Entanglement in Curved Spaces

Nikolaos Tetradis

University of Athens

Work with K. Boutivas, D. Katsinis, G. Pastras

Introduction

- Quantum entanglement denotes a phenomenon in which particles (like photons or electrons) become interconnected, sharing a single quantum state regardless of the distance separating them.
- Measuring a property of one particle instantly determines the state of the other.
- The 2022 Nobel Prize in Physics recognized experiments that confirmed this phenomenon, rejecting the idea of "hidden variables". (Alain Aspect, John F. Clauser, Anton Zeilinger)
- Entanglement is essential for building multi-qubit quantum gates and quantum computers. Important applications in cryptography and quantum sensing.
- Entanglement entropy is a quantitative measure of the quantum entanglement between two parts of a composite system.It tracks the information lost or the ”uncertainty” that arises when you can only observe one part of an entangled pair.

- Consider a quantum mechanical system with many degrees of freedom, such as a spin chain or a quantum field.
- Assume it is in the ground state $|\Psi\rangle$, which is a pure state.
- The density matrix of the total system is $\rho_{\rm tot} = |\Psi\rangle\langle\Psi|$.
- The von Neumann entropy $S_{\rm tot} = -{\rm tr}\rho_{\rm tot} \log \rho_{\rm tot}$ vanishes.
- Now divide the total system into subsystems A and B and assume that B is inaccessible to A.
- Trace out the part B of the Hilbert space in order to obtain the reduced density matrix of A: $\rho_A = {\rm tr}_B \rho_{\rm tot}$.
- The entropy $S_A = -{\rm tr}_A \rho_A \log \rho_A$ is a measure of the entanglement between A and B.
- It is nonvanishing and $S_A = S_B$.

- For spatially separated systems in the ground state in a static background, the leading contribution is proportional to the area of the entangling surface between A and B:

$$\mathrm{S_A} \sim \frac{\partial \mathrm{A}}{\epsilon^{\mathrm{d}-1}} + \text{subleading terms.}$$

- Massless scalar field in 3+1 dimensions (spherical entangling surface):

$$\mathrm{S_A} = \mathrm{s}\,(\mathrm{R}/\epsilon)^2 + \mathrm{a}\,\log(\mathrm{R}/\epsilon) + \mathrm{d}$$

$\mathrm{s} \simeq 0.3$ (scheme-dependent) (Srednicki 1993)
$\mathrm{a} = -\frac{1}{90}$ (universal)
(Solodukhin 2008, Lohmayer et al. 2009)

- Conformal field theory in 1+1 dimensions, with central charge c. System of length L, divided into pieces of lengths ℓ and $\mathrm{L} - \ell$:

$$\mathrm{S_A} = \frac{\mathrm{c}}{6}\log\left(\frac{2\mathrm{L}}{\pi\epsilon}\sin\frac{\pi\ell}{\mathrm{L}}\right) + \bar{\mathrm{c}}_1'$$

$\bar{\mathrm{c}}_1'$ scheme-dependent. (Korepin 2004, Calabrese, Cardy 2004)

- Entanglement entropy in curved backgrounds
  de Sitter space (Maldacena, Pimentel 2013)
- Relation between entanglement entropy and c-, F- and a-functions
  $1+1$ d: $\mathrm{R}\,\Delta\mathrm{S}'(\mathrm{R}) \equiv$ c-function. (Casini, Huerta 2012)
  $3+1$ d: $\mathrm{R}^2\,\Delta\mathrm{S}'' - \mathrm{R}\,\Delta\mathrm{S}' \equiv$ a-function. (Casini, Teste, Torroba 2017)

References

Entanglement and expansion

- K. Boutivas, G. Pastras and N. Tetradis
  "Entanglement and expansion"
  arXiv:2302.14666 [hep-th], JHEP 05 (2023) 199
- K. Boutivas, D. Katsinis, G. Pastras and N. Tetradis
  "Entanglement in Cosmology"
  arXiv:2310.17208 [gr-qc], JCAP 04 (2024) 017

Entanglement of squeezed states

- D. Katsinis, G. Pastras and N. Tetradis
  "Entanglement of Harmonic Systems in Squeezed States"
  arXiv:2304.04241 [hep-th], JHEP 10 (2023) 039
- D. Katsinis, G. Pastras and N. Tetradis
  "Entanglement Entropy of a Scalar Field in a Squeezed State"
  arXiv:2403.03136 [hep-th], JHEP 10 (2024) 173

Entanglement in de Sitter space

- K. Boutivas, D. Katsinis, G. Pastras and N. Tetradis
  "A Numerical Calculation of Entanglement Entropy in dS Space"
  arXiv:2407.07811 [hep-th], Phys. Rev. D 111 (2025) 6, 065010
- K. Boutivas, D. Katsinis, G. Pastras and N. Tetradis
  "Entanglement Entropy as a Probe Beyond the Horizon"
  arXiv:2407.07824 [hep-th], Phys. Rev. Lett. 135 (2025) 6, 061602

Entanglement in the Einstein universe

- K. Boutivas, D. Katsinis, G. Pastras, N. Prountzos and N. Tetradis
  "Entanglement on a Sphere"
  arXiv:2507.01174 [hep-th], Phys. Rev. D 112 (2025) 6, 065020

Entanglement in Anti-de Sitter space

- K. Boutivas, D. Katsinis, I. Papadimitriou, G. Pastras and N. Tetradis
  "Entanglement Entropy of a Scalar Field in Anti-de Sitter Space"
  arXiv:2505.15980 [hep-th], JHEP 11 (2025) 064

Computing c- and a-functions from entanglement

- K. Boutivas, D. Katsinis, G. Pastras and N. Tetradis
  "Computing c- and a-functions from Entanglement"
  arXiv:2509.03259 [hep-th], Phys. Rev. D 113 (2026) 10, 105026

Plan

- Quantum fields in cosmology
- Entanglement entropy in an expanding background
- Entanglement entropy in de Sitter space
- Various curved backgrounds
- c- and a-functions through entanglement
- Prospects

Expanding the field in momentum modes

- Consider a free scalar field $\phi(\tau, x)$ in a FRW background

$$ds^2 = a^2(\tau)\left(d\tau^2 - dr^2 - r^2 d\Omega^2\right).$$

- With the definition $\phi(\tau, x) = f(\tau, x)/a(\tau)$, the action becomes

$$S = \frac{1}{2}\int d\tau\, d^3x \left(f'^2 - (\nabla f)^2 + \left(\frac{a''}{a} - a^2 m^2\right) f^2\right).$$

  The field $f(\tau, x)$ has a canonically normalized kinetic term.
- For de Sitter: $a(\tau) = -1/(H\tau)$ with $-\infty < \tau < 0$, and

$$S = \frac{1}{2}\int d\tau\, d^3x \left(f'^2 - (\nabla f)^2 + \frac{2\kappa}{\tau^2} f^2\right),$$

  where $\kappa = 1 - m^2/2H^2$.

