this is the condition under which all tests of Einstein's GR, including the question of the fifth force, are fulfilled. if the dust is in the normal conditions there is a possibility to provide the desirable feature of the dust in GR: it must be -. This is realized provided that in normal conditions (n.c.) the following equality holds with extremely high accuracy:

$$\chi^{(n.c.)} \approx b_m - 2b_g \tag{35}$$

Remind that we have assumed $b_m > b_g$. Then $\chi^{(n.c.)} + b_g > 0$, and the transformation (25) and the subsequent equations in the Einstein frame are well defined. Inserting (35) in the last term of Eq. (38) we obtain the effective dust energy density in normal conditions

$$\rho_m^{(n.c.)} = 2\sqrt{b_m - b_g}\, m\tilde{n} \tag{36}$$

When we get only a slight deviation of from ζ from $b_m - 2b_g$, when the matter energy density is many orders of magnitude larger than the dilaton contribution to the dark energy density, we obtain an effective 5th force coupling f. For this look at the ϕ-equation in the form (32) and estimate the Yukawa type coupling constant in the r.h.s. of this equation. In fact, using the constraint (44) and representing the particle density in the form $\tilde{n} \approx N/v$ where N is the number of particles in a volume v, one can make the following estimation for the effective dilaton to matter coupling "constant" f defined by the Yukawa type interaction term $f\tilde{n}\phi$ (if we were to invent an effective action whose variation with respect to ϕ would result in Eq. (32)):

$$f \equiv \alpha \frac{m}{M_p} \frac{\chi - b_m + 2b_g}{2\sqrt{\zeta + b_g}} \approx \alpha \frac{m}{M_p} \frac{\chi - b_m + 2b_g}{2\sqrt{b_m - b_g}} \sim \frac{\alpha}{M_p} \frac{\rho_{vac}}{\tilde{n}} \approx \alpha \frac{\rho_{vac} v}{N M_p} \tag{37}$$

becomes less than the ratio of the "mass of the vacuum" in the volume occupied by the matter to the Planck mass. The model yields this kind of "Archimedes law" without any especial (intended for this) choice of the underlying action and without fine tuning of the parameters. The model not only explains why all attempts to discover a scalar force correction to Newtonian gravity were unsuccessful so far but also predicts that in the near future there is no chance to detect such corrections in the astronomical measurements as well as in the specially designed fifth force experiments on intermediate, short (like millimeter) and even ultrashort (a few nanometer) ranges. This prediction is alternative to predictions of other known models.

Ok, we have seen that when energy density of the dust, these dust particles behaves do not

Suffer from the 5th force.

The particles also become pressureless, which means, we recover a normal, canonical dust!, this mechanism is the opposite of the Camelion discussed by Lucila in the course last week, here is the coupling constant of the matter source to the scalar field what dissappears at high density, rather that the mass of the scalar becoming very big as the density becomes very big.

But what about the low density case, can this situation provide a matter- dark energy transition?,

Wait for a few preliminaries, and you withh see that for a fermionic dark matter this can be the case,

7.1. Flat potentials in the absence of matter, consequences for DE or Inflation

If we set To clarify what the theory is telling us only in the gravity, scalar field sector, we set the dust matter momentum tensor are as follows:

$$T_{00}^{eff} = \frac{\chi + b_\phi}{\chi + b_g}\left(\dot{\phi}^2 - \tilde{g}_{00}X\right) + \tilde{g}_{00}\left[V_{eff}(\phi;\chi,M) - \frac{\delta \cdot b_g}{\zeta + b_g}X\right] \tag{38}$$

$$T_{ij}^{eff} = \frac{\chi + b_\phi}{\chi + b_g}(\phi_{,k}\phi_{,l} - \tilde{g}_{kl}X) + \tilde{g}_{kl}\left[V_{eff}(\phi;\zeta,M) - \frac{\delta \cdot b_g}{\chi + b_g}X\right] \tag{39}$$

Here the following notations have been used:

$$X \equiv \frac{1}{2}\tilde{g}^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} \qquad and \qquad \delta = \frac{b_g - b_\phi}{b_g} \tag{40}$$

and the function $V_{eff}(\phi;\chi)$ is defined by

$$V_{eff}(\phi;\chi) = \frac{b_g\left[Me^{-2\alpha\phi/M_p} + V_1\right] + V_2}{(\chi + b_g)^2} \tag{41}$$

The dilaton ϕ field equation in the Einstein frame is as follows

$$\frac{1}{\sqrt{-\tilde{g}}}\partial_\mu\left[\frac{\chi + b_\phi}{\chi + b_g}\sqrt{-\tilde{g}}\tilde{g}^{\mu\nu}\partial_\nu\phi\right] - \frac{\alpha}{M_p}\frac{(\zeta + b_g)Me^{-2\alpha\phi/M_p} - (\zeta - b_g)V_1 + 2V_2 - \delta b_g(\chi + b_g)X}{(\chi + b_g)^2} = \tag{42}$$

In the above equations, the scalar field χ is determined as a function $\chi(\phi, X,)$ by means of the following constraint:

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$$\frac{(b_g - \chi)\left(Me^{-2\alpha\phi/M_p} + V_1\right) + 2V_2}{(\chi + b_g)^2} - \frac{\delta \cdot b_g X}{\chi + b_g} = 0 \tag{43}$$

which now becomes, after some simplifications a linear equation. The simplest case, resulting from the choice $b_g = 0$, studied first in [2] already gives a very simple result

$$\chi = \frac{2V_2}{e^{-2\alpha\phi/M_p} + V_1} \tag{44}$$

which then inserted into (41), gives

$$V_{eff}(\phi) = \frac{(e^{-2\alpha\phi/M_p} + V_1)^2}{4V_2} \tag{45}$$

which generalizes the effective potential used by Starobinsky for example. It also displays an infinite flat region, that can be used for inflation, but it does not display Dark energy in that case.

