The conformal realms of quantum electrodynamics

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Summary

I. Spin-orbit duality

II. QED_4 as CSM_3 on $\mathbb{S}^2 \times \mathbb{R}$

III. The conformal realms of QED_4

Part I

Spin-orbit duality

Part I – The duality

Noether's theorem on Lorentz invariance, $\dot{J}^{\mu\nu} = 0$, decomposes

$$J^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu} \tag{1}$$

This **kinematic/dynamic** complementarity is made geometric by $(1+3)$-decomposing wrt $\eta^{\mu\nu} = \frac{p^\mu p^\nu}{p^2} + h^{\mu\nu}$,

$$J^{\mu\nu} = E^{\mu\nu} + H^{\mu\nu} \quad : \qquad \left\{ p_\nu \star E^{\mu\nu} = 0 \ , \ H^{\mu\nu} p_\nu = 0 \right\} := \mathcal{H} \tag{2}$$

This is a **Hodge decomposition**, generalization of the $\mathbb{R}^3$ Helmholtz decomposition (into curl-free and divergence-free parts). For $J^{\mu\nu}$,

$$p_\nu \star L^{\mu\nu} = 0 \quad \text{and} \quad S^{\mu\nu} p_\nu = 0 \quad \text{(SSC)} \tag{3}$$

Algebraically, SSC $=$ $S^{\mu\nu}$ set as generators of the little group.

Part I – The duality

Hodge decomposition separates between **electric** and **magnetic** parts,

$$J^{\mu\nu} \;=\; E^{\mu} \wedge p^{\nu} \;+\; \star(H^{\mu} \wedge p^{\nu}) \tag{4}$$

where

$$\begin{aligned} E^{\mu} &= \frac{L^{\mu\nu} p_{\nu}}{p^2} = N^{\mu} && \text{spacelike coordinate} \\ H^{\mu} &= \frac{S^{\mu\nu} \star p^{\nu}}{p^2} = W^{\mu} && \text{Pauli-Lubanski (spin) coordinate} \end{aligned} \tag{5}$$

We call them electric/magnetic parts since, in the rest frame,

$$J^{\mu\nu} \;=\; m \begin{pmatrix} 0 & -N^{i} \\ N^{i} & \epsilon^{ijk} W_{k} \end{pmatrix}_{+} \tag{6}$$

Part I – The duality

Then, if $\boxed{p^\mu \mapsto p^\mu}$, spin-orbit duality is an **electric-magnetic duality**,

$$\begin{aligned} N^\mu &\mapsto W^\mu \\ W^\mu &\mapsto -N^\mu \end{aligned} \qquad \Leftrightarrow \qquad J \mapsto \star J$$

▷ Why is this a (meaningful) duality?

- It is an automorphism of structure $\mathcal{H}$ (original motivation).
- For $F^{\mu\nu}$, it is the usual U(1) electromagnetic duality.
- It preserves the Poincarè conservation laws $\dot{J} = \dot{p} = 0$.

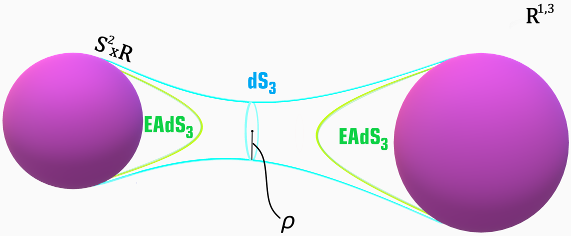
↪ SO(1,3) is preserved. However, spacetime transforms and translation generators are realized projectively: ISO(1,3) transforms.

The map $\boxed{N^\mu \mapsto W^\mu}$ implies:

- It becomes trivial at $N^\mu = W^\mu$, on the $\mathbb{S}^2 \times \mathbb{R}$ **particle worldtube**

$$\rho = \sqrt{W^2} = \frac{S}{m} \qquad \text{or} \qquad \hat{\rho} = \frac{\hbar\sqrt{s(s+1)}}{m} \qquad \textbf{(Møller radius)} \tag{7}$$

$\hookrightarrow$ ρ is natural boundary: Classically, encloses region of non-covariance. Quantum-mechanically, $\hat{\rho} \sim \lambda_C$, signifies pair production.



