Numerical implementation of integrand level reduction

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Amplitude Construction and Reduction options

- Qgraph → symbolic manipulation, dimensionally regularized amplitudes → IBP: FIRE Kira or numerical pySecDec
- Numerical unitarity → dimensionally regularized amplitudes by gluing tree amplitudes in different integer dimensions → D_s (Abreu, Cordero, Ita, Page and Sotnikov, 2021)
- OpenLoops → Feynman graph → openning the loops → amplitudes in $d = 4$ → coefficients of tensor integrals (Pozzorini, Schär and Zoller, 2022)

What do we need for HELAC 2-loop?

3 steps:

- 1. Amplitude Construction (*Costas Papadopoulos*)
- **2. Amplitude Reduction at 2-loops (My talk today)**
- 3. Do the integrals

Outline

- OPP Amplitude reduction at 1-loop
- 2-loop reduction analytic methodology
- Numerical Implementation benchmarks
- Outlook

Amplitude reduction OPP @ 1-loop

Reduction is done at the *integrand* level. OPP (Ossola, Papadopoulos and Pittau, 2007) master formula a numerator at 1 loop:

$$
\begin{aligned}
N(q) = & \sum_{i_0<i_1<i_2<i_3}^{m-1} d\left(i_0 i_1 i_2 i_3\right) + \tilde{d}\left(q; i_0 i_1 i_2 i_3\right) \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} D_i \\
& + \sum_{i_0<i_1<i_2}^{m-1} c\left(i_0 i_1 i_2\right) + \tilde{c}\left(q; i_0 i_1 i_2\right) \prod_{i \neq i_0, i_1, i_2}^{m-1} D_i \\
& + \sum_{i_0<i_1<i_2}^{m-1} b\left(i_0 i_1\right) + \tilde{b}\left(q; i_0 i_1\right) \prod_{i \neq i_0, i_1}^{m-1} D_i \\
& + \sum_{i_0}^{m-1} a\left(i_0\right) + \tilde{a}\left(q; i_0\right) \prod_{i \neq i_0}^{m-1} D_i
\end{aligned}
$$

where $\tilde{d}$, $\tilde{c}$, $\tilde{b}$, $\tilde{a}$ are terms which vanish upon integration.

Amplitude reduction OPP: Fitting

The system is solved by iteratively:

- Evaluate the numerator on values of k for which $D_i(k) = 0$, starting from the first line of the previous equation, where 4 propagators are put on shell
- This condition by default sets to zero all the rest of the terms, allowing us to calculate d and $\tilde{d}$
- Repeat for c, $\tilde{c}$ and so on until we fit all the coefficients

2-Loop Amplitude: Algebraic Reduction

The fit will be more complex!

A generic 2-loop integrand can be written using the following scalar product set:

$$\{p_i \cdot p_j, k_i \cdot k_j, k_i \cdot p_j, k_i \cdot \eta_j\}$$

as well as any masses inside the loops. For the rest of the discussion we will ignore the case of massive loops.

The integrand can be written in the general form

$$\mathcal{R} = \frac{\mathcal{N}}{\mathcal{D}} = \frac{\sum_a c_a (z_1^{(a)})^{\beta_1} \ldots (z_{n_a}^{(a)})^{\beta_N}}{D_1 \ldots D_{N_p}} \tag{1}$$

where the z_i are any of the scalar products in the set.

2-Loop Amplitude reduction

Using $\bar{z}_i$ which are only the scalar products which cannot be eliminated by being written as linear combinations of D_i, know as irreducible scalar products (ISPs) or the transverse $k_i \cdot \eta_j$ and σ is any subset of $\{1, \ldots, N_p\}$ with m elements.

Write the numerator of the integrand level amplitude as follows:

Numerator Formula

$$\mathcal{N} = \sum_{m=0}^{N_p} \sum_{\sigma} \sum_{a} \bar{c}_a \prod_{i}^{N_{T+ISP}} (\bar{z}_i)^{\alpha_i} \prod_{j} D_j \tag{2}$$

where N_{T+ISP} is the number of ISP and transverse scalar products and $j \neq \sigma_i$ for all i (Bevilacqua, Canko, Papadopoulos and Spourdalakis, 2025)

2-Loop Amplitude reduction

We include all D_i as long as the scalar products present in the numerator can be written as linear combinations of D_i.
This broader set is named *family*.
For each numerator, we

- Set all propagators of the family to zero and solve the equations that put all of them on shell simultaneously, AKA find *cut solutions*.
- Write a linear system for the coefficients $\mathbf{M} \cdot \vec{\bar{c}} = \vec{\mathcal{N}}$ where $\mathbf{M}$ is a matrix of all monomials $\bar{z}_i$ evaluated on different values of cut-solutions, $\vec{c}$ is all the $\bar{c}_a$ and $\vec{\mathcal{N}}$ is a vector of equal length with values of the numerator evaluated on the cut-solutions
- Solve the system of equations
- Subtract the result from the numerator

2-Loop Amplitude reduction

- Move to the next cut, where one less propagator is put on shell
- Do this till until fits can be performed with trivial polynomials
- Algebraic test that the reduced amplitude is equal to the original amplitude, known as the **N=N test**.
- The algebraic reduction is complete!

$$\mathcal{N} = P_{maxcut} + \sum_i P_{maxcut-1} D_i + \sum_{ij} P_{maxcut-2} D_i D_j + \ldots \tag{3}$$

- At the moment, we do this projection on a full family, i.e., a full set of inverse propagators, not individually looking at each graph.

