Exotic Spin chains from $\mathcal{N}=2$ Superconformal Field Theory

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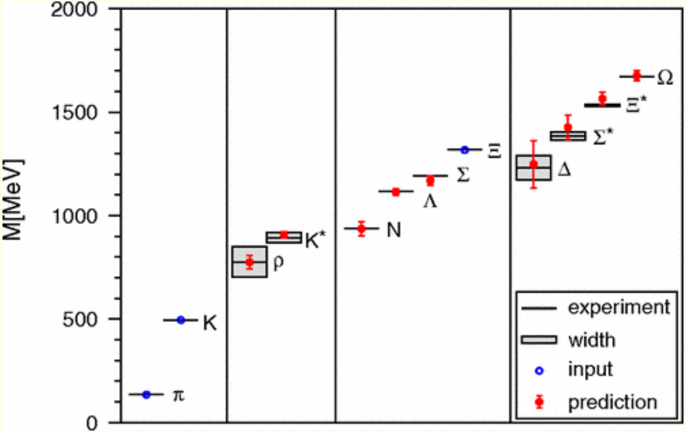
Based on 2602.22065 (w. J. Bath) and ongoing work with R. Klabbers, E. Pomoni, A. Retore and P. Ryan

The Standard Model of Particle Physics

	FERMIONS			BOSONS	
	Spin $\frac{1}{2}$			Spin 1	Spin 0
QUARKS	u	c	t	γ	H
	d	s	b	g	
LEPTONS	e	μ	τ	$W^{\pm}$	
	ν_e	ν_μ	ν_τ	Z^0	Weakly coupled description!
	Matter			Interactions	

The spectrum of light hadrons

[Dürr et al. '08]



- QCD at nuclear scales is out of reach of perturbation theory

The $\mathcal{N} = 4$ SYM theory

- Turn to the most symmetric 4d Quantum Field Theory:
- $\mathcal{N} = 4$ Super-Yang-Mills with $\mathrm{SU}(N)$ gauge group
- The hydrogen atom of High Energy Theory
- Supersymmetry: Bosons $\leftrightarrow$ Fermions
- No quarks - 6 scalars in the adjoint of $\mathrm{SU}(N)$, $\mathrm{SU}(4)$ "flavour" symmetry
- Conformal Field Theory - All masses are zero
- Strong constraints on correlation functions

$$\langle O(x) O(y) \rangle = \frac{1}{|x-y|^{2\Delta}}$$

- Still a nontrivial interacting 4d QFT

Spin chains from $\mathcal{N} = 4$ SYM

- Take planar limit $N \to \infty$ ['t Hooft '73]
- Observables are gauge invariant operators
- E.g. $\mathrm{SU}(2)$ scalar sector: X, Y

$$O(x) = Tr\,(\cdots XXXYXXXYXX \cdots)$$

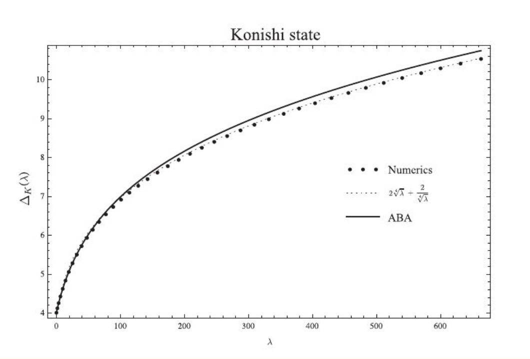
- Want to diagonalise the dilatation operator of the theory

$$DO(x) = (L + \gamma) O(x) \qquad \gamma\text{: anomalous dimension}$$

- Difficult because of operator mixing
- [Minahan,Zarembo '02]: At one-loop, D acts exactly like a spin-$\frac{1}{2}$, ferromagnetic XXX Heisenberg Hamiltonian $\Rightarrow$ Integrable!
- Long-range Hamiltonian at higher loops – Integrability persists!

Why is integrability useful?

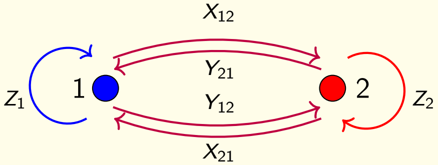
- Combined with AdS/CFT and symmetry arguments, provides the full non-perturbative solution of the (planar) $\mathcal{N} = 4$ SYM spectral problem
- E.g. Konishi operator $\mathcal{O}_K = \mathrm{Tr}\left(X\bar{X} + Y\bar{Y} + Z\bar{Z}\right)$
- [Gromov,Kazakov,Vieira '09] obtained its dimension for any 't Hooft coupling λ



- Can one hope for anything similar for more realistic theories?

$\mathbb{Z}_2$ orbifold of $\mathcal{N} = 4$ SYM

- Next simplest case is $\mathcal{N} = 2$ supersymmetry
- Orbifold: Impose invariance under a $\mathbb{Z}_2$ discrete group
- Represent using a quiver diagram:



- The $\mathrm{SU}(4)$ symmetry is (naively) broken to $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{U}(1)$.
- E.g. $\mathcal{R}^3{}_2 Z_1 = X_{12}$, $\mathcal{R}^2{}_3 X_{12} = Z_1$ are not (naively) sensible any more
- Integrable structure is not affected [Beisert,Roiban '05]
- Marginally deform by taking $\kappa = g_2/g_1 \neq 1$

XY sector: Hamiltonian

- $\mathcal{N} = 2$ picture

$$\mathcal{H}_{\ell,\ell+1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \kappa^{-1} & -\kappa^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\kappa^{-1} & \kappa^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \kappa & -\kappa & 0 \\ 0 & 0 & 0 & 0 & 0 & -\kappa & \kappa & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ on: } \begin{pmatrix} X_{12}X_{21} \\ X_{12}Y_{21} \\ Y_{12}X_{21} \\ Y_{12}Y_{21} \\ X_{21}X_{12} \\ X_{21}Y_{12} \\ Y_{21}X_{12} \\ Y_{21}Y_{12} \end{pmatrix}$$

- "Dynamical $\mathcal{N} = 4$" picture

$$\mathcal{H}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \kappa^{-1} & -\kappa^{-1} & 0 \\ 0 & -\kappa^{-1} & \kappa^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ \mathcal{H}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \kappa & -\kappa & 0 \\ 0 & -\kappa & \kappa & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ on: } \begin{pmatrix} XX \\ XY \\ YX \\ YY \end{pmatrix}_i$$

XZ sector: Hamiltonian

- $\mathcal{N} = 2$ picture

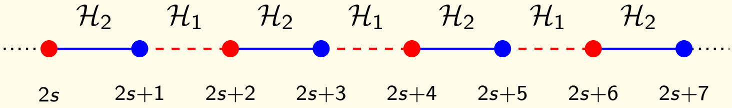
$$\mathcal{H}_{i,i+1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \kappa & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & \kappa^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \kappa^{-1} & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & \kappa & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ on: } \begin{pmatrix} X_{12}X_{21} \\ X_{12}Z_{22} \\ Z_{11}X_{12} \\ Z_{11}Z_{11} \\ X_{21}X_{12} \\ X_{21}Z_{11} \\ Z_{22}X_{21} \\ Z_{22}Z_{22} \end{pmatrix} .$$

- "Dynamical $\mathcal{N} = 4$" picture

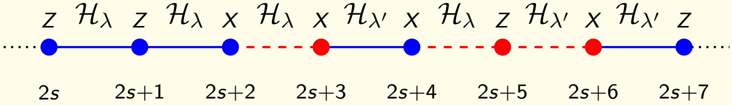
$$\mathcal{H}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \kappa & -1 & 0 \\ 0 & -1 & \kappa^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \mathcal{H}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \kappa^{-1} & -1 & 0 \\ 0 & -1 & \kappa & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ on: } \begin{pmatrix} XX \\ XZ \\ ZX \\ ZZ \end{pmatrix}_i$$

Dynamical spin chains

- The XY chain is strictly alternating:



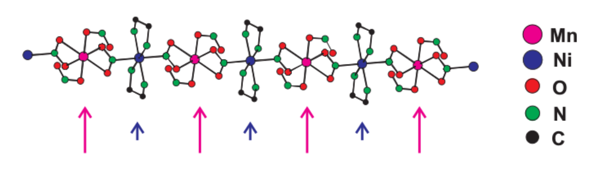
- The XZ chain is “dynamical”: The Hamiltonian depends on the number of X’s crossed.