II. ATTEMPTS TO OBTAIN FERMIONIC DARK ENERGY IN THE CONTEXT OF STANDARD FIELD THEORY

In GR the free fermion part of the action has the form $S_0^{(f)} = \int d^4x\sqrt{-g}L_f$ where L_f is the free fermion Lagrangian density in the Einstein-Cartan space-time. Then a potential cosmological-like term $\propto g_{\mu\nu}L_f$ naively appears but the Dirac equation forces L_f to vanish.

Here we would like to give some feeling of the mechanism by which a term in the energy-momentum tensor proportional to $g_{\mu\nu}f(\bar{\psi}\psi)$ (where $f(\bar{\psi}\psi)$ is a function of $\bar{\psi}\psi$) could be obtained in the framework of the standard (non TMT) field theories. Such a term may cause a negative fermion contribution to the pressure and therefore might play an important role for the total dark energy density. We give in this section two examples of models where on the classical level the fermion contributes a term in the energy-momentum tensor proportional to $g_{\mu\nu}f(\bar{\psi}\psi)$.

A. A model with coupling to topological density

If the action includes a topological density then the term proportional to $g_{\mu\nu}\bar{\psi}\psi$ can appear in the energy-momentum tensor. As an example of a topological density we consider here $\Omega \equiv \epsilon^{\alpha\beta\mu\nu} Tr(F_{\alpha\beta}F_{\mu\nu})$ in a gauge theory.

For illustration let us consider a model where in addition to L_f, which includes now a minimal coupling to gauge fields, there is also the interaction of the fermion to the scalar field ϕ which in its turn couples to the topological density Ω (the latter is parity violating):

$$S_1 = \int \sqrt{-g} d^4x \left[\frac{1}{2} g^{\mu\nu} \phi_\mu \phi_\nu - \frac{1}{2} m_\phi^2 \phi^2 + \lambda_f \phi \bar{\psi}\psi \right] + \int \lambda_{top} \phi \Omega d^4x. \tag{1}$$

By integrating out the ϕ-field one can get (assuming that ϕ is slowly varying) the following fermion part of the effective action

$$S_{eff}^{(ferm)} = \int \sqrt{-g} d^4x \left[L_f + \frac{\lambda_f^2}{2m_\phi^2} (\bar{\psi}\psi)^2 \right] + \int \frac{\lambda_f \lambda_{top}}{m_\phi^2} \Omega \bar{\psi}\psi d^4x. \tag{2}$$

Now the fermion equation yields

$$\sqrt{-g} \left[L_f + \frac{\lambda_f^2}{m_\phi^2} (\bar{\psi}\psi)^2 \right] + \frac{\lambda_f \lambda_{top}}{m_\phi^2} \Omega \bar{\psi}\psi = 0 \tag{3}$$

Varying the effective action (2) with respect to $g^{\mu\nu}$ and using Eq.(3) we obtain the following fermion contribution to the energy-momentum tensor proportional to $g_{\mu\nu}$:

$$\Delta T_{\mu\nu}^{(ferm)} = g_{\mu\nu} \left[\frac{\lambda_f \lambda_{top}}{m_\phi^2} \frac{\Omega}{\sqrt{-g}} \bar{\psi}\psi + \frac{\lambda_f^2}{2m_\phi^2} (\bar{\psi}\psi)^2 \right] \tag{4}$$

B. *A model with fermion selfinteraction*

Another model of interest is that of a fermion with a selfinteraction in curved space-time. The fermion part of the action we choose in the form

$$S = \int \sqrt{-g} d^4x \left[L_f + \lambda (\bar{\psi}\psi)^l \right] \tag{5}$$

where L_f is the free fermion Lagrangian density in the Einstein-Cartan space-time; l is a dimensionless constant different from one; λ is a coupling of the selfinteraction.

It follows from the fermion equation of motion that

$$L_f = -l\lambda(\bar{\psi}\psi)^l \tag{6}$$

Varying the action (5) with respect to $g^{\mu\nu}$ and using Eq.(6) we obtain the following fermion contribution to the energy-momentum tensor proportional to $g_{\mu\nu}$:

$$\Delta T_{\mu\nu}^{(ferm)} = (l-1)\lambda(\bar{\psi}\psi)^l g_{\mu\nu} \tag{7}$$

A topological density seems to do the job, so the introduction of a modified measure seems

Promising for the obtaining vacuum energy from fermions!

We formulate now then the generic case of fermions with a modified measure

II. GENERAL DESCRIPTION

In this section, we will present a brief analysis of the implications of considering both the scalar field and fermion fields. Thus, in order to describe our model, we consider that the general action can be written as [22]

$$
\begin{aligned}
S &= \int d^4x e^{\alpha\phi/M_P}(\Phi + b\sqrt{-g})\left[-\frac{1}{\kappa}R(\omega,e) + \frac{1}{2}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}\right] \\
&- \int d^4x e^{2\alpha\phi/M_P}[\Phi V_1 + \sqrt{-g}V_2] - \int d^4x\sqrt{-g}\frac{1}{4}g^{\alpha\beta}g^{\mu\nu}F_{\alpha\mu}F_{\beta\nu} \\
&+ \int d^4x e^{\alpha\phi/M_P}(\Phi + k\sqrt{-g})\frac{i}{2}\sum_i \overline{\Psi}_i\left(\gamma^a e_a^\mu \overrightarrow{\nabla}_\mu^{(i)} - \overleftarrow{\nabla}_\mu^{(i)}\gamma^a e_a^\mu\right)\Psi_i \\
&- \int d^4x e^{\frac{3}{2}\alpha\phi/M_P}\left[(\Phi + h_E\sqrt{-g})\mu_E\overline{E}E + (\Phi + h_n\sqrt{-g})\mu_n\overline{n}n\right], \qquad (1)
\end{aligned}
$$