- The eom (Mukhanov-Sasaki equation) in Fourier space is

$$\mathrm{f}_\mathrm{k}'' + \mathrm{k}^2 \mathrm{f}_\mathrm{k} - \frac{2\kappa}{\tau^2} \mathrm{f}_\mathrm{k} = 0.$$

- Its general solution is

$$\mathrm{f}_\mathrm{k}(\tau) = \mathrm{A}_1 \sqrt{-\tau}\, \mathrm{J}_\nu\left(-\mathrm{k}\tau\right) + \mathrm{A}_2 \sqrt{-\tau}\, \mathrm{Y}_\nu\left(-\mathrm{k}\tau\right) \qquad \nu = \frac{1}{2}\sqrt{1+8\kappa}.$$

- Bunch-Davies vacuum: $\mathrm{A}_1 = -\frac{\sqrt{\pi}}{2}$, $\mathrm{A}_2 = -\frac{\sqrt{\pi}}{2}\mathrm{i}$. For $\tau \to -\infty$

$$\mathrm{f}_\mathrm{k}(\tau) \simeq \frac{1}{\sqrt{2\mathrm{k}}} \mathrm{e}^{-\mathrm{i}\mathrm{k}\tau}.$$

- For $\kappa = 1$ (massless scalar), the full solution reads

$$\mathrm{f}_\mathrm{k}(\tau) = \frac{1}{\sqrt{2\mathrm{k}}} \mathrm{e}^{-\mathrm{i}\mathrm{k}\tau} \left(1 - \frac{\mathrm{i}}{\mathrm{k}\tau}\right).$$

For $\mathrm{k}\tau \to 0^-$ the mode becomes superhorizon and the oscillations stop. The mode freezes.

- The quantum field can be expressed as

$$\hat{f}(\tau, x) = \int \frac{d^3k}{(2\pi)^{3/2}} \left[f_k(\tau)\hat{a}_k + f_k^*(\tau)\hat{a}_k^\dagger \right] e^{ik\cdot x}$$

where $\hat{a}_k^\dagger$, $\hat{a}_k$ are standard creation and annihilation operators.

- For superhorizon modes with $k\tau \to 0^-$ the growing term dominates and

$$\hat{\pi}(\tau, x) \simeq -\frac{1}{\tau}\hat{f}(\tau, x).$$

- The field and its conjugate momentum commute.
- For most of its properties the field can be viewed as a classical stochastic field.
- However, the full quantum field and its conjugate always obey the canonical commutation relation.
- The quantum nature of the field is visible in the entanglement entropy.

Expanding the field in coordinate space

- For spherical entangling surfaces, expand in spherical harmonics.
- Discretize the radial coordinate as $r_j = j\epsilon$, $1 \leq j \leq N$.
- UV cutoff: $1/\epsilon$. IR cutoff: $1/L$ with $L = N\epsilon$.
- Trace out the oscillators with $j\epsilon > R$.
- ϵ, L, R are comoving scales.
- The 'ground state' of the system is the product of the 'ground states' of the modes that diagonalize the Hamiltonian.
- The discretized Hamiltonian for the free field during inflation is

$$H = \frac{1}{2\epsilon} \sum_{\ell,m} \sum_{j=1}^{N} \left[\tilde{\pi}^2_{\ell m,j} + \left(\omega^2_{\ell m,j} - \frac{2\kappa}{(\tau/\epsilon)^2} \right) \tilde{f}^2_{\ell m,j} \right],$$

where $\tilde{f}_{\ell m,j}$ are the canonical modes.

- In the Bunch-Davies vacuum, the 'ground state' is the solution of the Schrödinger equation that reduces to the usual simple harmonic oscillator ground state as $\tau \to -\infty$.

de Sitter era

- Oscillator with time-dependent frequency

$$\omega^2(\tau) = \omega_0^2 - \frac{2\kappa}{\tau^2}.$$

- Find the general solution of the Ermakov equation

$$\mathrm{b}''(\tau) + \omega^2(\tau)\mathrm{b}(\tau) = \frac{\omega_0^2}{\mathrm{b}^3(\tau)}.$$

- For the Bunch-Davies vacuum, $\mathrm{b}(\tau)$ must tend to 1 as $\tau \to -\infty$.

$$\mathrm{b}^2(\tau) = -\frac{\pi}{2}\omega_0\tau \left(\mathrm{J}_\nu^2\left(-\omega_0\tau\right) + \mathrm{Y}_\nu^2\left(-\omega_0\tau\right)\right).$$

- The solution of the Schrödinger equation can now be expressed as

$$\mathrm{F}(\tau, \mathrm{f}) = \frac{1}{\sqrt{\mathrm{b}(\tau)}} \exp\left(\frac{i}{2}\frac{\mathrm{b}'(\tau)}{\mathrm{b}(\tau)}\mathrm{f}^2\right) \mathrm{F}^0\left(\int \frac{d\tau}{\mathrm{b}^2(\tau)}, \frac{\mathrm{f}}{\mathrm{b}(\tau)}\right),$$

  where $\mathrm{F}^0(\tau, \mathrm{f})$ is a solution with constant frequency ω_0.
- For $\kappa > 0$ and $\tau \to 0^-$, we have $\Delta \mathrm{f}/\Delta\pi \to 0$: Squeezed state.

Transition from dS with $a(\tau) \sim -1/\tau$ to radiation domination (RD) with $a(\tau) \sim \tau$

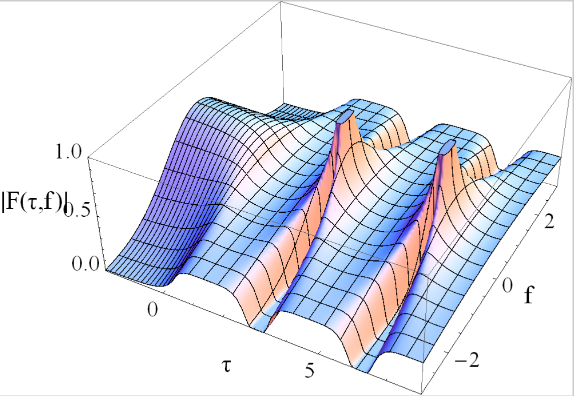

Figure: Left plot: The amplitude of the 'ground-state' wave function for the transition from a dS to a RD background at $\tau = 0.5$, for $\omega_0 = 1$, H = 2.

Transition from dS with $a(\tau) \sim -1/\tau$ to matter domination (MD) with $a(\tau) \sim \tau^2$

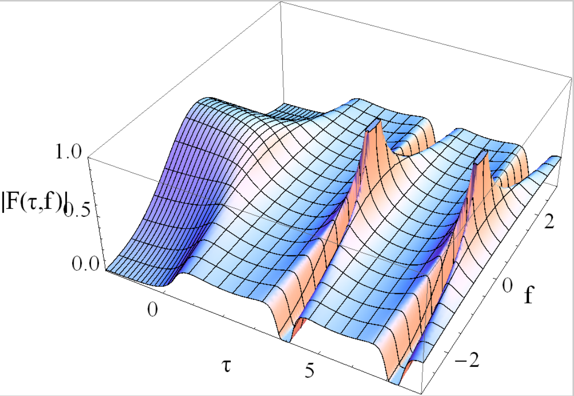

Figure: Left plot: The amplitude of the 'ground-state' wave function for the transition from a dS to a MD background at $\tau = 0.5$, for $\omega_0 = 1$, $H = 2$.

Entanglement entropy of two quantum oscillators

- Hamiltonian (we switched from f to x)

$$H = \frac{1}{2}\left[p_1^2 + p_2^2 + k_0(x_1^2 + x_2^2) + k_1(x_1 - x_2)^2 - \lambda(\tau)(x_1^2 + x_2^2)\right].$$

- The Hamiltonian can be rewritten as

$$H = \frac{1}{2}\left[p_+^2 + p_-^2 + w_+^2(\tau)x_+^2 + w_-^2(\tau)x_-^2\right],$$

$x_\pm = \frac{x_1 \pm x_2}{\sqrt{2}}$, $\omega_{0+}^2 = k_0, \omega_{0-}^2 = k_0 + 2k_1$, $w_\pm^2(\tau) = \omega_{0\pm}^2 - \lambda(\tau)$.