- Introduced a “dynamical” parameter taking two values λ, λ'
- $\lambda \leftrightarrow \lambda'$ when crossing X, Y, unchanged when crossing Z

Alternating chains

- Extensive literature on alternating chains (but mostly antiferromagnetic)
- A ferromagnetic alternating bond example is in [Sirker et al. '08]
- Alternating spin chains are mathematically very similar
- E.g. the bimetallic chain $\mathrm{MnNi(NO_2)_4(en)_2(en = ethylenediamide)}$

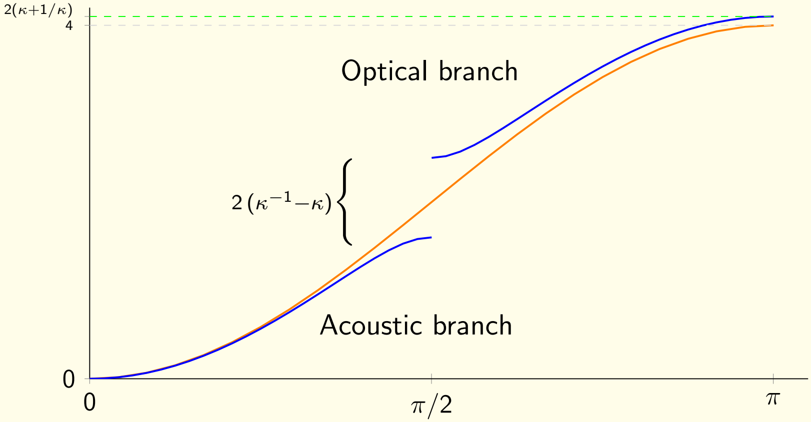


[Feyerherm, Mathonière, Kahn, J. Phys. Condens. Matter 13, 2639 (2001)]

- Have been studied with various techniques such as the recursion method [Southern, Cuellar, Lavis '98], also long-wavelength approximations [Huang et al. '91]

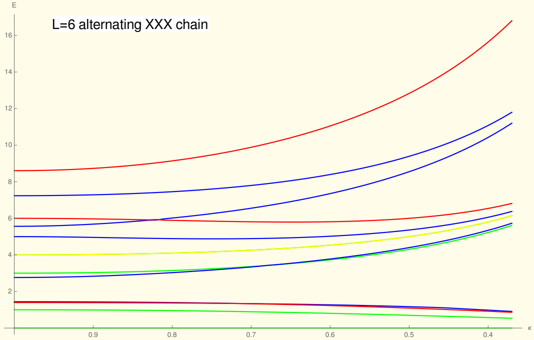
Dispersion Relation

$$E(p) = 2 - 2\cos(p) \quad \rightarrow \quad E(p)\frac{1}{\kappa} + \kappa \pm \frac{1}{\kappa}\sqrt{(1+\kappa^2)^2 - 4\kappa^2 \sin^2 p}$$



XY sector: Spectrum

- Explicitly diagonalise the Hamiltonian for short chains



Red: 3 magnons
Blue: 2 magnons
Green: 1 magnon

- 2-magnon Bethe ansatz in [Pomoni, Rabe, KZ '21]
- For XZ sector also Z-vacuum Bethe Ansatz - dynamical Temperley-Lieb
- Recent progress on 3 and 4 magnon problems [Bozkurt,Kong,Nieto-Garcia,Pomoni]

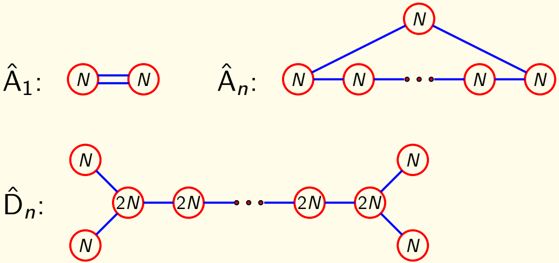
General ADE Orbifolds

- $\mathbb{Z}_2$ is still quite special – in general we don't have the $\mathrm{SU}(2)_L$ symmetry.
- We would like to study spin chains for the most general $\mathcal{N} = 2$ gauge theories
- Orbifold $\mathcal{N} = 4$ SYM by any discrete subgroup Γ of $\mathrm{SU}(2)$
- Cyclic, binary dihedral, tetrahedral, octahedral and icosahedral groups
- McKay correspondence: Related to $\hat{A}_k, \hat{D}_k, \hat{E}_6, \hat{E}_7, \hat{E}_8$
- Impose Γ invariance:

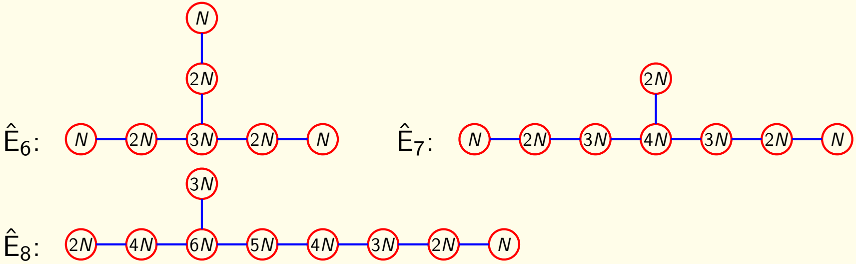
$$\Phi^I = \mathcal{R}^I{}_J(g)\gamma_{\text{reg}}(g)\Phi^J\gamma_{\text{reg}}^{-1}(g) \ \text{, for } \ g \in \Gamma$$

- Integrable when all the gauge couplings are equal [Beisert, Roiban '05, Solovyov '07]
- Marginally deform by varying the couplings of each node independently

ADE Quiver theories



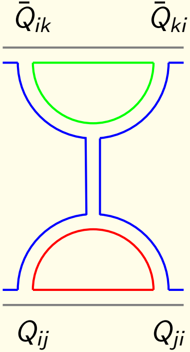
Quiver	$\hat{A}_k$	$\hat{D}_k$	$\hat{E}_6$	$\hat{E}_7$	$\hat{E}_8$
Γ	$\mathbb{Z}_{k+1}$	$\hat{\mathrm{D}}_k$	2T	2O	2I
M	$k+1$	$k+1$	7	8	9
H	$k+1$	k	6	7	8



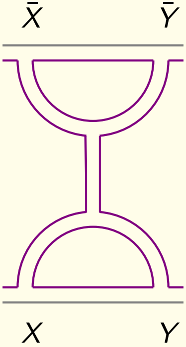
Superspace diagrams

- Two holomorphic bifundamentals:

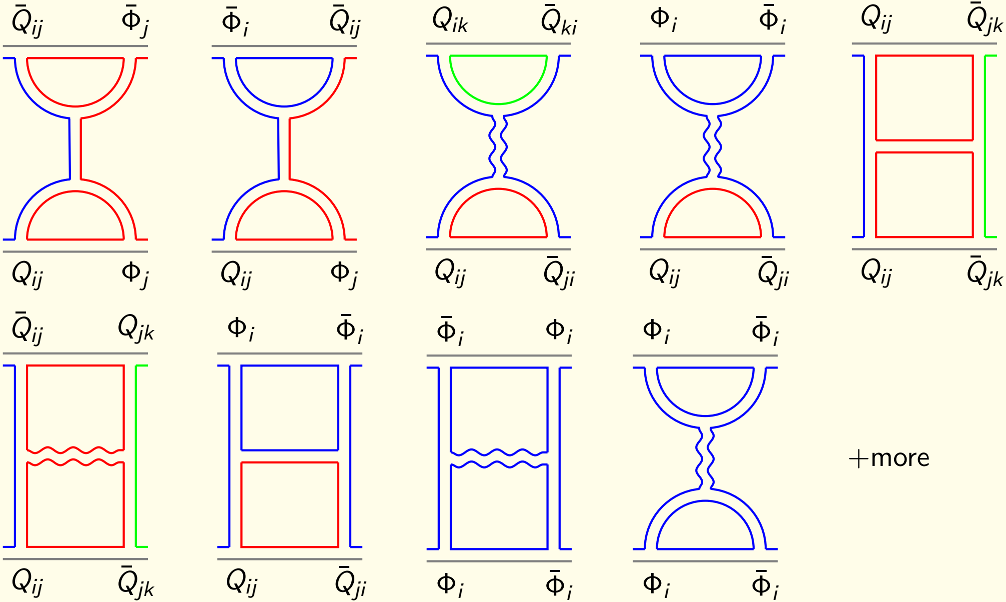
$\mathcal{N} = 2$ $\qquad\qquad$ $\mathcal{N} = 4$



$$= \kappa_i^2 \frac{\bar{d}_{ji} d_{ki}}{n_j n_i^3} \times$$



Superspace diagrams

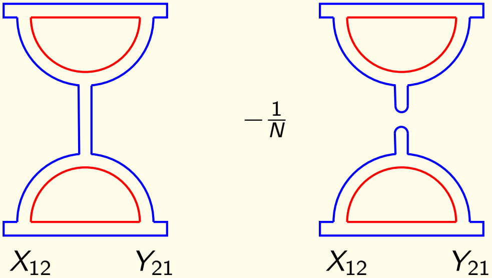


Superspace diagrams - subleading corrections

- We are in $\mathrm{SU}(N)$

$$\langle (A_\mu)^a{}_b (A_\nu)^c{}_d \rangle = \frac{\eta_{\mu\nu}}{p^2} \left(\delta^a{}_d \delta^c{}_b - \frac{1}{N} \delta^a{}_b \delta^c{}_d \right)$$

- The subleading part contributes at $L = 2$



- Takes certain anomalous dimensions to zero

The ADE Hamiltonian

- Holomorphic bifundamentals

$$\mathcal{H}_{\ell,\ell+1} = \frac{2\kappa_i^2}{n_i^3} \begin{pmatrix} \frac{\bar{d}_{ij_1} d_{ij_1}}{n_{j_1}} & \frac{\bar{d}_{ij_1} d_{ij_2}}{n_{j_1}} & \cdots & \frac{\bar{d}_{ij_1} d_{ij_{H_i}}}{n_{j_1}} \\ \frac{\bar{d}_{ij_2} d_{ij_1}}{n_{j_2}} & \frac{\bar{d}_{ij_2} d_{ij_2}}{n_{j_2}} & \cdots & \frac{\bar{d}_{ij_2} d_{ij_{H_i}}}{n_{j_2}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\bar{d}_{ij_{H_i}} d_{ij_1}}{n_{j_{H_i}}} & \frac{\bar{d}_{ij_{m_i}} d_{ij_2}}{n_{j_{H_i}}} & \cdots & \frac{\bar{d}_{ij_{H_i}} d_{ij_{H_i}}}{n_{j_{H_i}}} \end{pmatrix} \text{ in the basis } \begin{pmatrix} Q_{ij_1} Q_{j_1 i} \\ Q_{ij_2} Q_{j_2 i} \\ \vdots \\ Q_{ij_{H_i}} Q_{j_{H_i} i} \end{pmatrix}$$

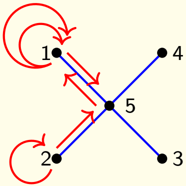
- Holomorphic bifundamental + adjoint

$$\mathcal{H}_{\ell,\ell+1} = \begin{pmatrix} 2\kappa_i^2 & -2\kappa_i\kappa_j \\ -2\kappa_i\kappa_j & 2\kappa_j^2 \end{pmatrix} \text{ in the basis } \begin{pmatrix} Z_i Q_{ij} \\ Q_{ij} Z_j \end{pmatrix}.$$

- + antiholomorphic + mixed sectors

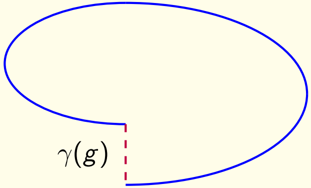
The ADE spin chains

- The ADE spin chains have "$\mathrm{SU}(4)^{\kappa}_{\Gamma}$ groupoid" symmetry
- Restricted Hilbert Space: Follow the quiver path algebra



$$\longrightarrow \qquad \cdots Z_2 Y_{25} X_{51} Z_1 Z_1 Y_{15} \cdots$$

- Twisted boundary conditions



$$\longrightarrow \qquad \mathrm{Tr}(\gamma(g) ZZXYZYX \cdots) \qquad g \in \Gamma$$

The $\mathbb{Z}_3$ Orbifold

- Definition:

$$\{a | a^3 = 1\}$$

- Action on $\mathrm{SU}(3N)$

$$\gamma(e) = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \quad \gamma(a) = \begin{pmatrix} 1 & & \\ & \omega_3 & \\ & & \omega_3^2 \end{pmatrix}, \quad \gamma(a^2) = \begin{pmatrix} 1 & & \\ & \omega_3^2 & \\ & & \omega_3 \end{pmatrix}$$

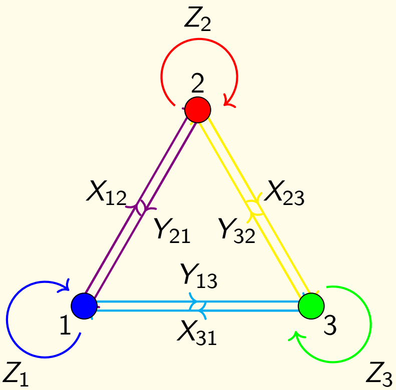
- Fields

$$Z = \begin{pmatrix} Z_1 & & \\ & Z_2 & \\ & & Z_3 \end{pmatrix}, X = \begin{pmatrix} & X_{12} & \\ & & X_{23} \\ X_{31} & & \end{pmatrix}, Y = \begin{pmatrix} & & Y_{13} \\ Y_{21} & & \\ & Y_{32} & \end{pmatrix}$$