in which the measure fields φ_a are utilized for definition of the measure $\Phi = \varepsilon^{\mu\nu\alpha\beta}\varepsilon_{abcd}\partial_\mu\varphi_a\partial_\nu\varphi_b\partial_\alpha\varphi_c\partial_\beta\varphi_d$. Therefore, the quantity Ψ_i ($i = n, E$ neutrinos, electrons) is the general notation for the primordial fermion fields n and E; the variable $F_{\alpha\beta}$ is defined as $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ and the parameters μ_n and μ_E are the mass parameters. Besides, we have $\overrightarrow{\nabla}_\mu^{(n)} = \vec{\partial} + \frac{1}{2}\omega_\mu^{cd}\sigma_{cd}$, $\overrightarrow{\nabla}_\mu^{(E)} = \vec{\partial} + \frac{1}{2}\omega_\mu^{cd}\sigma_{cd} + ieA_\mu$; $R(\omega, V) = e^{a\mu}e^{b\nu}R_{\mu\nu ab}(\omega)$ is the scalar curvature; e_a^μ and ω_μ^{ab} are the vierbein and spin-connection; $g^{\mu\nu} = e_a^\mu e_b^\nu\eta^{ab}$ and $R_{\mu\nu ab}(\omega) = \partial_\mu\omega_{\nu ab} + \omega_{\mu a}^c\omega_{\nu cb} - (\mu \leftrightarrow \nu)$. The quantities V_1 and V_2 are constants with the dimensionality $(mass)^4$. In addition, the constants b, k, h_n, h_E are non specified dimensionless real parameters. In relation to the parameter α, we have that it is a real parameter which is positive and dimensionless. We will determine the relative signs of the kinetic terms and mass terms in the above lagrangian

comparing only the terms that are multiplied by the conventional measure, and since k will be determined to negative, we will choose the corresponding coefficients of the mass terms , like μ_E and μ_n to be negative.

We note that the general action given by Eq.(1) is invariant under the global scale transformations

$$e^a_\mu \to e^{\theta/2} e^a_\mu, \quad \omega^\mu_{ab} \to \omega^\mu_{ab}, \quad \varphi_a \to \lambda_{ab}\varphi_b \quad A_\alpha \to A_\alpha,$$
$$\phi \to \phi - \frac{M_p}{\alpha}\theta, \quad \Psi_i \to e^{-\theta/4}\Psi_i, \quad \overline{\Psi}_i \to e^{-\theta/4}\overline{\Psi}_i, \quad \det(\lambda_{ab}) = e^{2\theta}. \tag{2}$$

In relation to the two measure theory we have that it is a generally coordinate invariant theory, where the action becomes [9]-[11]

$$S = \int L_1 \Phi d^4x + \int L_2 \sqrt{-g} d^4x, \tag{3}$$

where there are two Lagrangians L_1 and L_2 and two measures of integration: the standard $\sqrt{-g}$ and the new one Φ. Here, we have that the Lagrangian L_1 satisfies the relation [9]

$$B^\mu_a \partial_\mu L_1 = 0, \quad \text{and then} \quad L_1 = M^4 = \text{const.} \tag{4}$$

where $B^\mu_a = \varepsilon^{\mu\nu\alpha\beta}\varepsilon_{abcd}\partial_\nu\varphi_b\partial_\alpha\varphi_c\partial_\beta\varphi_d$ and since the determinant of the matrix B^μ_a is proportional to the measure, it follows that for a non singular measure we have that $\partial_\mu L_1 = 0$.

From the action given by Eq.(1), we have a first order formalism that contain terms proportional to $\partial_\mu\chi$, where $\chi = \Phi/\sqrt{-g}$. Thus, we obtain that the space-time non-Riemannian and equations of motion - non canonical.

In order to obtain the Einstein frame, we can define a new set of variables (ϕ and A_μ remain unchanged) such that

$$\tilde{e}_{a\mu} = e^{\frac{1}{2}\alpha\phi/M_p}(\chi+b)^{1/2}e_{a\mu}, \quad \tilde{g}_{\mu\nu} = e^{\alpha\phi/M_p}(\chi+b)g_{\mu\nu},$$
$$\Psi_i' = e^{-\frac{1}{4}\alpha\phi/M_p}\frac{(\chi+k)^{1/2}}{(\chi+b)^{3/4}}\Psi_i, \quad \text{with} \quad i = n, E. \tag{5}$$

Note that $\tilde{e}_{a\mu}$, $\tilde{g}_{\mu\nu}$, n' and E' are invariant under the scale transformations (2).

In the Einstein frame the equations take the standard General Relativity form given by

$$G_{\mu\nu}(\tilde{g}_{\alpha\beta}) = \frac{\kappa}{2}T^{eff}_{\mu\nu}, \tag{6}$$

in which the tensor $G_{\mu\nu}(\tilde{g}_{\alpha\beta})$ corresponds to the Einstein tensor in the Riemannian space-time with the metric $\tilde{g}_{\mu\nu}$ and the quantity $\kappa = 8\pi/(M_p^2)$ with M_p the Planck mass. Also, the energy-momentum tensor $T^{eff}_{\mu\nu}$ is defined as

$$T^{eff}_{\mu\nu} = \phi_{,\mu}\phi_{,\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{g}^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} + \tilde{g}_{\mu\nu}V_{eff}(\phi;\chi) + T^{(em)}_{\mu\nu} + T^{(ferm,can)}_{\mu\nu} + T^{(ferm,noncan)}_{\mu\nu}, \tag{7}$$

where the effective potential associated to scalar field and χ is given by

$$V_{eff}(\phi,\chi) = \frac{b\left(M^4e^{-2\alpha\phi/M_p} + V_1\right) - V_2}{(\chi+b)^2}. \tag{8}$$