- The 'ground state' is the tensor product of the 'ground states' of the two decoupled canonical modes:

$$\psi_0(x_+, x_-) = \left(\frac{\Omega_+\Omega_-}{\pi^2}\right)^{\frac{1}{4}} \exp\left[-\frac{1}{2}\left(\Omega_+x_+^2 + \Omega_-x_-^2\right) + \frac{i}{2}\left(G_+x_+^2 + G_-x_-^2\right)\right.$$

$$\Omega_\pm(\tau) \equiv \frac{\omega_{0\pm}}{b^2(\tau;\omega_{0\pm})}, \quad G_\pm(\tau) \equiv \frac{b'(\tau;\omega_{0\pm})}{b(\tau;\omega_{0\pm})}.$$

- Express the wave function in terms of x_1, x_2.
- The reduced density matrix is given by

$$\rho(x_2, x_2') = \int_{-\infty}^{+\infty} dx_1 \psi_0(x_1, x_2)\psi_0^*(x_1, x_2').$$

- The Gaussian integration gives

$$\rho(x_2, x_2') = \sqrt{\frac{\gamma - \beta}{\pi}} \exp\left(-\frac{\gamma}{2}(x_2^2 + x_2'^2) + \beta x_2 x_2'\right) \exp\left(i\frac{\delta}{2}(x_2^2 - x_2'^2)\right),$$

where γ, β, δ are functions of $\Omega_\pm$, $G_\pm$.
- The eigenfunctions of the reduced density matrix satisfy

$$\int_{-\infty}^{+\infty} dx_2' \rho(x_2, x_2') f_n(x_2') = p_n f_n(x_2).$$

- One finds

$$f_n(x) = H_n(\sqrt{\alpha}x) \exp\left(-\frac{\alpha}{2}x^2\right) \exp\left(i\frac{\delta}{2}x^2\right),$$

where $\alpha = \sqrt{\gamma^2 - \beta^2}$ and H_n is a Hermite polynomial.

- The eigenvalues p_n are

$$p_n = \sqrt{\frac{2(\gamma-\beta)}{\gamma+\alpha}}\left(\frac{\beta}{\gamma+\alpha}\right)^n = (1-\xi)\xi^n,$$

where

$$\xi = \frac{\beta}{\gamma+\alpha}.$$

- The entanglement entropy can be calculated as

$$S = -\sum_{n=0}^{\infty}(1-\xi)\xi^n \log\left[(1-\xi)\xi^n\right] = -\log(1-\xi) - \frac{\xi}{1-\xi}\log\xi.$$

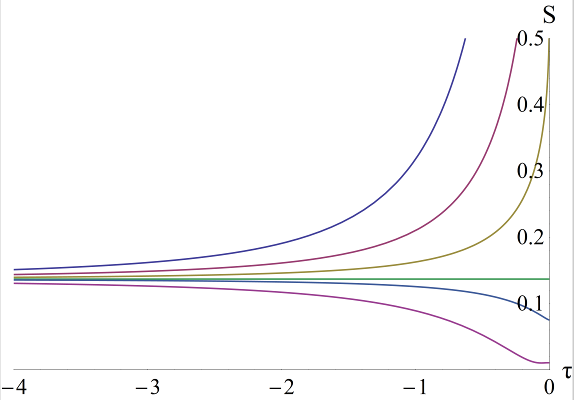

Figure: Left plot: The entanglement entropy in a dS background as a function of conformal time τ for $\omega_+ = 1$, $\omega_- = 2$ and κ =1, 0.5, 0.2, 0, −0.1, −0.5 (from top to bottom), where $\kappa = 1 - \mathrm{m}^2/2\mathrm{H}^2$.

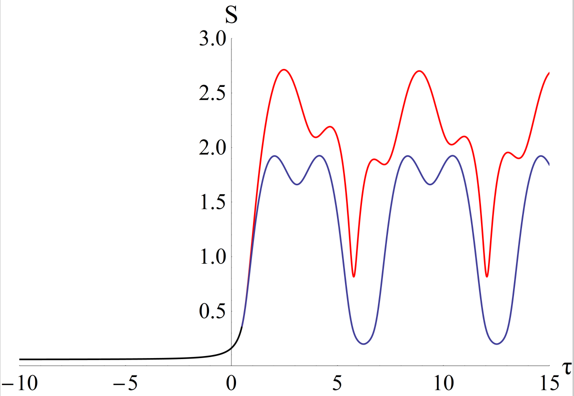

Figure: Left plot: The entanglement entropy as a function of conformal time τ for $\omega_+ = 1$, $\omega_- = 1.5$, H = 2 and $\tau_0 = 0.5$. The black line corresponds to a dS background, with a transition at τ_0 to either a RD era (blue line) or to a MD era (red line).

Entanglement entropy of many quantum oscillators

- Consider N coupled oscillators with a Hamiltonian

$$H = \frac{1}{2}\sum_{i}^{N} p_i^2 + \frac{1}{2}\sum_{i,j=1}^{N} x_i K_{ij} x_j .$$

- Diagonalize $K = O^T K_D O$ through an orthogonal matrix O. $\Omega_D = K_D^{1/2}$ contains the eigenfrequencies of the corresponding canonical modes $\tilde{X} = OX$.
- The wave function of the system is the product of the wave functions of the canonical modes.
- The system is assumed to lie in the ground state of each canonical mode in the asymptotic past (Bunch-Davies vacuum).
- At late times this becomes a squeezed state.

- When the density matrix is expressed in terms of the original coordinates x_i it has the form

$$\rho(X, X') \sim \mathsf{exp}\left[-\frac{1}{2}\left(X^T W X + X'^T W^* X'\right)\right].$$

- One can introduce the block notation

$$W = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}, \qquad X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

where A is a complex symmetric $n \times n$ matrix, C a complex symmetric $(N-n) \times (N-n)$ matrix, X_1 an n-dimensional vector, and so on.

- Oscillators 1 to n comprise the subsystem A, while oscillators $n+1$ to N comprise the complementary subsystem C. By computing the density matrix and tracing out the subsystem C, one can calculate the entanglement entropy of the subsystem A through the reduced density matrix.

- When the $N-n$ oscillators in X_2 are traced out, the reduced density matrix is

$$\rho_1(X_1,X_1') \sim \exp\left(-\frac{1}{2}X_1^T\gamma\, X_1 - \frac{1}{2}X_1'^T\gamma\, X_1' + X_1'^T\beta\, X_1 + \frac{i}{2}X_1^T\delta\, X_1 - \frac{i}{2}X_1'^T\delta\, X_1'\right),$$

where

$$\gamma - i\delta = A - \frac{1}{2}B\,\mathrm{Re}\,(C)^{-1}\,B^T,$$
$$\beta = \frac{1}{2}B^*\mathrm{Re}\,(C)^{-1}\,B^T.$$

- γ and δ are $n\times n$ real symmetric matrices, while β is a $n\times n$ Hermitian matrix. (In a static background it would be real symmetric.)
- The eigenvalues of the density matrix do not depend on δ.

- In a static background, the matrices γ and β can be diagonalized through real orthogonal transformations in order to identify the eigenvalues of the reduced density matrix. (Srednicki 1993)
- In a time-dependent background, this is not possible. The eigenvalues are real. A method has been developed for their computation. A detailed presentation is given in the publications.
- It has been shown (analytically and numerically) that the results are identical to those obtained through the covariance matrix, built from correlation functions.