The $\mathbb{Z}_3$ Quiver

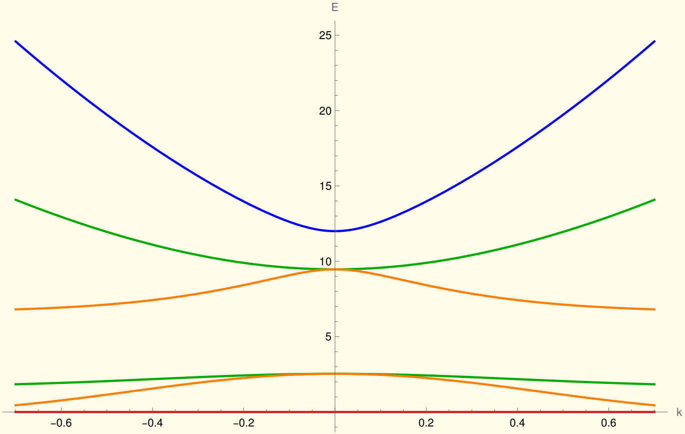
$\mathrm{SU}(N)^3$, 3 Hypermultiplets

1 untwisted, 2 twisted sectors



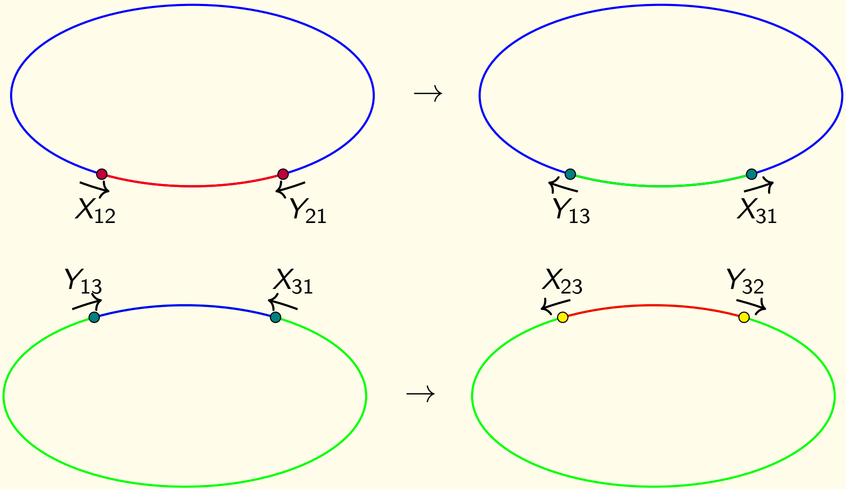
$\mathbb{Z}_3$: $L = 3$ Holomorphic Spectrum

$$\kappa_1 = 1 - k, \kappa_2 = 1, \kappa_3 = 1 + k$$

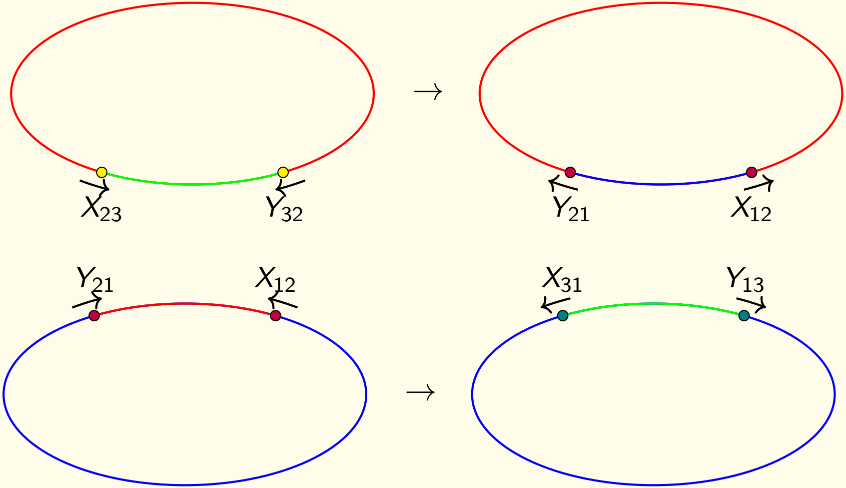


- Reproduce using Bethe ansatz?

Scattering on the $\mathbb{Z}_3$ chain



Scattering on the $\mathbb{Z}_3$ chain



$\mathbb{Z}_3$ Bethe Ansatz - Open Chain

- Single-magnon wavefunction: $|\psi\rangle^{(1)}_{i,j} = \sum_\ell e^{ip} |\ell\rangle_{i,j}$, where

$$|\ell\rangle^X_{i,i+1} = \cdots Z_i Z_i \underset{\ell}{X_{i,i+1}} Z_{i+1} Z_{i+1} \cdots \quad , \quad |\ell\rangle^Y_{i,i-1} = \cdots Z_i Z_i \underset{\ell}{Y_{i,i-1}} Z_{i-1} Z_{i-1} \cdots \quad ,$$

- Dispersion relation

$$E_{i,j} = 2(\kappa_i^2 + \kappa_j^2) - 2\kappa_i \kappa_j (e^{ip} + e^{-ip})$$

- One X and one Y magnon - same exterior, different interior vacuum

$$|\ell_1, \ell_2\rangle^{XY}_i = \cdots Z_i Z_i \underset{\ell_1}{X_{i,i+1}} Z_{i+1} \cdots Z_{i+1} \underset{\ell_2}{Y_{i+1,i}} Z_i Z_i \cdots$$

$$|\ell_1, \ell_2\rangle^{YX}_i = \cdots Z_i Z_i \underset{\ell_1}{Y_{i,i-1}} Z_{i-1} \cdots Z_{i-1} \underset{\ell_2}{X_{i-1,i}} Z_i Z_i \cdots$$

- Bethe Ansatz (note different momenta for XY and YX):

$$|\psi\rangle^{\text{2-mag}}_1 = \sum_{1 \le \ell_1 < \ell_2 \le L} \Big[\big(A_1 e^{ip_1\ell_1 + ip_2\ell_2} + B_1 e^{ip_2\ell_1 + ip_1\ell_2} \big) |\ell_1, \ell_2\rangle^{XY}_i$$
$$+ \big(C_1 e^{iq_2\ell_1 + iq_1\ell_2} + D_1 e^{iq_1\ell_1 + iq_2\ell_2} \big) |\ell_1, \ell_2\rangle^{YX}_i \Big]$$

$\mathbb{Z}_3$ S-matrix

$$\begin{pmatrix} A_1 \\ D_1 \end{pmatrix} = \begin{pmatrix} S_{AB} & S_{AC} \\ S_{DB} & S_{DC} \end{pmatrix} \begin{pmatrix} B_1 \\ C_1 \end{pmatrix}$$

$$S_{AB} = 1 + (e^{ip_1} - e^{ip_2})(\kappa_1\kappa_3(\kappa_1^2 - 2\kappa_2^2)(1 + e^{i(q_1+q_2)}) - 2e^{iq_2}(\kappa_1^2(\kappa_2^2 + \kappa_3^2) - 2\kappa_2^2\kappa_3^2))/\mathcal{D} ,$$

$$S_{AC} = \kappa_1^3\kappa_3 e^{i(q_1+q_2)} \left(e^{iq_1} - e^{-iq_1} - e^{iq_2} + e^{-iq_2}\right)/\mathcal{D}$$

$$S_{DB} = \kappa_1^3\kappa_2 e^{i(p_1+p_2)} \left(e^{ip_1} - e^{-ip_1} - e^{ip_2} + e^{-ip_2}\right)/\mathcal{D}$$

$$S_{DC} = 1 + (e^{iq_1} - e^{iq_2})(\kappa_1\kappa_2(\kappa_1^2 - \kappa_3^2)(1 + e^{i(p_1+p_2)}) + e^{ip_2}(4\kappa_2^2\kappa_3^2 - 2\kappa_1^2(\kappa_2^2 + \kappa_3^2)))/\mathcal{D}$$

$$\begin{aligned}\mathcal{D} = {} & (1 + e^{i(p_1+p_2)})(1 + e^{i(q_1+q_2)})\kappa_1^2\kappa_2\kappa_3 + e^{ip_2}(1 + e^{iq_1+iq_2})\kappa_1(\kappa_1^2 - 2\kappa_2^2)\kappa_3 \\ & + e^{iq_2}(1 + e^{ip_1+ip_2})\kappa_1\kappa_2(\kappa_1^2 - 2\kappa_3^3) - 2e^{ip_2+iq_2}(\kappa_1^2(\kappa_2^2 + \kappa_3^2 - 2\kappa_2^2\kappa_3^2)\end{aligned}$$