Besides, the quantity $T^{(em)}_{\mu\nu}$ corresponds to the canonical energy momentum tensor of the electromagnetic field and the tensor $T^{(ferm,can)}_{\mu\nu}$ represents the canonical energy momentum tensor for primordial fermions n' and E' in curved space-time. The quantity $T^{(ferm,noncan)}_{\mu\nu}$ denotes the noncanonical contribution of the fermions into the energy momentum tensor and it is defined as[22]

$$T^{(ferm,noncan)}_{\mu\nu} = -\tilde{g}_{\mu\nu}\Lambda^{(ferm)}_{dyn}, \tag{9}$$

where $\Lambda^{(ferm)}_{dyn}$ is given by

$$\Lambda^{(ferm)}_{dyn} \equiv Z_n(\chi)m_n(\chi)\overline{n'}n' + Z_E(\chi)m_E(\chi)\overline{E'}E', \qquad Z_i(\chi) \equiv \frac{(\chi-\chi_1^{(i)})(\chi-\chi_2^{(i)})}{2(\chi+k)(\chi+h_i)}, \qquad m_i(\chi) = \frac{\mu_i(\chi+h_i)}{(\chi+k)(\chi+b)^{1/2}},$$

and $Z_i(\chi)$ and $m_i(\chi)$ $(i = n', E')$ are defined as

where

$$\chi_{1,2}^{(i)} = \frac{1}{2}\left[k - 3h_i \pm \sqrt{(k-3h_i)^2 + 8b(k-h_i) - 4kh_i}\right]. \tag{12}$$

It is important to mention that the noncanonical contribution given by $T_{\mu\nu}^{(ferm,noncan)}$ of the fermions into the energy momentum tensor produces a cosmological constant like term but it is proportional to fermion densities $\overline{\Psi}_i'\Psi_i'$ $(i = n', E')$, and this is why we will refer to it as "dynamical fermionic Λ term".

The scalar field χ given by $\chi = \Phi/\sqrt{-g}$, is determined in terms of the scalar field ϕ and $\overline{\Psi}_i'\Psi_i'$ $(i = n', E')$ by the following equation[22]

$$\frac{1}{(\chi+b)^2}\left\{(b-\chi)\left[M^4 e^{-2\alpha\phi/M_P} + V_1\right] - 2V_2\right\} = \Lambda_{dyn}^{(ferm)}. \tag{13}$$

In the following, we will study the inflationary stage in which we will consider the absence of fermions during this epoch.

III. INFLATION

In this section we will analyze the inflationary era where we do not consider fermions so that $\Lambda_{dyn}^{(ferm)} = 0$. Thus, from Eq.(13) we find that the quantity χ in terms of the scalar field becomes

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In this section we will analyze the inflationary era where we do not consider fermions so that $\Lambda_{dyn}^{(ferm)} = 0$. Thus, from Eq.(13) we find that the quantity χ in terms of the scalar field becomes

$$\chi = \chi_0(\phi) \equiv b - \frac{2V_2}{V_1 + M^4 e^{-2\alpha\phi/M_p}}. \tag{14}$$

Now replacing this equation into Eq.(8) we obtain that the effective potential of the scalar field ϕ in absence of fermions results

$$V_{eff}(\phi) \equiv V_{eff}(\phi;\chi_0)|_{\overline{\psi'}\psi'=0} = \frac{[V_1 + M^4 e^{-2\alpha\phi/M_p}]^2}{4[b\,(V_1 + M^4 e^{-2\alpha\phi/M_p}) - V_2]}. \tag{15}$$

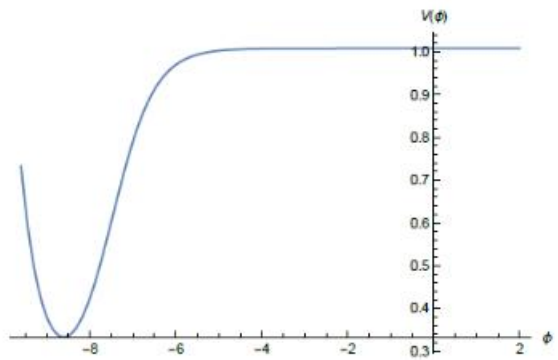


FIG. 1: A schematic representation of the effective potential $V(\phi)$ as a function of the scalar field ϕ, given by Eq. (15). The flat region corresponds to the inflationary phase described by Eq. (16). At the minimum, the mechanism of reheating, and also the production of fermions occurs.

FIG. 1: A schematic representation of the effective potential $V(\phi)$ as a function of the scalar field ϕ, given by Eq. (15). The flat region corresponds to the inflationary phase described by Eq. (16). At the minimum, the mechanism of reheating, and also the production of fermions occurs.

In order to analyze the inflationary period, we can consider that the scalar field start from very high positive values ($\phi \to +\infty$), with a flat region given by

$$V_{eff}(\phi) \simeq \frac{V_1^2}{4(bV_1 - V_2)}, \quad \text{with} \quad bV_1 > V_2. \tag{16}$$

Fig.1 shows the effective potential as a function of the scalar field defined by Eq.(15). For this potential, we note that the flat region refers to the inflationary phase defined by Eq.(16). Furthermore, at the minimum of this effective potential, the mechanism of reheating occurs. This process involves the perturbative decay of the oscillating field ϕ at the end of the inflationary scenario, which generates the reheating of the universe[23]. In this context, various

particles, including fermions, are produced at the minimum of the effective potential. Here we have used the values; $\alpha = 1$, $V_1 = (1.1/3)M_p^4$, $V_2 = (1/3)M_p^4$, $M = 0.01M_p$ and $b = 1$, respectively.

Now, expanding the effective potential (15), we obtain that the next term is given by

$$V_{eff}(\phi) \simeq \frac{V_1^2}{4(bV_1 - V_2)} + \frac{V_1 M^4}{2(bV_1 - V_2)}\left[1 - \frac{bV_1}{2(bV_1 - V_2)}\right] e^{-2\alpha\phi/M_p} + ..., \tag{17}$$

with $bV_1/2 < V_2$, in order to obtain a decrease of the effective potential as we move to smaller values of the scalar field during the inflationary epoch. However, we also have that $bV_1 > V_2$, then the condition on the parameter V_2 becomes $bV_1/2 < V_2 < bV_1$.

This choice $bV_1/2 < V_2$ differs from that of Ref.[22], where inflation was not considered and the evolution was in the other direction. In contrast, here, the high values of the scalar field will be used twice, once for inflation and then, after reheating, the same scalar field goes back in the other direction, but now feeling a modified potential modified by the particles created during the reheating.