- The entanglement entropy of Gaussian states can be calculated by making use of correlation functions. (Sorkin 2012)
- Define the $2N \times 2N$ matrix $\mathcal{M} = 2iJ\mathrm{Re}\,(M)$, where

$$M = \begin{pmatrix} \langle x_i x_j \rangle & \langle x_i \pi_j \rangle \\ \langle x_i \pi_j \rangle^T & \langle \pi_i \pi_j \rangle \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

- The $2n \times 2n$ block A of this matrix for subsystem A is

$$\mathcal{M}_A = i \begin{pmatrix} 0 & (\Omega)_A \\ -\left(\Omega^{-1}\right)_A & 0 \end{pmatrix}.$$

- Its eigenvalues come in pairs $\pm\lambda_i$ and satisfy $|\lambda_i| \geq 1$.
- The entanglement entropy can be computed as

$$S = \sum_{i=0}^{n} \left(\frac{\lambda_i + 1}{2} \log \frac{\lambda_i + 1}{2} - \frac{\lambda_i - 1}{2} \log \frac{\lambda_i - 1}{2} \right),$$

where the sum is performed only over the positive eigenvalues.
- The two methods are equivalent.

Entanglement entropy of a quantum field in cosmology

- Consider a massless scalar field in $3+1$ dimensions.
- Expand the field in real spherical harmonic moments.
- The radial coordinate is discretized on a lattice of concentric spherical shells with radii $\mathrm{r_j} = \mathrm{j}\epsilon$, where $\mathrm{j} = 1, 2, \ldots, \mathrm{N}$.
- The Hamiltonian reads

$$\mathrm{H} = \frac{1}{2\epsilon} \sum_{\ell,\mathrm{m}} \sum_{\mathrm{j}=1}^{\mathrm{N}} \Bigg[\pi^2_{\ell\mathrm{m,j}} - \mathrm{f}_{\ell\mathrm{m,j}} \left(\mathrm{f}_{\ell\mathrm{m,j+1}} - 2\mathrm{f}_{\ell\mathrm{m,j}} + \mathrm{f}_{\ell\mathrm{m,j-1}} \right) + \left(\frac{\ell(\ell+1)}{\mathrm{j}^2} - \frac{2\kappa}{(\tau/\epsilon)^2} \right) \mathrm{f}^2_{\ell\mathrm{m,j}} \Bigg],$$

with $\kappa = 1$ for dS and $\kappa = 0$ for RD.

- Trace out the oscillators with $j\epsilon > R$.
- Sectors with different angular momenta do not interact. The dynamics depends on ℓ solely. The entanglement entropy reads

$$S(n, N) = \sum_{\ell=0}^{\infty} (2\ell + 1)\, S_\ell(n, N).$$

- For inflation, the entanglement of interest is between modes with wavelengths above a UV cutoff $\epsilon \sim 1/k_s \sim 1/H_{infl}$ equal to the wavelength of the mode that crossed outside the horizon at the end of inflation.

A volume effect

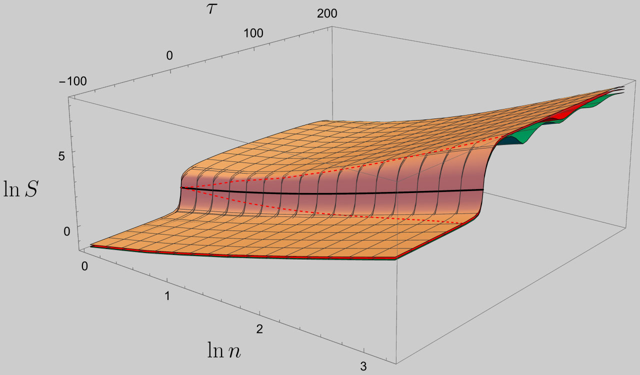

Figure: The entanglement entropy for a spherical region as a function of the entangling radius at various times for $H\epsilon = 1$. The radius of the spherical lattice is $L = N\epsilon$. Results for $N = 200$ (brown), $N = 100$ (red), $N = 50$ (green). We indicate the entropy at the dS to RD transition (black curve) and the location of the comoving horizon (dashed, red curve).

Fit the result with a function ($\epsilon = 1$)

$$\mathrm{S} = \mathrm{s}(\tau)\,\mathrm{R}^2 + \mathrm{c}(\tau)\,\mathrm{R}^3.$$

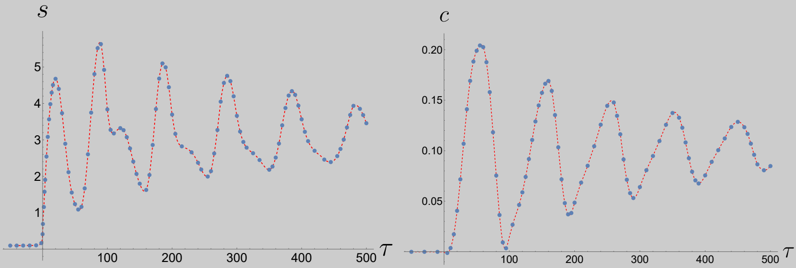

Figure: The coefficents s (left plot) and c (right plot) of the quadratic and cubic term, respectively, as a function of time.

Summary

- Modes that start as quantum fluctuations in the Bunch-Davies vacuum are expected to freeze upon horizon exit and transmute into classical stochastic fluctuations.
- This is only part of the picture. Even though its classical features are dominant, the field never loses its quantum nature.
- The various modes evolve into squeezed states.
- The squeezing triggers an enhancement of quantum entanglement. The effect is visible in the entanglement entropy.
- The entanglement entropy survives during the eras of radiation or matter domination. A volume effect appears during these eras.
- Observable consequences? A look behind the horizon?
- Weakly interacting, very light fields that stay coherent during the cosmological evolution (gravitational waves).
- Interpretation of the entropy as thermodynamic? A quantum-mechanical realization of reheating after inflation?
- Can the entropy of the Universe be attributed to the presence of the cosmological horizon?

Entanglement entropy in de Sitter space in 3+1 dimensions

- Conjectured form of the entanglement entropy in de Sitter space (Maldacena, Pimentel 2013):

$$S = \frac{c_1}{4\pi}\frac{\mathcal{A}}{\epsilon_p^2} + \frac{c_2}{2}\log\left(\frac{1}{H^2\epsilon_p^2}\right) + \frac{c_4}{4\pi}H^2\mathcal{A}\log\left(\frac{1}{H\epsilon_p}\right) + \frac{c_5}{4\pi}H^2\mathcal{A} + \frac{c_6}{2}\log(H^2\mathcal{A}).$$

- $\mathcal{A} = 4\pi a^2(\tau)R^2 = 4\pi R_p^2$: proper area of the entangling surface
- $\epsilon_p = a(\tau)\epsilon$: physical UV cutoff
- c_1 (scheme-dependent) (Srednicki 1993)
- $c_2 = -1/90$ (Solodukhin 2008, Lohmayer et al. 2009)
- $c_6 = -1/90$ (Maldacena, Pimentel 2013)
- No volume term.

Infrared effect in de Sitter space

- In the Bunch-Davies vacuum of de Sitter space, expressed in planar coordinates, consider an $1/\tau$ expansion around $\tau = -\infty$ for a subsystem of radius R within a system of total size R^{tot}.
- The numerical analysis, verified by an analytical calculation, indicates that the $\ell = 0$ sector generates an IR correction

$$\Delta S = \frac{R^2}{\tau^2}\left[\frac{1}{3}\log\left(\frac{R^{tot}}{2\pi R}\right)+\frac{4}{9}\right] = H^2 R_p^2\left[\frac{1}{3}\log\left(\frac{R_p^{tot}}{2\pi R_p}\right)+\frac{4}{9}\right]$$

  for $R_p \ll 1/H \ll R_p^{tot}$.
- Unexpected dependence on the size of the total system, linked to the IR singularities of dS space.
- The IR term persists when all the ℓ-sectors are taken into account in 3+1 dimensions.