$\mathbb{Z}_3$ Bethe Ansatz - Closed chain

- Impose cyclicity on the spin chain wavefunction, e.g. for $L = 3$:

$$X_{12}Y_{21}Z_1 \to A_1 e^{ip_1+2ip_2} + B_1 e^{ip_2+2ip_1}$$
$$Z_1 X_{12} Y_{21} \to A_1 e^{2ip_1+3ip_2} + B_1 e^{2ip_2+3ip_1}$$
$$Y_{21} Z_1 X_{12} \to C_2 e^{ip_2+3ip_1} + D_2 e^{ip_1+3ip_2}$$

- Imposes the centre-of-mass condition $p_2 = -p_1$, $q_2 = -q_1$
- Can solve the scattering problem and obtain the energies

$(\kappa_1, \kappa_2, \kappa_3) = (1, 1, 1)$		$(\kappa_1, \kappa_2, \kappa_3) = (0.8, 0.9, 1)$	
p_1	E	p_1	E
"0"	0	$-0.1179i$	0
2.0944	12	2.4737	10.3225
0.8189	2.5359	0.8463	1.9827
0.8189	2.5359	0.8562	2.02549
1.7538	9.4641	1.9207	7.7745
1.7538	9.4641	1.8333	7.2948

- All values agree with explicit diagonalisation

$\mathbb{Z}_3$ Bethe Ansatz - Orbifold point

- In general the S-matrix satisfies $S^* S = I$
- Becomes symmetric at the orbifold point, and then $S^\dagger S = I$

$$S_{\text{o.p.}} = \frac{1}{1-2e^{ip_2} + e^{i(p_1+p_2)}} \begin{pmatrix} -(1-e^{ip_1})(1-e^{ip_2}) & e^{ip_1} - e^{ip_2} \\ e^{ip_1} - e^{ip_2} & -(1-e^{ip_1})(1-e^{ip_2}) \end{pmatrix} ,$$

p_1	E	Sector
"0"	0	Untwisted
$\frac{2\pi}{3}$	12	Untwisted
$i \ln \frac{1+\sqrt{3}-i\sqrt{12-2\sqrt{3}}}{4}$	$2(3-2\sqrt{3})$	a-twisted
$i \ln \frac{1+\sqrt{3}-i\sqrt{12-2\sqrt{3}}}{4}$	$2(3-2\sqrt{3})$	a^2-twisted
$i \ln \frac{1-\sqrt{3}-i\sqrt{12+2\sqrt{3}}}{4}$	$2(3+2\sqrt{3})$	a-twisted
$i \ln \frac{1-\sqrt{3}-i\sqrt{12+2\sqrt{3}}}{4}$	$2(3+2\sqrt{3})$	a^2-twisted

- Twisted sectors distinguished by signs in $p_1 = \pm q_1 = \pm r_1$.

The $\hat{D}_4$ orbifold

- Definition of the binary dihedral group:

$$\{a, b | a^{2n} = 1, b^2 = a^n, bab^{-1} = a^{-1}\} \ .$$

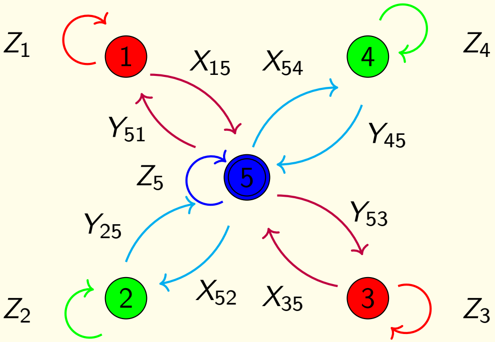
- Action on $\mathrm{SU}(8N)$

$$\gamma(a) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & iI_{2\times 2} & 0 \\ 0 & 0 & 0 & 0 & 0 & -iI_{2\times 2} \end{pmatrix} \ , \ \gamma(b) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{2\times 2} \\ 0 & 0 & 0 & 0 & -I_{2\times 2} & 0 \end{pmatrix} \ ,$$

The $\hat{D}_4$ quiver

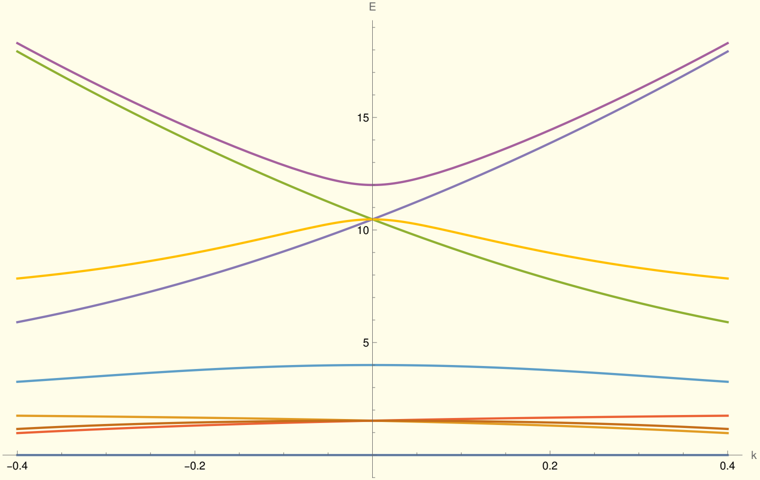
$\mathrm{SU}(2N) \times \mathrm{SU}(N)^4$, 4 Hypermultiplets

1 untwisted, 4 twisted sectors

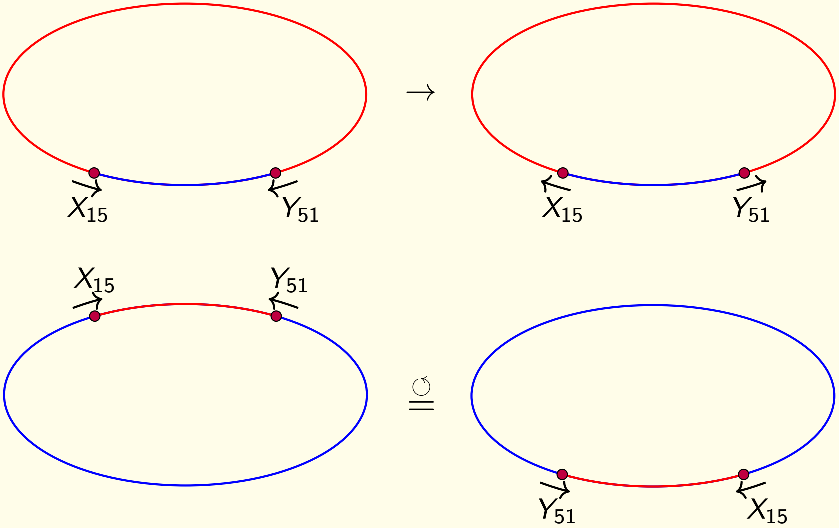


$\hat{D}_4$: $L = 3$ Holomorphic spectrum

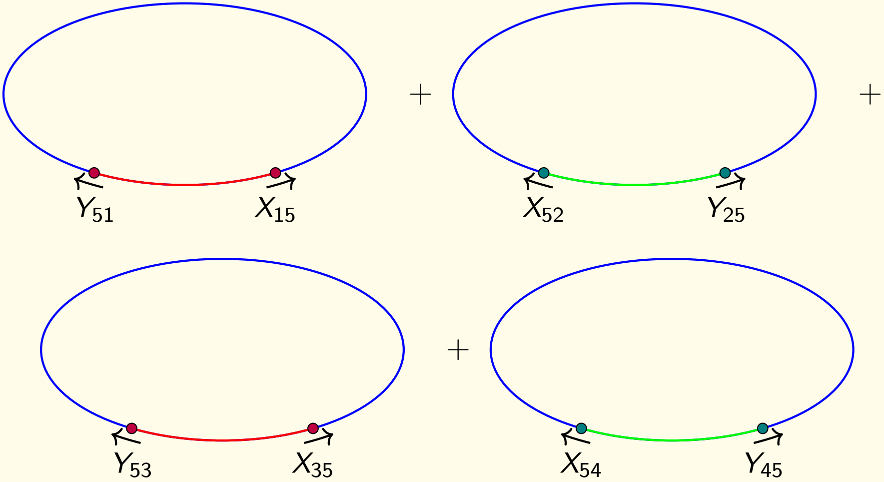
$$\kappa_1 = \kappa_3 = 1 - k, \kappa_2 = \kappa_4 = 1 + k, \kappa_5 = 1$$