In the context of the slow roll approximation and considering a flat Friedmann-Roberson-Walker metric, we have that the equations of motion are reduced to

$$3H^2 = \kappa\rho \simeq \kappa V_{eff}, \quad \text{and} \quad 3H\dot{\phi} \simeq -\frac{\partial V_{eff}}{\partial\phi}, \tag{18}$$

where $H = \dot{a}/a$ denotes the Hubble parameter and the quantity a the scale factor. Here, in the absence of fermions, we have considered Eq.(6) together with the conservation of the energy-momentum tensor, defined by Eq. (7).

Thus, under the slow roll approximation from Eq.(17), we find that the scalar field in terms of the cosmological time can be written as

$$e^{\alpha\phi/M_P}\sqrt{(V_{01}+V_{02}e^{-2\alpha\phi/M_P})}\left(e^{\alpha\phi/M_P}+\frac{\sqrt{V_{02}}\text{ArcSinh}[e^{\alpha\phi/M_P}\sqrt{V_{01}/V_{02}}]}{\sqrt{V_{01}}\sqrt{1+e^{2\alpha\phi/M_P}V_{01}/V_{02}}}\right)=\frac{4V_{02}(\alpha/M_p)^2}{\sqrt{3\kappa}}t+C_0, \tag{19}$$

where C_0 corresponds to an integration constant. Here the constants V_{01} and V_{02} are defined as

$$V_{01}=\frac{V_1^2}{4(bV_1-V_2)}, \quad \text{and} \quad V_{02}=\frac{V_1M^4}{2(bV_1-V_2)}\left[1-\frac{bV_1}{2(bV_1-V_2)}\right]<0, \tag{20}$$

respectively. From Eq.(19), we note that for $\phi\to\infty$, the solution for the scalar field in terms of time is approximated to

$$e^{2\alpha\phi/M_P}\sim\frac{4V_{02}(\alpha/M_p)^2}{\sqrt{3\kappa V_{01}}}t+C_0',$$

where $C_0'=C_0/\sqrt{V_{01}}$.

By introducing the slow roll parameters ϵ and η, we have

$$\epsilon=\frac{1}{2\kappa}\left[\frac{\partial V_{eff}/\partial\phi}{V_{eff}}\right]^2, \quad \text{and} \quad \eta=\frac{1}{\kappa}\left[\frac{\partial^2 V_{eff}/\partial\phi^2}{V_{eff}}\right]. \tag{21}$$

Thus, considering that the end of inflation taken place when $\epsilon(\phi=\phi_e)=1$, we find the value at the end of inflation becomes

$$e^{-2\alpha\phi_e/M_p} = -\left[\frac{V_{01}}{V_{02}}\right]\frac{1}{[\frac{\sqrt{2}\alpha}{\sqrt{\kappa}M_p}+1]}. \tag{22}$$

Here ϕ_e denotes the value of the scalar field at the end of inflationary epoch and recalled that $V_{02} < 0$.

Also, the number of e-folds N between two different values of the times t and t_e results

$$N = \int_t^{t_e} H\,dt \simeq \kappa \int_{\phi_e}^{\phi} \frac{V_{eff}}{(\partial V_{eff}/\partial\phi)}\,d\phi, \tag{23}$$

where t_e denotes the end of the inflationary epoch. Thus, we find that the number of $e-$folds as a function of the scalar field can be written as

$$N = \left(\frac{\kappa M_p}{2\alpha|V_{02}|}\right)\left(\left[\frac{M_pV_{01}e^{2\alpha\phi/M_p}}{2\alpha}+V_{02}\phi\right]-\left[\frac{M_pV_{01}e^{2\alpha\phi_e/M_p}}{2\alpha}+V_{02}\phi_e\right]\right). \tag{24}$$

In this context, we can determine the scalar field as a function of the number of $e-$ folds as

$$\phi(N) = \frac{\tilde{N}}{V_{02}} - \frac{M_p\text{ProducLog}[(V_{01}/V_{02})e^{2\alpha\tilde{N}/(M_pV_{02})}]}{2\alpha}, \tag{25}$$

in which the quantity $\tilde{N}$ is defined as

$$\tilde{N} = \left(\frac{2\alpha|V_{02}|}{\kappa M_p}\right)N + \left[\frac{M_pV_{01}e^{2\alpha\phi_e/M_p}}{2\alpha}+V_{02}\phi_e\right],$$

where the scalar field at the end of inflation ϕ_e is given by Eq.(22). Besides, the ProductLog function corresponds to the product logarithm, also called the Omega function or Lambert W function. This function is a multivalued function, and it satisfy the converse relation of the function $f(w) = we^w$, see Ref.[24]

On the other hand, the power spectrum $\mathcal{P}_R$ of the curvature perturbations, in the context of the slow roll approximation becomes[25]

$$\mathcal{P}_R = \left(\frac{H^2}{\dot{\phi}^2}\right)\left(\frac{H}{2\pi}\right)^2 \simeq \frac{\kappa^3}{12\pi^2}\left[\frac{V_{eff}^3}{(\partial V_{eff}/\partial\phi)^2}\right], \tag{26}$$

and the scalar spectral index n_s results [25]

$$n_s - 1 = -6\epsilon + 2\eta, \tag{27}$$

and the tensor to scalar ratio r is defined as [25]

$$r = \frac{8}{\kappa}\left[\frac{\partial V_{eff}/\partial\phi}{V_{eff}}\right]^2. \tag{28}$$

From Eq.(26) we find that the scalar power spectrum results

$$\mathcal{P}_R \simeq k_1\, e^{4\alpha\phi/M_{\rm P}}\left[V_{01} + V_{02}e^{-2\alpha\phi/M_{\rm P}}\right]^3, \tag{29}$$

where the constant k_1 is defined as

$$k_1 = \frac{\kappa^3 M_p^2}{48\pi^2\alpha^2 V_{02}^2}.$$

Also, we determine that the scalar spectral index in terms of the scalar field from Eq.(27) results

$$n_s - 1 = -\left[\frac{4\alpha^2|V_{02}|}{\kappa M_p^2}\right](|V_{02}| + 2V_{01}e^{2\alpha\phi/M_p})\,(V_{01} + V_{02}e^{2\alpha\phi/M_p})^{-2}. \tag{30}$$