- The numerical analysis indicates with high accuracy that $c_4 = \frac{1}{3}$.
- The entanglement entropy of a massless field in $(3+1)$-dimensional dS space in the range $R_p \ll 1/H \ll R_p^{tot}$ takes the form:

$$S = \frac{c_1}{4\pi}\frac{\mathcal{A}}{\epsilon_p^2} + \frac{c_2}{2}\log\left(\frac{\mathcal{A}}{\epsilon_p^2}\right) + \frac{c_4}{4\pi}H^2\mathcal{A}\log\left(\frac{1}{H\epsilon_p}\right) + \frac{c_{IR}}{4\pi}H^2\mathcal{A}\log\left(HR_p^{tot}\right) + \text{finit}$$

with

$$c_1 \simeq 0.2954, \quad c_2 = -\frac{1}{90}, \quad c_4 = \frac{1}{3}, \quad c_{IR} = \frac{1}{3}.$$

- Entanglement entropy acts as a probe beyond the horizon.

Divergent terms for massive fields in curved backgrounds

- Consider curved backgrounds consistent with spherical symmetry.
- Define a canonically normalized field by absorbing the curvature-dependent coefficient of the kinetic term.
- Expand in real spherical harmonics $\mathrm{Y}_{\ell \mathrm{m}}(\theta, \phi)$ in order to obtain a system of $(1+1)$-dimensional, decoupled fields $\phi_{\ell \mathrm{m}}(\mathrm{t}, \mathrm{r})$.
- Discretize the radial direction in order to obtain a system of coupled harmonic oscillators for each sector.
- Derive the Hamiltonian of the system and deduce the coupling matrix $\mathrm{K}_{\mathrm{i},\mathrm{j}}$.
- Define the quantum state by selecting the appropriate vacuum.
- Impose Dirichlet boundary conditions for the entire system, with the fields vanishing at the origin and a radius $\mathrm{R}^{\mathrm{tot}}$.
- Compute the reduced density matrix and its eigenvalues for a subsystem extending from the origin to a radius R.
- Repeat for every (ℓ, m)-sector.
- Compute the entanglement entropy by adding all the sectors.

Massive scalar field in Anti-de Sitter space

- The metric of AdS space in global coordinates is

$$ds^2 = \frac{1}{\cos^2\frac{w}{a}}\left(-dt^2 + dw^2 + a^2\sin^2\frac{w}{a}d\Omega^2_{d-1}\right).$$

- $w = 0$ corresponds to the center of AdS and $w = a\pi/2$ to the conformal boundary.
- The action of a free massive scalar field is

$$\mathcal{S} = \frac{1}{2}\int dt \int_0^{a\frac{\pi}{2}} dw \int_{S^{d-1}} d\Omega_{d-1}\left(a\tan\frac{w}{a}\right)^{d-1} \left[\dot{\phi}^2 - (\partial_w\phi)^2 + \frac{\phi\Delta_{d-1}\phi}{a^2\sin^2\frac{w}{a}} - \frac{\mu^2\phi^2}{\cos^2\frac{w}{a}}\right].$$

- Expand the field in hyper-spherical harmonics

$$\phi(t, w, \hat{r}) = \frac{1}{a^{\frac{d-1}{2}}\tan^{\frac{d-1}{2}}\frac{w}{a}}\sum_{\ell,\vec{m}}\phi_{\ell\vec{m}}(t, w)\, Y_{\ell\vec{m}}(\hat{r}).$$

- The Hamiltonian of each sector reads

$$H_{\ell\vec{m}} = \frac{1}{2}\int_0^{a\frac{\pi}{2}} dw \left[\pi_{\ell\vec{m}}^2 + (\partial_w \phi_{\ell\vec{m}})^2 + \frac{1}{a^2}\left(\frac{\nu^2 - \frac{1}{4}}{\sin^2\frac{w}{a}} + \frac{\kappa^2 - \frac{1}{4}}{\cos^2\frac{w}{a}}\right)\phi_{\ell\vec{m}}^2\right],$$

with a the AdS length and

$$\nu = \ell + \frac{d}{2} - 1\,, \qquad \kappa = \sqrt{\mu^2 a^2 + \frac{d^2}{4}}.$$

- The Breitenlohner-Freedman stability bound

$$\mu^2 \geq -\frac{d^2}{4a^2}$$

is saturated for $\kappa = 0$.
- A non-minimal coupling $\frac{1}{2}\xi R\phi^2$ results in an effective mass

$$\mu_{\text{eff}}^2 = \mu^2 - \xi\frac{d(d+1)}{a^2}\,.$$

- A massless scalar with a conformal coupling $\xi = \frac{d-1}{4d}$ corresponds to $\kappa = 1/2$ for any d.

- The Dirichlet conditions imposed on $\phi_{\ell \vec{m}}$ determine uniquely the solution only for $\kappa \geq 1/2$. We focus on this regime.
- In the window $1/2 > \kappa \geq 0$, between the conformal point and the BF bound, Dirichlet conditions on $\phi_{\ell \vec{m}}$ at the conformal boundary do not fix the asymptotics. A separate analysis is required and the numerical implementation is still unclear.
- We use a lattice of $\mathrm{N}+2$ points for the interval $[0, \pi/2]$. The UV cutoff is given by $\frac{\mathrm{a}}{\epsilon} = \frac{2}{\pi}(\mathrm{N}+1)$.
- When an infinite number of ℓ-sectors are included, the entropy in $3+1$ dimensions takes the form

$$\mathrm{S}_\infty(\mathrm{n}, \mathrm{N}, \mu^2 \mathrm{a}^2) = \frac{\mathrm{a}^2}{\epsilon^2} \mathrm{S}^{(2)}(\mathrm{n}, \mu^2 \mathrm{a}^2) + \mathrm{S}_{\mathrm{l}}^{(0)}(\mathrm{n}, \mu^2 \mathrm{a}^2) \log \frac{\mathrm{a}}{\epsilon}$$
$$+\mathrm{S}^{(0)}(\mathrm{n}, \mu^2 \mathrm{a}^2) + \mathrm{R}(\mathrm{n}, \mathrm{N}, \mu^2 \mathrm{a}^2),$$

where the remainder $\mathrm{R}(\mathrm{n}, \mathrm{N}, \mu^2 \mathrm{a}^2)$ vanishes for $\epsilon \to 0$.

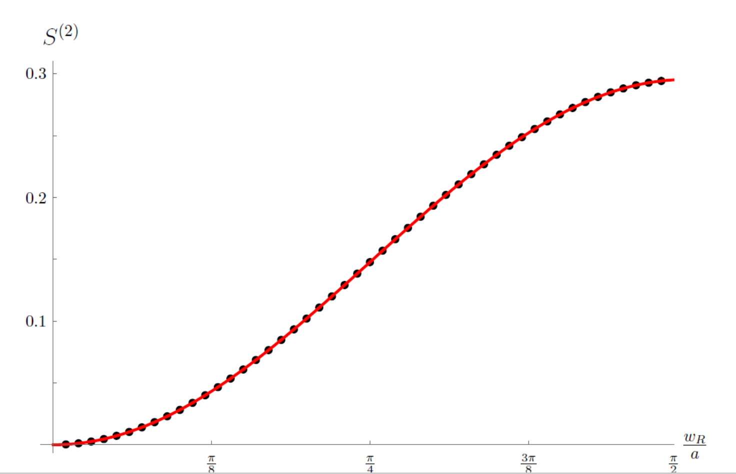

Figure: Numerical fit for the determination of $S^{(2)}$ as a function of $\frac{w_R}{a} = \frac{\pi}{2}\frac{n+\frac{1}{2}}{N+1}$ for $\mu^2 a^2 = 0$. The data indicate that $S^{(2)} = d_1 \sin^2 \frac{w_R}{a}$, with $d_1 \simeq 0.29543145$ for all values of $\mu^2 a^2$.