Each family has *sectors*, defined by which D_i are actually present in each numerator

2-Loop Amplitude: Final Form

Final Form:

$$\mathcal{A} = \sum_i \bar{c}_i F_i \tag{4}$$

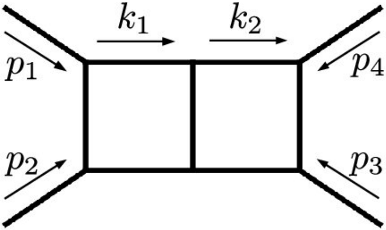
where:

$$F_i \equiv F_{a_1 \ldots a_N} = \int d^d k \frac{\overbrace{(D_{m+1})^{a_{m+1}} \cdots (D_N)^{a_N}}^{ISP}}{\underbrace{(D_1)^{a_1} \cdots (D_m)^{a_m}}_{RSP}}$$

Use Integration by Parts (IBP) and/or other methods to further simplify the integral basis.

2-Loop reduction example

$2 \to 2$ gluon-topology



Maximal cut: 9 propagators on shell.

2-Loop reduction example: D-dimensions

Begin with 11 free parameters: 8 from the components of the two 4-momenta, and 3 $\mu_{11}, \mu_{22}, \mu_{12}$ the ϵ components of k_1^2, k_2^2 and $k_1 \cdot k_2$ respectively.
The maximal cut has 9 cut equations, we have a remainder of 2 free parameters, enough to find solutions which can construct a full rank **M**

How do we make the polynomials?

$$\mathcal{N} = P_{maxcut} + \sum_i P_{maxcut-1} D_i + \sum_{ij} P_{maxcut-2} D_i D_j + \ldots \tag{5}$$

What's inside the P_i's?

- Use BasisDet or just build a polynomial using all ISP combinations with an educated guess for the maximum monomial power.

Analytic Results

- Algebraic reduction completed analytically for all $2 \to 2$ topologies (both in $d = 4$ and in $d = 4 - 2\epsilon$) ✓

Results

We have completed an *analytic* Mathematica simulation of this fit. Agreement with known results: Caravel (Abreu, Cordero, Ita, Page and Sotnikov, 2021). Completed cut+fit for all subtologies with gluons and ghosts

Numerical Reduction Implementation

Numerical reduction recipe!

- Topology and sectors definition
- Cut-equation solver
- Monomials corresponding to each cut
- Linear System Solver (to fit the polynomials)

Instructions:

- Solve cut equations for each cut and generate enough independent solutions to fit the polynomial
- Use the solutions to populate the **M** and $\vec{\mathcal{N}}$ in the equation: $\mathbf{M} \cdot \vec{c} = \vec{\mathcal{N}}$
- Do this for all numerators in a given sector
- Iterate until polynomials are trivial

You will get a set of integral coefficients!

Numerical Reduction Implementation

- Started with a hardcoded set of cut solutions, dependent on the form of the external momenta.
- Moved to cut equations obtained for general external momenta using the iterative Levenberg–Marquardt method
- Used Lapack (Anderson, Bai, Bischof, Blackford, Demmel, Dongarra, Du Croz, Greenbaum, Hammarling, McKenney and Sorensen, 1999) compiled in both double and quadraple precision both for cut-equation solving and for the polynomial fitting
- We can now use a d-dimensional version of HELAC for gluons (see talk by C.P.)

Preliminary Numerical Benchmarks

For a $gg \to gg$ processes. Times are for my laptop, M1 Apple Core. No parallelization applied.

- Double precision N=N between 1 and 8 digits for most numerators
- Double precision time for all $2 \to 2$ numerators $\sim$ 6 minutes
- Quad precision N=N at least 15 digits for all numerator, most around 20.
- Quad precision time for all $2 \to 2$ numerators $\sim$ 2.5 hours

Timing constraints breakdown (Preliminary)

- Cut-equation solving: Done or each monomial.
  In Double: $\sim 2.2 * 10^{-5}$. In Quad: $\sim 4 * 10^{-4}$ seconds
- Linear System Solving done once per cut:
  Double: $\sim 2.5 * 10^{-5}$, Quad: $\sim 1.2 * 10^{-3}$ seconds
- HELAC. Called once per matrix row:
  Double $\sim 2.3 * 10^{-3}$, Quad: $\sim 1.9 * 10^{-2}$ seconds

Order of magnitude estimates to gauge where the main issue is

Precision constraints breakdown (Preliminary)

- Loss due to solving cut-equations (minimal)
- Loss due to solving linear system (minimal)
- Loss due to cancelatiton between large numbers at each step of the recursion (Largest contribution)

Outlook

- Implement the fully D-dimensional fit @ 1 and 2 loops numerically
- Numerical D-dimensional reduction at 2-loops for all $2 \to 2$ topologies (including quarks)
- $2 \to 3$ topology reduction
- Generalizing for further topologies!

Structure is simple to generalize!

Conceptually the algorithm structure does not change when moving to higher topologies

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