From Eq.(28) we find that the tensor to scalar ratio r as a function of the scalar field becomes

$$r = r_0\, e^{-4\alpha\phi/M_p}\,(V_{01} + V_{02}e^{2\alpha\phi/M_p})^{-2}, \tag{31}$$

where the constant r_0 is defined as

$$r_0 = \frac{32\alpha^2 V_{02}^2}{\kappa M_p^2}.$$

In particular assuming the value of the parameter $\alpha = 1$ together with the cosmological parameters $\mathcal{P}_R = 2.2 \times 10^{-9}$ and $n_s = 0.968$ at $N = 60$ (see [26]), then numerically we obtain that the parameters V_{01} and V_{02} result

$$V_{01} \simeq 4,034 \times 10^{-4} M_p^4, \quad \text{and} \quad |V_{02}| = 5,286 \times 10^{-5} M_p^4, \tag{32}$$

respectively. In addition, using these values, we find that from these values the tensor to scalar ratio r at $N = 60$, becomes $r \simeq 0.01299$.

In relation to the original parameters, we note that by substituting the constraints given by Eq.(32) into Eq. (20), we find that the parameter M, obtained from the condition (4), can be expressed in terms of the parameters V_1 and b as follows:

$$M \simeq \left(\frac{6.3 \times 10^{-3} V_1}{\frac{b}{125 V_1} - 1} \right)^{1/4}, \tag{33}$$

with the condition $b > 125V_1$ to ensure a real solution for the parameter M. Besides, we note that for the special situation in which the ratio $b/125V_1 \gg 1$, the parameter M becomes $M \simeq V_1^{1/2} b^{-1/4}$. Additionally, we find that the equation for V_2 in terms of V_1 and b is given by

$$V_2 \simeq bV_1 - 625V_1^2. \tag{34}$$

Here we need to consider that $b > 625V_1$ to obtain a positive value of the parameter V_2. Thus, this condition on the parameter b is more strong that the condition $b > 125V_1$ obtained from Eq.(33) to find a real solution of the parameter M. As before, here we have considered Eqs.(20) and (32), respectively. In addition, we note that for the case in which the parameter $V_1 \ll 1$, we have that $V_2 \sim bV_1$.

After of the inflationary stage, the scalar field oscillates above the minimum of the effective potential and during this process occurs the particles production and in particular the fermions. In particular, we will look at the effect of the fermions as they get diluted and how they affect the dark energy in the late universe.

In the following, we will analyze the effect of the fermions as dark energy and dark matter.

IV. DARK ENERGY FROM FERMIONS

In this section we will study the accelerated expansion of the present universe produced from the effective potential associated to the fermions generated during the inflationary stage.

For dust, the neutrino has zero momenta, and then we have $\overline{n}'n' = u^{\dagger}u$ where u corresponds to the large component of the Dirac spinor n'. From the space components of the 4-current $\tilde{e}_a^{\mu}\overline{n}'\gamma^a n'$ equal zero and assuming the 4-current conservation we have $\overline{n}'n' = u^{\dagger}u = \frac{const}{a^3}$ where $a = a(t)$ as before is the scale factor. Thus, from Eq. (13) we have

$$\frac{(b-\chi)\left[M^4 e^{-2\alpha\phi/M_p} + V_1\right] - 2V_2}{(\chi+b)^{3/2}} = \frac{(\chi-\chi_1^{(N)})(\chi-\chi_2^{(N)})}{(\chi+k)^2}\mu_N \frac{const}{a^3}. \tag{35}$$

We note that there is a solution where the decaying fermion contribution $u^{\dagger}u \sim \frac{const}{a^3}$ is not trivial to the right hand side of Eq.(35). In this sense, to generate the dark energy from the fermions, we can consider the asymptotic behavior $\chi \to -k$ in such a way that $(\chi+k)^{-2} \propto a^3$, then the r.h.s. of (35) became a constant. We note that the described regime denotes a very unexpected state of the primordial fermions.

To obtain the total effective potential $U_{eff}(\phi)$ in terms of the scalar field associated with fermions during the current universe, we have the following

$$U_{eff} = V_{eff} - \Lambda_{dyn}^{(ferm)} \simeq \frac{V_2 - kV_1}{(b-k)^2} - \frac{k}{(b-k)^2}M^4 e^{-2\alpha\phi/M_p} + \mathcal{O}(\chi+k). \tag{36}$$

Here we have utilized Eqs.(8) and (13) together with $\chi \to -k$. Notice that this vacuum energy can be very small if $-k$ is very big.

We note from this total effective potential U_{eff} we have a modified flat region for positive large ϕ associated to dark energy at the present given by

$$U_{eff}(\phi \to \infty) = U_{DE} \simeq \frac{V_2 - kV_1}{(b-k)^2}. \tag{37}$$

More exact version of the above equations 36, 37, for any value of the scalar field is

The *remarkable result* consists in the fact that

$$V^{(0)}_{eff}(\phi) - U^{(tot)}_{eff}(\phi) \equiv \frac{\left[(b+k)\left(V_1 + M^4 e^{-2\alpha\phi/M_p}\right) - 2V_2\right]^2}{4(b-k)^2\left[b\left(V_1 + M^4 e^{-2\alpha\phi/M_p}\right) - V_2\right]} > 0. \qquad (61)$$

where the conditions (34) and (38) have been used. This means that *the universe with the gas of uniformly distributed non-relativistic neutrinos in the CLEP state is energetically more preferable than the one in the absence of fermions case.*

Which means that after the reheating the vacuum energy has been lowered

- This is a consistency condition, since the difference of vacuum energies has been used for Dark Matter production, etc.

n addition, we can define the density parameter associated to the dark energy $\Omega_{DE} = \kappa\rho_{DE}/3H^2 \simeq \kappa U_{eff}/3H^2$. Thus, evaluating the parameter Ω_{DE} at the present time i.e., Ω_{DE_0}, we find

$$\Omega_{DE_0} \simeq \frac{\kappa}{3H_0^2}\left(\frac{V_2 - kV_1}{(b-k)^2}\right). \tag{38}$$

By considering that the Hubble parameter at the present time, $H_0 \approx 10^{-61} M_p$, we obtain the constraint

$$\frac{V_2 - kV_1}{(b-k)^2} \approx 2 \times 10^{-122}\frac{M_P^2}{\kappa}, \tag{39}$$

where we have used that at the present the parameter $\Omega_{DE_0} \simeq 0.7$.

where we have used that at the present the parameter $\Omega_{DE_0} \simeq 0.7$.