Scattering on the D_4 chain



Scattering on the $\hat{D}_4$ chain



The $\hat{D}_4$ Bethe Ansatz

- We can set up a similar Bethe Ansatz to the $\mathbb{Z}_3$ case.
- Imposing periodicity one can reproduce the energies from explicit diagonalisation

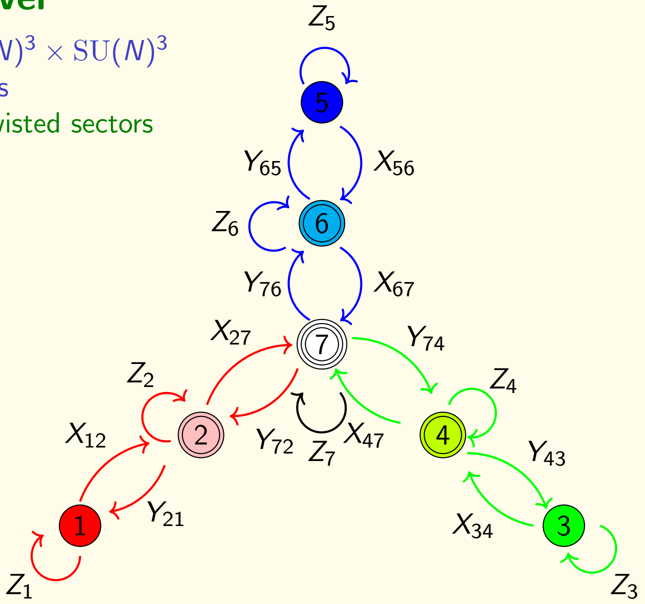
$(1,1,1,1,1)$		$(0.9,0.9,0.9,0.9,1)$		$(0.9,0.8,0.93,0.99,1)$	
p_1	E	$p_1^{(1)}$	E	$p_1^{(1)}$	E
2.0944	12	2.1072	10.9193	2.1689	11.2942
1.885	10.4721	1.8246	9.0476	1.9473	9.8874
1.885	10.4721	1.8246	9.0476	1.8493	9.2192
1.885	10.4721	1.8246	9.0476	1.6932	8.1193
1.0472	4	1.0344	3.5607	1.0332	3.5528
0.6284	1.5279	0.6324	1.4324	0.6475	1.4972
0.6284	1.5279	0.6324	1.4324	0.6357	1.4465
0.6284	1.5279	0.6324	1.4324	0.6111	1.3433

The $\hat{E}_6$ quiver

$\mathrm{SU}(3N) \times \mathrm{SU}(2N)^3 \times \mathrm{SU}(N)^3$

6 Hypermultiplets

1 untwisted, 6 twisted sectors



Back to $\mathbb{Z}_2$: Energy splittings

- The $\mathrm{SU}(2)$ symmetry is broken in the XZ sector
- E.g. $L = 3$: $\ydiagram{1}^{\,3} = \ydiagram{3} + 2 \times \ydiagram{2,1}$
- Doublet with $E = 1$ at orbifold point

$$\left|\frac{1}{2}, \frac{1}{2}\right\rangle_1 = -X_{12}X_{21}Z_1 + Z_1X_{12}X_{21} \quad , \quad E = \frac{1}{\kappa}$$

$$\left|\frac{1}{2}, -\frac{1}{2}\right\rangle_1 = -\kappa X_{12}Z_2Z_2 + (1-\kappa)Z_1X_{12}Z_2 + Z_1Z_1X_{12} \quad , \quad E = \frac{1}{\kappa} - 1 + \kappa$$

- Doublet with $E = 3$ at orbifold point

$$\left|\frac{1}{2}, \frac{1}{2}\right\rangle_3 = X_{12}X_{21}Z_1 - 2\kappa X_{12}Z_2X_{21} + Z_1X_{12}X_{21} \quad , \quad E = \frac{1}{\kappa} + 2\kappa$$

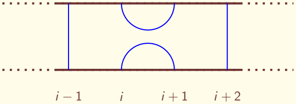
$$\left|\frac{1}{2}, -\frac{1}{2}\right\rangle_3 = \kappa X_{12}Z_2Z_2 - (1+\kappa)Z_1X_{12}Z_2 + Z_1Z_1X_{12} \quad , \quad E = \frac{1}{\kappa} + 1 + \kappa$$

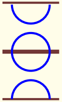
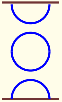
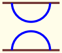
Some questions

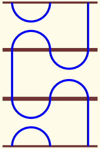
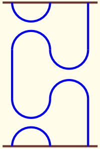
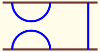
- How exactly does the breaking happen?
- Can we still raise/lower states within these broken multiplets?
- Is there a mathematically rigorous "$\mathrm{SU}(2)_\kappa$" quantum group associated to the XZ sector?
- Some aspects discussed in [Bertle,Pomoni,Zhang,KZ '24]: Groupoid symmetries (restricted Hilbert space), Twists, Quasi-Hopf algebras $\rightarrow$ Elli's IGST '25 talk
- Here: A dynamical version of the Temperley-Lieb algebra

Standard Temperley-Lieb algebra

$$e_i^2 = -(\kappa + \kappa^{-1})e_i \quad , \quad e_i e_{i\pm 1} e_i = e_i \quad , \quad e_i e_j = e_j e_i \ , \ |i-j| > 1$$

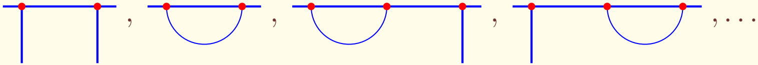
$e_i =$ 

$e_i^2 =$  $=$  $= \ -(\kappa + \kappa^{-1})$  $= -(\kappa + \kappa^{-1})e_i$

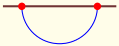
$e_1 e_2 e_1 =$  $=$  $=$  $= e_1$

Standard Temperley-Lieb algebra

- TL acts on link states

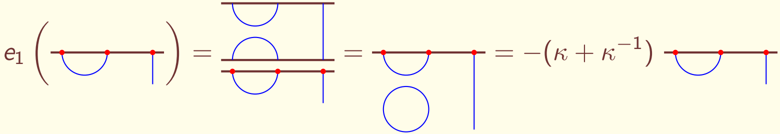


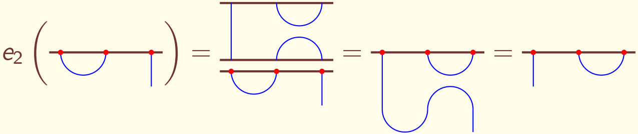
- Map to the spin-chain state space

 $= (XY - \kappa^{-1} YX)$,  $= \{XX, (XY + \kappa YX), YY\}$

Standard Temperley-Lieb algebra

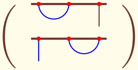
- Length-3





- The Hamiltonian $\mathcal{H} = -e_1 - e_2$ acts as

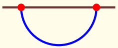
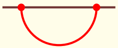
$$\mathcal{H} = \begin{pmatrix} \kappa + \kappa^{-1} & -1 \\ -1 & \kappa + \kappa^{-1} \end{pmatrix}$$ in the basis