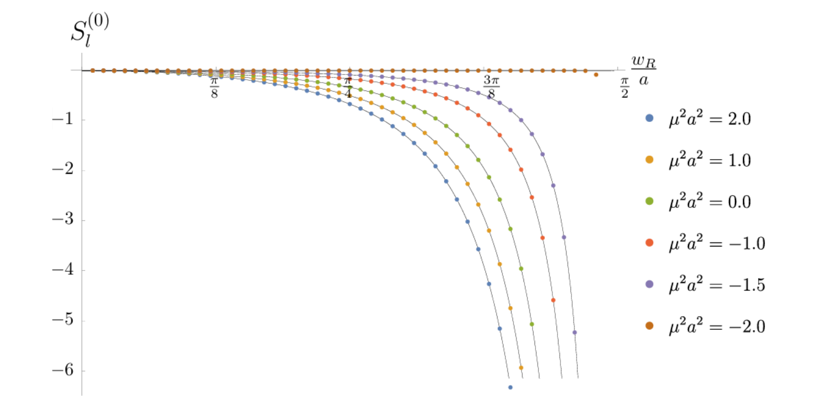

Figure: Numerical fits for the determination of $S_l^{(0)}$ as a function of w_R/a for various values of $\mu^2 a^2$. The data indicate with high precision that $S_l^{(0)} = a_l(\mu^2 a^2) + b_l(\mu^2 a^2)\tan^2 \frac{w_R}{a}$.

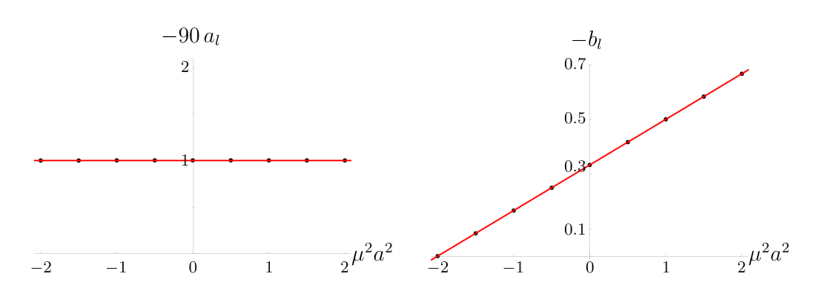

Figure: The coefficients $\mathrm{a_l}$ and $\mathrm{b_l}$ for various values of $\mu^2\mathrm{a}^2$. The left plot demonstrates that $\mathrm{a_l} = -1/90$, independently of $\mu^2\mathrm{a}^2$, with high precision. The right plot demonstrates that $\mathrm{b_l}$ has a linear dependence on $\mu^2\mathrm{a}^2$ with slope equal to $-1/6.0004$ and y−intercept equal to $-1/3.0003$. We can confidently deduce that $\mathrm{b_l} = -\mu^2\mathrm{a}^2/6 - 1/3$.

- Our results can be summarized in the expression

$$S_{AdS} = \frac{d_1}{\epsilon^2} \sin^2 \frac{w_R}{a} + \left(d_2 + \left(d_3\, \mu^2 a^2 + d_4\right) \tan^2 \frac{w_R}{a}\right) \log \frac{a}{\epsilon} + \text{finite},$$

  with $d_1 \simeq 0.29543145$, $d_2 = -\frac{1}{90}$, $d_3 = -\frac{1}{3}$, $d_4 = -\frac{1}{6}$.
- The coefficient d_1 is identical to that in flat space. This is expected, as the leading divergence arises from short-distance correlations.
- d_2 is linked to the A-type anomaly of the conformal theory.
- d_3 and d_4 are universal coefficients, for deformations of the conformal theory through a mass term or the curvature.
- There is a discrepancy, as the proper area of the entangling surface is $A = 4\pi a^2 \tan^2 \frac{w_R}{a}$. This is due to the extreme regulator dependence of the first term. There is a relative factor $1/\cos^2 \frac{w}{a}$ for the number of degrees of freedom in the discretization of the coordinate $r = a \tan \frac{w}{a}$ relative to the tortoise coordinate w. When this is taken into account, the discrepancy disappears.

Massive scalar field in the Einstein universe

- The metric of the $\mathrm{R} \times \mathrm{S}^{\mathrm{d}}$ geometry is

$$\mathrm{ds}^2 = -\mathrm{dt}^2 + \mathrm{dw}^2 + \mathrm{a}^2 \sin^2 \frac{\mathrm{w}}{\mathrm{a}} \mathrm{d}\Omega_{\mathrm{d}-1}.$$

- The metric is conformal to that of AdS space in global coordinates. The comparison of the results in the two cases confirms the conclusion about the structure of the leading UV-divergent term.
- The free scalar field has a zero mode on this background, which generates IR divergences. The effect on the entanglement entropy can be studied and compared with the de Sitter case.

Summary

- In $3+1$ dimensions, the divergent terms in the entanglement entropy within a spherical entangling surface of proper area A are

$$S_{EE} = \frac{c_1}{4\pi}\frac{A}{\epsilon^2} + \left(c_2 + c_3 \frac{A}{4\pi}\mu^2 + c_4 \frac{A}{4\pi a^2}\right)\log\frac{a}{\epsilon} + c_{IR}\frac{A}{4\pi a^2}\log\frac{R^{tot}}{a} + \text{finite},$$

  with a the curvature scale of the background.
- For dS space $a = 1/H$, for AdS space a is the AdS length a_{AdS}, for the Einstein universe it is the radius of the spatial sphere.
- A dependence on the size of the entire system R^{tot} appears in dS space and in the Einstein universe.
- The universal coefficients are

$$R\times S^3: \quad c_2 = -\frac{1}{90}, \quad c_3 = -\frac{1}{6}, \quad c_4 = \frac{1}{6}, \quad c_{IR} = \frac{1}{6}$$

$$dS: \quad c_2 = -\frac{1}{90}, \quad c_3 = -\frac{1}{6}, \quad c_4 = \frac{1}{3}, \quad c_{IR} = \frac{1}{3}$$

$$AdS: \quad c_2 = -\frac{1}{90}, \quad c_3 = -\frac{1}{6}, \quad c_4 = -\frac{1}{3}, \quad c_{IR} = 0.$$

Comments

- The values of c_3 and c_4 are such that the corresponding contributions cancel for a field whose mass arises from a non-minimal coupling to gravity $\frac{1}{6}\mathcal{R}\phi^2$. The coefficient of the logarithmic term for the resulting Weyl-invariant theory is given by c_2 and is related to the A-type conformal anomaly. (Solodukhin 2008)
- The values of c_4 for dS and AdS are opposite, as they are related through the analytic continuation $H^2 \to -1/a^2_{AdS}$.
- In all spaces, $c_3 = -\frac{1}{6}$. (Hertzberg, Wilczek 2010)
- No infrared effect in AdS.

c-, F- and a- functions

- Consider the entanglement entropy for a spherical entangling surface.
- The evolution from the UV to the IR corresponds to R changing from 0 to ∞.
- $1+1$ d: $\mathrm{R}\,\Delta\mathrm{S}'(\mathrm{R}) \to$ c-function. (Casini, Huerta 2012)
  $3+1$ d: $\mathrm{R}^2\,\Delta\mathrm{S}''(\mathrm{R}) - \mathrm{R}\,\Delta\mathrm{S}'(\mathrm{R}) \equiv$ a-function. (Casini, Teste, Torroba 2017)
- The contribution to the entropy from the UV fixed point, as well as UV divergent terms related to the deformations of the conformal theory must be subtracted.
- The relation is also valid in the static patch of dS space (Abate, Torroba 2024) and in AdS space (Abate, Salazar, Torroba 2026).
- The relation makes use of the finite part of the entanglement entropy.