Besides, for very large values of k and considering Eq.(37), we can obtain an approximation for the parameter k and it becomes of order

$$|k| \sim \frac{V_1}{U_{DE}} \sim 10^{119}. \tag{40}$$

Here, we have assumed that $V_1 \sim V_2$, $b \sim \mathcal{O}(1)$, and, from Eq.(32) $V_{01} \sim V_1 \sim 10^{-4} M_p^4$, obtained during the inflationary epoch, together with the value of the dark energy at the present epoch, $U_{DE} \sim 10^{-123} M_p^4$.

V. DARK MATTER FROM FERMIONS

In this section, we will analyze the dark matter associated with the high-density fermions (localized) in our present universe generated during the inflationary epoch.

From the constraint equation given by Eq.(13), the right hand size is proportional to the fermions densities. In this context, $\Lambda_{dyn}^{(ferm)}$ given by Eq.(10) becomes very large unless the coefficient Z_i defined by Eq.(11) goes to zero. Thus, to obtain the coefficient $Z_i \to 0$, we need to consider that $\chi \to \chi_{1,2}^{(i)}$ where $\chi_{1,2}^{(i)}$ are defined by Eq.(12).

By considering that in the studied regime masses (11) of the non-relativistic primordial fermions are very large, we can neglect their kinetic energy. Thus, after averaging on the microscopic contribution to the energy-momentum tensor $T_{\mu\nu}^{(n,can)}$ we can written [22]

$$< T_{\mu\nu}^{(n',can)} >_{cosm.av.} \simeq \delta_\mu^0 \delta_\nu^0 \frac{h_n - k}{(b-k)^{1/2}} \mu_n \frac{c}{(\chi + k)a^3}, \tag{41}$$

where c is a constant. In order to obtain that the coefficient $Z_i \to 0$ and considering only the neutrinos ($i = n'$), in which $\chi \to \chi_{1,2}^{n'}$ and then we have that the canonical energy momentum tensor for fermions n' results

$$< T_{\mu\nu}^{(n',can)} >_{cosm.av.} \simeq \delta_{\mu}^{0}\delta_{\nu}^{0}\frac{h_n - k}{(b-k)^{1/2}}\mu_n \frac{c_{1,2}}{(\chi_{1,2}^{n'} + k)a^3} \propto \frac{1}{a^3}. \tag{42}$$

Here we note that there are two cases for the canonical energy momentum associated to the coefficients $\chi_{1,2}^{n'}$ (solutions) defined by Eq.(12).

Thus, for the first solution $\chi_1^{n'}$ from Eq.(42) we find that $< T_{\mu\nu}^{(n',can)} >_{cosm.av.}$ becomes

$$< T_{\mu\nu}^{(n',can)} >_{cosm.av.} \simeq \delta_{\mu}^{0}\delta_{\nu}^{0}\frac{h_n - k}{(b-k)^{1/2}}\mu_n \frac{c_{1}}{(\chi_{1}^{n'} + k)a^3} \propto \frac{1}{a^3}. \tag{43}$$

In the case when consider the second solution for the coefficient $\chi_2^{n'}$ we have that the canonical energy momentum tensor for fermions is given by

$$< T_{\mu\nu}^{(n',can)} >_{cosm.av.} \simeq \delta_{\mu}^{0}\delta_{\nu}^{0}\frac{h_n - k}{(b-k)^{1/2}}\mu_n \frac{c_{2}}{(\chi_{2}^{n'} + k)a^3} \propto \frac{1}{a^3}. \tag{44}$$

In this context, we consider the energy density related to dark matter to be defined as $\rho_{DM} =< T_{00}^{(n',can)(1)} >_{cosm.av.} + < T_{00}^{(n',can)(2)} >_{cosm.av.}$. In this sense, we find that the energy density associated to dark matter for large-k can be written as

$$\rho_{DM} \simeq \frac{h_n - k}{(b-k)^{1/2}}\mu_n \left[\frac{c_1}{(\chi_1^{n'} + k)} + \frac{c_2}{(\chi_2^{n'} + k)}\right] \frac{1}{a^3} \propto \frac{1}{a^3}. \tag{45}$$

Defining the density parameter related to the dark matter $\Omega_{DM} = \kappa\rho_{DM}/3H^2$, then we find that at the present time the density parameter becomes

$$\Omega_{DM_0} = \kappa\frac{h_n - k}{3H_0^2(b-k)^{1/2}}\mu_n\left[\frac{c_1}{(\chi_1^{n'}+k)} + \frac{c_2}{(\chi_2^{n'}+k)}\right], \tag{46}$$

where we have used that scale factor at present $a_0 = 1$. Now considering the density parameter $\Omega_{DM_0} \simeq 0.3$ and the Hubble parameter in the present $H_0 \approx 10^{-61}M_p$, we have

$$\frac{h_n - k}{(b-k)^{1/2}}\mu_n\left[\frac{c_1}{(\chi_1^{n'}+k)} + \frac{c_2}{(\chi_2^{n'}+k)}\right] = \frac{3H_0^2\Omega_{DM_0}}{\kappa} \approx \left(\frac{10^{-122}M_p^2}{\kappa}\right), \tag{47}$$

here we have utilized the value of k given by Eq.(40).