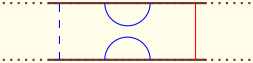
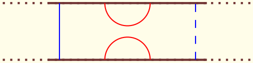
- Agrees with XXZ energies

Dilute dynamical Temperley-Lieb algebra

- Generalise the link states to κ-(anti)symmetrisation

 $= \left(X_{12}Z_2 - \frac{1}{\kappa} Z_1 X_{12}\right)$,  $= \left(X_{21}Z_1 - \kappa Z_2 X_{21}\right)$

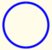
- Colour indicates the first index

$e_i^{(1)} =$  , $e_i^{(2)} =$ 

- Satisfies the correct ddTL relations

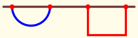
$$e_i^2 = -(\kappa + \kappa^{-1}) e_i \ , \ e_i e_{i\pm 1} e_i = \kappa^{s_L(1)+s_R(3)} e_i \ , \ e_i e_j = e_j e_i \ , |i-j| > 1$$

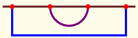
- Graphically:

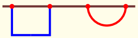
 $=$  $= -(\kappa + \kappa^{-1})$

Efficient graphical computation of the spectrum

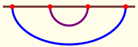
- 5 symmetry structures: $\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline\end{array}$, $\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2\\\cline{1-1}\end{array}$, $\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3\\\cline{1-1}\end{array}$, $\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4\\\cline{1-1}\end{array}$, $\begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline\end{array}$, $\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array}$
- In the link basis we have (apart from BPS)

 $= (X_{12}Z_2 - \frac{1}{\kappa}Z_1X_{12})(X_{21}Z_1 + \frac{1}{\kappa}Z_2X_{21})$,

 $= X_{12}(X_{21}Z_1 - \kappa Z_2X_{21})Z_1 + \kappa Z_1(X_{12}Z_2 - \kappa^{-1}Z_1X_{12})X_{21}$,

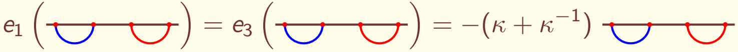
 $= (X_{12}Z_2 + \kappa Z_1X_{12})(X_{21}Z_1 - \kappa Z_2X_{21})$,

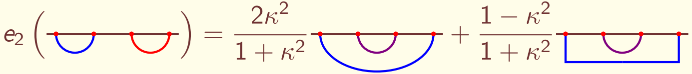
 $= (X_{12}Z_2 - \frac{1}{\kappa}Z_1X_{12})(X_{21}Z_1 - \kappa Z_2X_{21})$

 $= X_{12}(X_{21}Z_1 - \kappa Z_2X_{21})Z_1 - \frac{1}{\kappa}Z_1(X_{12}Z_2 - \frac{1}{\kappa}Z_1X_{12})X_{21}$.

Efficient graphical computation of the spectrum

- The action is different to TL due to the need to preserve the κ (anti) symmetrisations



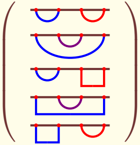


Efficient graphical computation of the spectrum

- Acting on the basis we obtain the mixing matrix:

$$\mathcal{H} = \begin{pmatrix} 2(\kappa+\kappa^{-1}) & -\frac{2}{\kappa+\kappa^{-1}} & 0 & -\frac{\kappa^{-1}-\kappa}{\kappa+\kappa^{-1}} & 0 \\ -2 & \kappa+\kappa^{-1} & 1-\kappa^2 & 0 & \\ 0 & \frac{\kappa^{-1}-\kappa}{\kappa+\kappa^{-1}} & \kappa+\kappa^{-1} & -\frac{2}{1+\kappa^2} & 0 \\ 0 & 0 & -\kappa^2 & \kappa+\kappa^{-1} & -1 \\ 0 & -\frac{\kappa^2-\kappa^4}{1+\kappa^2} & 0 & -\frac{1+\kappa^4}{1+\kappa^2} & \kappa+\kappa^{-1} \end{pmatrix}$$

in the basis



- Gives correct eigenvalues and eigenstates
- At $\kappa \neq 1$, mixing between singlets and triplets

$$\lambda^5\left(\kappa^2-\kappa\lambda+1\right)^2\left(2\kappa^2\lambda-\kappa\lambda^2-2\kappa+2\lambda\right)\left(2\kappa^3\lambda-\kappa^2\lambda^2-\kappa^2+2\kappa\lambda-1\right)$$
$$\times\left(\kappa^4-2\kappa^3\lambda+\kappa^2\lambda^2-2\kappa\lambda+1\right)\left(4\kappa^5\lambda-4\kappa^4\lambda^2-6\kappa^4+\kappa^3\lambda^3+9\kappa^3\lambda-4\kappa^2\lambda^2-4\kappa^2+5\kappa\lambda-2\right)$$

$$\downarrow^{\kappa\to 1}$$

$$\lambda^5(\lambda-2)^2\left(\lambda^2-4\lambda+2\right)^3(\lambda-2)(\lambda^2-6\lambda+6)$$

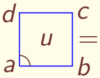
Summary and outlook

- Constructed the general ADE quiver Hamiltonian
- Studied the spectrum (a) explicitly, (b) using the Bethe ansatz and (c) via indices
- In the $\mathbb{Z}_2$ case, identified a dynamical version of the usual TL algebra

- Understand the groupoid-like hidden symmetries through a dynamical version of Schur-Weyl duality
- Gravity side of the correspondence $\Rightarrow \mathrm{AdS}_5 \times \mathrm{S}^5/\Gamma_{\kappa_i}$
- Integrability?

Thanks!

RSOS models



$$= \; W\left(\begin{array}{cc|c} d & c & \\ a & b & u \end{array}\right) .$$

- Lattice models where the Boltzmann weights depend on labels around each face (called heights)
- Introduced in [Andrews, Baxter, Forrester '84]
- Original model related to XYZ spin chain

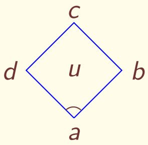
$$W\left(\begin{array}{cc|c} a & a+1 & \\ a+1 & a+2 & u \end{array}\right) = W\left(\begin{array}{cc|c} a & a-1 & \\ a-1 & a-2 & u \end{array}\right) = \frac{\theta_1(2\eta - u)}{\theta_1(2\eta)}$$

$$W\left(\begin{array}{cc|c} a & a+1 & \\ a-1 & a & u \end{array}\right) = W\left(\begin{array}{cc|c} a & a-1 & \\ a+1 & a & u \end{array}\right) = \frac{\sqrt{\theta_1(2\eta(a-1)+w_0)\theta_1(2\eta(a+1)+w_0)}}{\theta_1(2\eta a + w_0)} \frac{\theta_1(u)}{\theta_1(2\eta)}$$

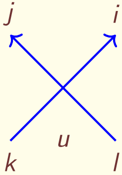
$$W\left(\begin{array}{cc|c} a & a+1 & \\ a+1 & a & u \end{array}\right) = \frac{\theta_1(2\eta a + w_0 + u)}{\theta_1(2\eta a + w_0)} \;,\quad W\left(\begin{array}{cc|c} a & a-1 & \\ a-1 & a & u \end{array}\right) = \frac{\theta_1(2\eta a + w_0 - u)}{\theta_1(2\eta a + w_0)}$$

Face-vertex map

- RSOS and vertex models can be mapped to each other
- Graphically:



$= \lambda$

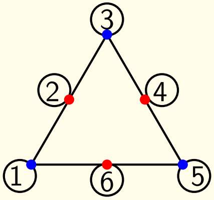


$= R(u,\lambda)^{ij}_{kl}$

- For $\mathrm{SU}(2)$ ABF case introduced by [Felder '94]
- The R-matrix depends on dynamical parameter $\lambda = -2\eta d$
- Satisfies a dynamical Yang-Baxter equation

$$\begin{aligned} &R_{23}(u_2 - u_3; \lambda) R_{13}(u_1 - u_3; \lambda + 2\eta h^{(2)}) R_{12}(u_1 - u_2; \lambda) \\ &= R_{12}(u_1 - u_2; \lambda + 2\eta h^{(3)}) R_{13}(u_1 - u_3; \lambda) R_{23}(u_2 - u_3; \lambda + 2\eta h^{(1)}) \, . \end{aligned}$$

$\mathbb{Z}_2$ orbifold as a dilute CSOS model



- Corresponds to a dynamical 15-vertex model
- The allowed heights are $d = 1, \ldots, 6$
- Dynamical: $R(u, \kappa)$ for d odd, $R(u, 1/\kappa)$ for d even
- Expect elliptic dependence on u
- Can the star-triangle relation be satisfied with this input?