Methodology

- We consider a free, massive, scalar field in (3+1)-dimensional flat space.
- We expand the field in real spherical harmonic moments
- The radial coordinate is discretized on a lattice of concentric spherical shells with radii $\mathrm{r_j} = \mathrm{j}\epsilon$, where $\mathrm{j} = 1, 2, \ldots, \mathrm{N}$.
- The Hamiltonian reads

$$\mathrm{H} = \frac{1}{2\epsilon}\sum_{\ell,\mathrm{m}}\sum_{\mathrm{j}=1}^{\mathrm{N}}\left[\pi_{\ell\mathrm{m,j}}^2 + \left(\phi_{\ell\mathrm{m,j+1}} - \phi_{\ell\mathrm{m,j}}\right)^2 + \frac{\ell\left(\ell+1\right)}{\mathrm{j}^2}\phi_{\ell\mathrm{m,j}}^2 + \bar{\mu}^2\,\phi_{\ell\mathrm{m,j}}^2\right].$$

  with $\bar{\mu} = \mu\epsilon$. We set $\epsilon = 1$.
- This form implements Dirichlet boundary conditions, with vanishing field at the origin and at a radius $\mathrm{r} = \mathrm{L}$. Possible zero modes are thus eliminated.
- The $\ell = 0$ sector corresponds to the discretized Hamiltonian of the $(1+1)$-dimensional field theory.

A c-function in $1 + 1$ dimensions

- Consider a free, massive scalar field in $1 + 1$ dimensions.
- In the deep UV the mass becomes irrelevant and we have one massless degree of freedom.
- In the deep IR the massive field decouples and there are no degrees of freedom.
- The c-function must interpolate between the two limits in a monotonic fashion.
- Consider a system of length L, divided into subsystems of lengths R and $L - R$.
- Impose Dirichlet boundary conditions, with the field vanishing at the ends. No zero modes in the spectrum.
- For $L \to \infty$, the entanglement entropy in the massless theory is

$$S = \frac{1}{6} \log\left(\frac{2R}{\epsilon}\right) + \bar{c}'_1$$

(Holzhey, Larsen, Wilczek 1994, Korepin 2004, Calabrese, Cardy 2004)

- If the contribution from the UV fixed point is subtracted, the finite part of the entropy must depend only on μR for $\mu\epsilon \to 0$. The constant part does not contribute to the entropy.
- The form of the entropy is

$$S(R,\mu) = \frac{1}{6}\ln\frac{R}{\epsilon} + S_{fin}(\mu R)$$

- For $x = \mu R \to 0$

$$S_{fin} = \frac{1}{6}x^2\log(x) + \left(\frac{\gamma_E}{6} - \frac{5}{36}\right)x^2.$$

- For $x = \mu R \to \infty$

$$S_{fin} = -\frac{1}{6}\log(x).$$

- An effective c-function can be defined as

$$c(x) = 6x\,S'(x,\mu\epsilon) \to 1 + 6x\,S'_{fin}(x).$$

- It interpolates between 1 and 0.

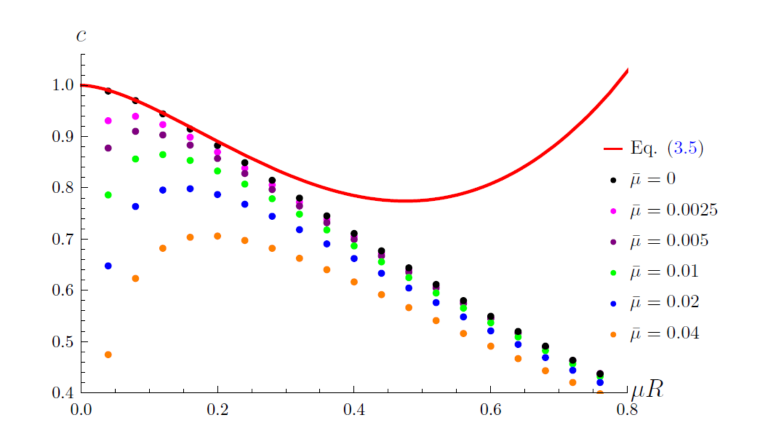

Figure: The numerical results for the c-function for various values of $\bar{\mu} = \mu\epsilon$. The black points correspond to the extrapolation to the continuum limit $\epsilon \to 0$. The solid red line corresponds to the analytical expression valid for small μR.

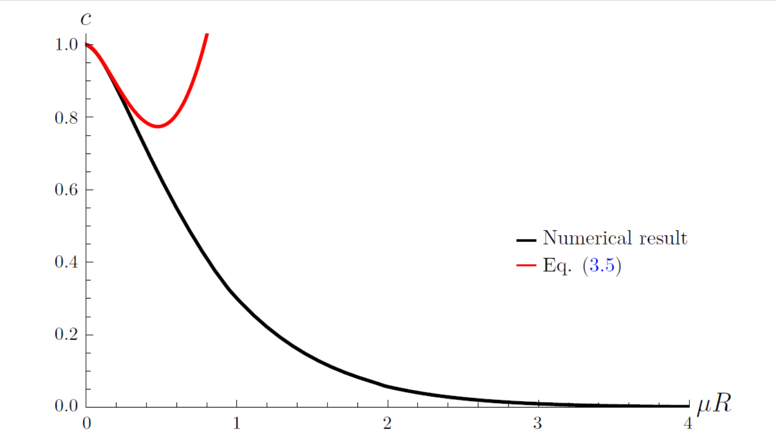

Figure: The black line depicts the numerical result for the c-function in the continuum. The red line corresponds to the analytical expression valid for small μR.

An a-function in $3+1$ dimensions

- The entropy has the form

$$S(R,\mu) = c\frac{R^2}{\epsilon^2} - \frac{1}{90}\ln\frac{R}{\epsilon} + \frac{1}{6}\mu^2 R^2 \ln \mu\epsilon + S_{\mathrm{fin}}(\mu R).$$

- An effective a-function can be defined as

$$a(x) = -45\left(x^2\, S''(x,\mu\epsilon) - x\, S'(x,\mu\epsilon)\right) \to 1-45\left(x^2\, S''_{\mathrm{fin}}(x) - x\, S'_{\mathrm{fin}}(x)\right).$$

- It interpolates between 1 and 0.

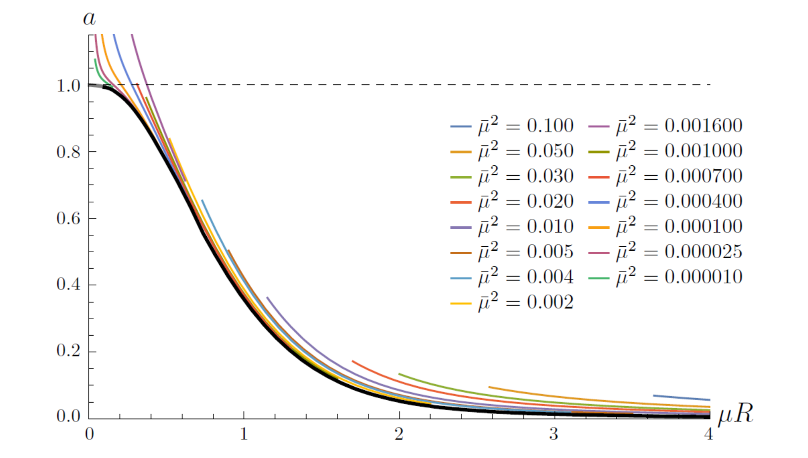

Figure: The colored lines depict the numerical results for the a-function for various values of $\bar{\mu} = \mu\epsilon$. The black line corresponds to the extrapolation to the continuum limit $\epsilon \to 0$.