In Euclidean signature:

▷ N^μ ($N^\mu N_\mu > 0$) is an SO(4) rep, foliating $\mathbb{R}^4$ into concentric $\mathbb{S}^3$'s,

▷ $\mathbb{S}^3 \cong$ SU(2) is a U(1)-bundle, since the homogeneous $\mathbb{S}^2 \cong$ SU(2)/U(1),

$$\boxed{N^\mu \mapsto W^\mu \qquad = \qquad \mathbb{S}^3 \xrightarrow{\mathbb{S}^1} \mathbb{S}^2 \quad \textbf{(1st Hopf map)}}$$

► Since $\mathbb{R}^4 \setminus \{0\}$ is flat space minus a timelike curve, the duality induces the **conformal immersion** $[\mathbb{R}^4 \setminus \{0\} \cong \mathbb{S}^3 \times \mathbb{R}] \rightarrow \mathbb{S}^2 \times \mathbb{R}$, and thus

$$\boxed{\mathbb{R}^{1,3} \quad \mapsto \quad \mathbb{S}^2 \times \mathbb{R}}$$

Part I – Duality and the conformal group

$\mathbb{R}^{1,3} \mapsto \mathbb{S}^2 \times \mathbb{R}$ yields that the bulk $\mathbf{G} = \mathbf{ISO(1,3)}$ **transforms**:

▷ SO(1,3) subgroup is preserved,

▷ $\mathcal{T}_4$ translations ($\dot{p} = 0$ preserved) are realized projectively,

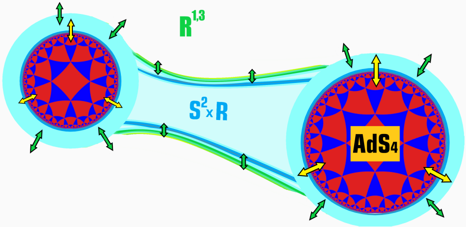
$$\boxed{\tilde{G} \;=\; \mathrm{SO}(2,3)}$$

- $\mathrm{SO}(2,3) \cong \mathrm{Conf}(\mathbb{S}^2 \times \mathbb{R})$ and $\mathrm{SO}(1,3) \cong \mathrm{Conf}(\mathbb{S}^2)$.
- $[\tilde{G} \mapsto G]/[G \mapsto \tilde{G}]$ is seen as an Inonu-Wigner contraction/expansion,

$$\boxed{\mathbb{R}^{1,3} \quad \mapsto \quad \mathrm{Conf}[\mathbb{S}^2 \times \mathbb{R}] \quad = \quad \partial(\mathrm{AdS}_4)}$$

The dual theory lives on $\mathrm{Conf}[\mathbb{S}^2 \times \mathbb{R}]$, with $\tilde{G} = \mathrm{SO}(2,3) = \mathrm{Isom}(\mathrm{AdS}_4)$. This is the **conformal boundary** of AdS_4. Through AdS/CFT duality, we look for a **triality**:

$$\begin{array}{c} \text{massive QFT} \\ \text{in } \mathbb{R}^{1,3} \end{array} \xrightleftharpoons[\text{duality}]{\text{spin-orbit}} \begin{array}{c} \text{(massless) CFT} \\ \text{on } \mathbb{S}^2 \times \mathbb{R} \end{array} \xrightleftharpoons[\text{duality}]{\text{AdS/CFT}} \begin{array}{c} \text{superstring} \\ \text{in } \mathrm{AdS}_4 \times \mathcal{M}^6 \end{array}$$



Part II

$\mathbf{QED}_4$ as $\mathbf{CSM}_3$ on $\mathbb{S}^2 \times \mathbb{R}$

Part II – QED_4 as CSM_3

In field theory, the simplest system with mass and spin is QED_4,

$$S = \int_{\mathbb{R}^4} \bar{\Psi}(i\not{D} - m)\Psi - \frac{F^2}{4} \tag{8}$$

▶ In analogy, we understand the duality to:

▷ Shift A^μ magnetic vacua into **monopoles** (Hopf map).
▷ Transform somehow the **mass term** ($\tilde{G} = \mathrm{SO}(2,3)$).
▷ Leave the **kinetic terms** invariant ($p^\mu \mapsto p^\mu$).

Part II – QED_4 as CSM_3

There is an elegant way to realize the duality. The generalized momenta $\Pi_\mu = i\partial_\mu\psi, \bar{\Pi}_\mu = i\partial_\mu\bar{\psi}$ define a kind of **generalized field coordinates**,

$$\Psi^\mu := \frac{\gamma^\mu\Psi}{2\sqrt{-p^2}} \quad \text{and} \quad \overline{\Psi}^\mu := -\frac{\bar{\Psi}\gamma^\mu}{2\sqrt{-p^2}}\ , \tag{9}$$

We may even extract a **spacelike coordinate** and define an OAM analog,

$$\mathrm{N}^\mu := \frac{\mathbf{S}^{\mu\nu}\partial_\nu\Psi}{p^2} \quad : \qquad \mathfrak{L}^{\mu\nu} := \mathrm{N}^\mu \wedge \partial^\nu \tag{10}$$