The $\hat{D}_4$ Bethe Ansatz - Open chain

- Now need to distinguish betwen outer vacuum being 5 or $i = 1, 2, 3, 4$

$$|\ell\rangle_{i5} = \cdots Z_i Z_i \underset{\ell}{Q_{i5}} Z_5 Z_5 \cdots \quad |\ell\rangle_{5i} = \cdots Z_5 Z_5 \underset{\ell}{Q_{5i}} Z_i Z_i \cdots \quad ,$$

- Dispersion relation: $E_{i5} = E_{5i} = 2(\kappa_i^2 + \kappa_5^2) - 2\kappa_i\kappa_5(e^{ip^{(i)}} + e^{-ip^{(i)}})$
- Two magnons: External i vacuum

$$|\ell_1, \ell_2\rangle_i = \cdots Z_i Z_i \underset{\ell_1}{Q_{i5}} Z_5 \cdots Z_5 \underset{\ell_2}{Q_{5i}} Z_i Z_i \cdots$$

$$|\psi\rangle_i = \sum_{\ell} \left(A_i e^{ip_1^{(i)} + ip_2^{(i)}} + B_i e^{ip_2^{(i)} + ip_1^{(i)}} \right) |\ell_1, \ell_2\rangle$$

- S-matrix satisfies $S^* S = I$

$$S_i = \frac{B_i}{A_i} = -\frac{\kappa_i\kappa_5(1 + e^{i(p_1^{(i)} + p_2^{(i)})}) + 2e^{ip_2^{(i)}}(\kappa_i^2 - \kappa_5^2)}{\kappa_i\kappa_5(1 + e^{i(p_1^{(i)} + p_2^{(i)})}) + 2e^{ip_1^{(i)}}(\kappa_i^2 - \kappa_5^2)}$$

The $\hat{D}_4$ Bethe Ansatz - Open chain

- Two-magnons: Exterior 5 vacuum

$$|\ell_1, \ell_2\rangle_i^{(5)} = \cdots Z_5 Z_5 \underset{\ell_1}{Q_{5i}} Z_i \cdots Z_i \underset{\ell_2}{Q_{i5}} Z_5 Z_5 \cdots$$

- Bethe ansatz

$$|\psi\rangle_i^{(5)} = \sum_{i=1}^{4} (-1)^{i+1} \left(C_i e^{ip_2^{(i)} + ip_1^{(i)}} + D_i e^{ip_1^{(i)} + ip_2^{(i)}} \right) |\ell_1, \ell_2\rangle_i^{(5)}$$

with

$$p_1^{(i)} + p_2^{(i)} = p_1^{(j)} + p_2^{(j)} \text{ and } E_i^{(2)}(p_1^{(i)}, p_2^{(i)}) = E_j^{(2)}(p_1^{(j)}, p_2^{(j)})$$

- S-matrix

$$D_i = \sum_j S_{ij} C_j$$

The $\hat{D}_4$ S-matrix

- First define:

$$n_k^{(i)} = \kappa_5 - 2\kappa_i e^{ip_k^{(i)}} + \kappa_5 e^{i(p_1^{(i)}+p_2^{(i)})}$$

$$m_k^{(i)} = 8e^{ip_k^{(i)}}\kappa_{i+1}\kappa_{i+2}\kappa_{i+3}\kappa_5^6 n_k^{(i+1)} n_k^{(i+2)} n_k^{(i+3)}$$

and

$$\mathcal{N}(k^1,k^2,k^3,k^4) = m_{k^1}^{(1)} + m_{k^2}^{(2)} + m_{k^3}^{(3)} + m_{k^4}^{(4)} + 16\kappa_1\kappa_2\kappa_3\kappa_4\kappa_5^4 n_{k^2}^{(1)} n_{k^2}^{(2)} n_{k^3}^{(3)} n_{k^4}^{(4)}$$

- The S-matrix is

$$S_{11} = -\frac{\mathcal{N}(1,2,2,2)}{\mathcal{N}(2,2,2,2)}\,, S_{22} = -\frac{\mathcal{N}(2,1,2,2)}{\mathcal{N}(2,2,2,2)}\,, S_{33} = -\frac{\mathcal{N}(2,2,1,2)}{\mathcal{N}(2,2,2,2)}\,, S_{44} = -\frac{\mathcal{N}(2,2,2,1)}{\mathcal{N}(2,2,2,2)}$$

$$S_{ij} = -\frac{8}{\mathcal{N}(2,2,2,2)}(\prod_{j\neq i}\kappa_i)\kappa_5^7\left(e^{ip_1^{(j)}} - e^{ip_2^{(j)}} + e^{i(2p_1^{(j)}+p_2^{(j)})} + e^{i(2p_2^{(j)}+p_1^{(j)})}\right)\prod_{m\neq i,j} n_2^{(m)}$$

- Again $S^*S = I \to S^\dagger S = I$ at the orbifold point

The $\hat{D}_4$ Bethe ansatz - Closed chain

- For $L = 3$ impose e.g. $Z_5 Y_{51} X_{15} = Y_{51} X_{15} Z_5 = X_{15} Z_5 Y_{51}$

$$p_2^{(i)} = -p_1^{(i)} \text{ and } C_i = A_i e^{-3ip_1^{(i)}} \ , \ D_i = B_i e^{3ip_1^{(i)}}$$

- Orbifold point

$p_1^{(1)}$	E	Sector
$\frac{2\pi}{3}$	12	Untwisted
$3\pi/5$	$2(3+2\sqrt{5})$	a-twisted
$3\pi/5$	$2(3+2\sqrt{5})$	b-twisted
$3\pi/5$	$2(3+2\sqrt{5})$	ab-twisted
$\pi/3$	4	a^2-twisted
$\pi/5$	$2(3-2\sqrt{5})$	a-twisted
$\pi/5$	$2(3-2\sqrt{5})$	b-twisted
$\pi/5$	$2(3-2\sqrt{5})$	ab-twisted

- Note no $E = 0$ state

The $\hat{D}_4$ Bethe Ansatz - Closed chain

- For general $(\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5)$:

$(1,1,1,1,1)$		$(0.9,0.9,0.9,0.9,1)$		$(0.9,0.8,0.93,0.99,1)$	
p_1	E	$p_1^{(1)}$	E	$p_1^{(1)}$	E
2.0944	12	2.1072	10.9193	2.1689	11.2942
1.885	10.4721	1.8246	9.0476	1.9473	9.8874
1.885	10.4721	1.8246	9.0476	1.8493	9.2192
1.885	10.4721	1.8246	9.0476	1.6932	8.1193
1.0472	4	1.0344	3.5607	1.0332	3.5528
0.6284	1.5279	0.6324	1.4324	0.6475	1.4972
0.6284	1.5279	0.6324	1.4324	0.6357	1.4465
0.6284	1.5279	0.6324	1.4324	0.6111	1.3433