Prospects

- Open questions
  - Mutual information in de Sitter.
  - Entanglement entropy in the window between the conformal point and the BF bound of AdS.
  - Entanglement and holography.
  - Calculation of the F-function in (2+1)-dimensional flat space.
  - Calculation of c-, F- and a-functions in de Sitter.
- Real challenges
  - Entanglement entropy of interacting theories.
  - Calculation of a-, F- and a-functions interpolating between nontrivial fixed points.
  - Entanglement during collapse (e.g. Vaidya background).
  - Entanglement entropy in geometries with event horizons (e.g. Schwarzschild background).
- How to measure entanglement entropy?
  - Cosmology
  - Analogue gravity

How to measure entanglement entropy

Verification of the area law of mutual information in a quantum field simulator #3

Mohammadamin Tajik (Vienna, Tech. U., Atominst.), Ivan Kukuljan (Munich, Max Planck Inst. Quantenopt. and MCQST, Munich), Spyros Sotiriadis (Freie U., Berlin), Bernhard Rauer (Vienna, Tech. U., Atominst. and Paris, Lab. Kastler Brossel), Thomas Schweigler (Vienna, Tech. U., Atominst.) et al. (Jun 21, 2022)

Published in: *Nature Phys.* 19 (2023) 7, 1022-1026 • e-Print: 2206.10563 [cond-mat.quant-gas]

Experimental verification of the area law of mutual information in a quantum field simulator

Mohammadamin Tajik,[1,*] Ivan Kukuljan,[2,3] Spyros Sotiriadis,[4] Bernhard Rauer,[1,5]
Thomas Schweigler,[1] Federica Cataldini,[1] João Sabino,[1,6,7] Frederik Møller,[1]
Philipp Schüttelkopf,[1] Si-Cong Ji,[1] Dries Sels,[8,9] Eugene Demler,[10] and Jörg Schmiedmayer[1,†]

[1] *Vienna Center for Quantum Science and Technology, Atominstitut, TU Wien, Stadionallee 2, 1020 Vienna, Austria*
[2] *Max-Planck Institute of Quantum Optics, Hans-Kopfermann-Str. 1, 85748 Garching, Germany*
[3] *Munich Center for Quantum Science and Technology (MCQST), Schellingstr. 4, D-80799 München*
[4] *Dahlem Center for Complex Quantum Systems, Freie Universität Berlin, Berlin, Germany*
[5] *Laboratoire Kastler Brossel, ENS-Université PSL, CNRS, Sorbonne Université, Collége de France, 24 rue Lhomond, 75005 Paris, France*
[6] *Instituto de Telecomunicações, Physics of Information and Quantum Technologies Group, Av. Rovisco Pais 1, 1049-001, Lisbon, Portugal*
[7] *Instituto Superior Técnico, Universidade de Lisboa, Av. Rovisco Pais 1, 1049-001, Lisbon, Portugal*
[8] *Department of physics, New York University, New York City, NY 10003, USA*
[9] *Center for Computational Quantum Physics, Flatiron Institute, 162 5th Ave, New York, NY 10010, USA*
[10] *Institute for Theoretical Physics, ETH Zurich, Wolfgang-Pauli-Str. 27, 8093 Zurich, Switzerland*

(Dated: March 8, 2023)

Theoretical understanding of the scaling of entropies and the mutual information has led to significant advances in the research of correlated states of matter, quantum field theory, and gravity. Measuring von Neumann entropy in quantum many-body systems is challenging as it requires complete knowledge of the density matrix. In this work, we measure the von Neumann entropy of spatially extended subsystems in an ultra-cold atom simulator of one-dimensional quantum field theories. We experimentally verify one of the fundamental properties of equilibrium states of gapped quantum many-body systems, the area law of quantum mutual information. We also study the dependence of mutual information on temperature and the separation between the subsystems. Our work is a crucial step toward employing ultra-cold atom simulators to probe entanglement in quantum field theories.

Quantum field simulator for dynamics in curved spacetime

#13

Celia Viermann (Kirchhoff Inst. Phys.), Marius Sparn (Kirchhoff Inst. Phys.), Nikolas Liebster (Kirchhoff Inst. Phys.), Maurus Hans (Kirchhoff Inst. Phys.), Elinor Kath (Kirchhoff Inst. Phys.) et al. (Feb 21, 2022)

Published in: *Nature* 611 (2022) 7935, 260-264 • e-Print: 2202.10399 [cond-mat.quant-gas]

Quantum field simulator for dynamics in curved spacetime

Celia Viermann,[1] Marius Sparn,[1] Nikolas Liebster,[1] Maurus Hans,[1] Elinor Kath,[1] Álvaro Parra-López,[2,3] Mireia Tolosa-Simeón,[2] Natalia Sánchez-Kuntz,[2] Tobias Haas,[2] Helmut Strobel,[1] Stefan Floerchinger,[2,4] and Markus K. Oberthaler[1]

[1]*Kirchhoff-Institut für Physik, Universität Heidelberg, Im Neuenheimer Feld 227, 69120 Heidelberg, Germany*
[2]*Institut für Theoretische Physik, Universität Heidelberg, Philosophenweg 16, 69120 Heidelberg, Germany*
[3]*Departamento de Física Teórica and IPARCOS, Facultad de Ciencias Físicas, Universidad Complutense de Madrid, Ciudad Universitaria, 28040 Madrid, Spain*
[4]*Theoretisch-Physikalisches Institut, Friedrich-Schiller-Universität Jena, Max-Wien-Platz 1, 07743 Jena, Germany*

The observed large-scale structure in our Universe is seen as a result of quantum fluctuations amplified by spacetime evolution [1]. This, and related problems in cosmology, asks for an understanding of the quantum fields of the standard model and dark matter in curved spacetime. Even the reduced problem of a scalar quantum field in an explicitly time-dependent spacetime metric is a theoretical challenge [2–4] and thus a quantum field simulator can lead to new insights. Here, we demonstrate such a quantum field simulator in a two-dimensional Bose-Einstein condensate with a configurable trap [5, 6] and adjustable interaction strength to implement this model system. We explicitly show the realisation of spacetimes with positive and negative spatial curvature by wave packet propagation and confirm particle pair production in controlled power-law expansion of space. We find quantitative agreement with new analytical predictions for different curvatures in time and space. This benchmarks and thereby establishes a quantum field simulator of a new class. In the future, straightforward upgrades offer the possibility to enter new, so far unexplored, regimes that give further insight into relativistic quantum field dynamics.

densate of potassium-39 with configurable de
bution and additional dynamic control of a
actions. With that we implement curved me
phononic field of the form (see methods: 'Cu
metric')

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left(\frac{\mathrm{d}u^2}{1-\kappa u^2} + u^2\mathrm{d}\varphi\right.$$

This corresponds to the standard cosmolo
of a 2 + 1 dimensional homogeneous and is
verse, the Friedmann-Lemaître-Robertson-V
ric (FLRW) in reduced circumference coordi
This metric is parametrised by intrinsic and e
vature: the intrinsic curvature, κ, is the curv
spatial part of the metric, while the extrins
arises from the time dependence of the scale
In our atomic implementation both parame
ture and scale factor, can be adjusted indep

Phonons in the central region of a h
trapped Bose-Einstein condensate experien
with $\kappa < 0$. In cosmological settings, this
bolic two-dimensional spatial geometry with
ordinate of infinite range, as depicted in
Through the Poincaré transformation, the in
bolic space is mapped to a finite disc, perf
to be implemented in finite-size ultracold