By assuming that the parameter k is very large and the mass parameter μ_n negative, we find that the constraint given by Eq.(47) is reduced to

$$|\mu_n|\left[\frac{c_1}{2} + c_2\right] \approx \left(\frac{10^{-62}M_p^2}{\kappa}\right). \tag{48}$$

Here we have used that for large-k, the first solution of Eq.(12) corresponds to $\chi_1^{n'} \to k$ and the second solution $\chi_2^{n'} \to 0$. In addition, we have considered the valor of k found in Eq.(40).

VI. DISCUSSION ON POSSIBLE FURTHER DEVELOPMENTS

In this paper we have discussed a unified picture of inflation and the late DE and DM period of the universe based on a scalar field together with the creation of fermions that in a model with a dilaton field and spontaneously broken scale invariance in the context of a Two Measures Theory.

During the inflationary stage the scalar field produces the accelerated expansion of the universe before the production of exotic fermions. In inflation we have found different constraints related to the effective potential of the scalar field in absence of fermions from the observational parameters.

In the case of the late universe, the fermions, when diluted can have exotic behavior and can substantially modified the scalar field potential, reducing the very large inflation to the very low DE in the late universe. In order to describe the late universe, we have considered that after reheating the scalar field retraces its trajectory in field space but now the scalar field potential can be drastically modified by the effect of the fermions. From the noncanonical contribution of the fermions associated to the energy momentum tensor we have obtained a part of the dark energy responsible of the present accelerated universe. The other contribution to the dark energy is related to the scalar field contribution which is also modified by the fermions because of the solution of the constraint equation given by Eq.(35). We have considered that the present dark energy with its very small value in comparison to the inflationary phase which can be adjusted by choosing appropriately the parameter space of couplings of the Riemannian and non Riemannian measures to the fermions.

To generate the dark matter we have considered the canonical contribution of fermions related to energy momentum tensor. In these clumped regions, we have that the matter is more dense. For the dark matter we have assumed two solutions associated to the constraint $\chi \to \chi_{1,2}^{n'}$ that contributes to the energy density of the dark matter ρ_{DM}. In the limit in which the parameter k is very large, we have obtained a constraint on the mass parameter and the coefficients c_1 and c_2 from the present density parameter Ω_{DM_0}, see Eq.(48).

Of course in the future, there are many issues to be studied, for example, what are these fermions?, may be sterile neutrinos?. How this theory works for structure formation?. Since the final value of the DE takes some time as the universe needs to expand so that the asymptotic value of the DE sets in, does this evolution of the DE helps to explain the H_0 tension through some kind of Early DE scenario?.

Appendix A: Connection in the original and Einstein frames

In this appendix, we show the dependence of the spin connection ω_{μ}^{ab} on e_{μ}^{a}, χ, Ψ and $\overline{\Psi}$. Thus, varying the Eq.(1) with respect to ω_{μ}^{ab} and considering

$$R(V,\omega) \equiv -\frac{1}{4\sqrt{-g}}\varepsilon^{\mu\nu\alpha\beta}\varepsilon_{abcd}e_{\alpha}^{c}e_{\beta}^{d}R_{\mu\nu}^{ab}(\omega) \tag{A1}$$

we find

$$\varepsilon^{\mu\nu\alpha\beta}\varepsilon_{abcd}\left[(\chi+b)e_{\alpha}^{c}D_{\nu}e_{\beta}^{d}+\frac{1}{2}e_{\alpha}^{c}e_{\beta}^{d}\left(\chi_{,\nu}+\frac{\alpha}{M_{p}}\phi_{,\nu}\right)\right]+\frac{\kappa}{4}\sqrt{-g}(\chi+k)e^{c\mu}\varepsilon_{abcd}\overline{\Psi}\gamma^{5}\gamma^{d}\Psi=0, \tag{A2}$$

in which

$$D_{\nu}e_{a\beta} \equiv \partial_{\nu}e_{a\beta}+\omega_{\nu a}^{d}e_{d\beta}. \tag{A3}$$

The solution of Eq. (A2) can be written as

$$\omega_{\mu}^{ab} = \omega_{\mu}^{ab}(e)+K_{\mu}^{ab}(e,\overline{\Psi},\Psi)+K_{\mu}^{ab}(\chi,\phi), \tag{A4}$$

where

$$\omega_{\mu}^{ab}(e) = e_{\alpha}^{a}e^{b\nu}\{^{\alpha}_{\mu\nu}\}-e^{b\nu}\partial_{\mu}e_{\nu}^{a}, \tag{A5}$$

and it corresponds to the Riemannian part of the spin-connection. Besides, the quantity

$$K_{\mu}^{ab}(e,\overline{\Psi},\Psi) = \frac{\kappa}{8}\frac{\chi+k}{\chi+b}\eta_{cn}e_{d\mu}\varepsilon^{abcd}\overline{\Psi}\gamma^{5}\gamma^{n}\Psi, \tag{A6}$$

denoted the fermionic contribution that differs from the standard expression [27] by the factor $\frac{\chi+k}{\chi+b_g}$. Also, the parameter $K_{\mu}^{ab}(\chi,\phi)$ is defined as

$$K_{\mu}^{ab}(\chi,\phi) = \frac{1}{2(\chi+b)}\left(\chi_{,\alpha}+\frac{\alpha}{M_{p}}\phi_{,\alpha}\right)(e_{\mu}^{a}e^{b\alpha}-e_{\mu}^{b}e^{a\alpha}), \tag{A7}$$

and it corresponds to the non-Riemannian part of the spin-connection obtained by specific features of TMT.

In the Einstein frame, we determine that the variables defined by Eq.(5), the spin-connection read

$$\omega_{\mu}^{\prime ab} = \omega_{\mu}^{ab}(\tilde{e})+\frac{\kappa}{8}\eta_{cn}\tilde{e}_{d\mu}\varepsilon^{abcd}\overline{\Psi}^{\prime}\gamma^{5}\gamma^{n}\Psi^{\prime}, \tag{A8}$$

which is equivalent to the spin-connection of the Einstein-Cartan space-time[27] with the vierbein $\tilde{e}_{\mu}^{a}$. In the solution of this paper, the last term given by Eq.(A8) is zero.

Some references on Avoidance of 5th force versus vacuum energy from matter in Modified Measure Theory

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