Then, the total angular momentum generator, $\mathfrak{J}^{\mu\nu} = \mathfrak{L}^{\mu\nu} + \mathfrak{S}^{\mu\nu}$, where $\mathfrak{S}^{\mu\nu} = \mathbf{S}^{\mu\nu}/2$, satisfies the Lorentz algebra. Hence, duality $J^{\mu\nu} \mapsto \star J^{\mu\nu}$ is (in this representation) $\mathfrak{J}^{\mu\nu} \mapsto \star\mathfrak{J}^{\mu\nu}$. Equally,

$$\boxed{\mathrm{N}^\mu \quad \mapsto \quad \mathrm{W}^\mu}$$

where $\mathrm{W}^\mu = (i\overrightarrow{\partial}_\nu \star \mathbf{S}^{\mu\nu}\Psi)/p^2$.

Part II – QED$_4$ as CSM$_3$

In fact, the algebraic hard core of the spin-orbit duality is really between the **transverse projector**,

$$N^{\mu\nu} := \eta^{\mu\nu}\overleftrightarrow{\partial}^2 - \overleftrightarrow{\partial}^\mu \overleftrightarrow{\partial}^\nu \tag{11}$$

i.e. a Laplace-de Rham operator $\Delta = d\delta + \delta d$, and **topological operator**

$$W^{\mu\nu} := \frac{1}{4}\,\epsilon^{\mu\sigma\nu\rho}\overleftarrow{\partial}_\sigma \overrightarrow{\partial}_\rho \tag{12}$$

under the map

$$\boxed{\mathrm{N}^{\mu\nu} \quad \mapsto \quad \mathrm{W}^{\mu\nu}}$$

▶ The duality is a **morphism between geometric/topological data**, where analytic/topological operations between $\mathbb{R}^{1,3}$ and $\mathbb{S}^2 \times \mathbb{R}$ are equal descriptions of the underlying de-Rham cohomology.

Overall, the map transforms the fermion mass term,

$$\boxed{m\,\bar{\Psi}\Psi \quad \mapsto \quad \frac{i}{4}\bar{\psi}_i\,\gamma^{\alpha}\,\omega_{\alpha}{}^{ab}\,\sigma_{ab}\,\psi_i}$$

into a **spin connection** on $\mathbb{S}^2 \times \mathbb{R}$, and the gauge kinetic term,

$$\boxed{F^2 \quad \mapsto \quad A \wedge \mathrm{d}A}$$

into an Abelian **Chern-Simons** term. Also, the Hopf map on the Dirac operator, for $\mathbb{S}^1$-invariant sections, imply **unit monopole charge** on $\mathbb{S}^2$,

$$\boxed{\not{D}_{\mathbb{S}^3} \quad \mapsto \quad \not{D}_{\mathbb{S}^2} - \frac{1}{2}}$$

Part II – QED_4 as CSM_3

Applying the map and reducing on the Hopf fiber, we look the IR and only keep the zero modes. The **dual actions** are

$$S = \int_{\mathbb{R}^4} \bar{\Psi}(i\not{D} - m)\Psi - \frac{F^2}{4}$$

$$\downharpoonleft\upharpoonleft$$

$$\tilde{S} = \int_{\mathbb{S}^2 \times \mathbb{R}} i\bar{\psi}_i \not{D} \psi_i \; - \; \phi(\bar{\psi}_1\psi_1 - \bar{\psi}_2\psi_2) \; + \; \frac{k}{4\pi} A \wedge \mathrm{d}A \; + \; \frac{1}{g^2} \mathrm{d}\phi \wedge \mathrm{d}A$$

- The $2N_f$ fermions imply global $SU(N_f)$ symmetry.
- ϕ is auxiliary, with EOM: $\bar{\psi}_1\psi_1 - \bar{\psi}_2\psi_2 = dF = J_m$, for J_m source.
- The dual theory is (classically) SO(2,3) invariant and should be [Kapustin&Pronin94',etc] a strongly-coupled **IR fixed point** CFT.

Part II – Magnetic vacuum

In the **non-relativistic limit**, in a uniform field $< B^i >= \epsilon^{ijk}\partial_j < A_k >$, the $\mathbb{R}^{1,3}$ fermion propagator implies the **Landau levels**,

$$E_4 \; = \; \omega(m) \left(n + \frac{1}{2} \right) \; , \qquad n \in \mathbb{Z} \tag{13}$$

On dual $\mathbb{S}^2 \times \mathbb{R}$, fermions acquire a 'magnetic' mass $< A_3 >=< \phi >= \mu$, and the uniform field (Hopf) maps to a **monopole** problem [Haldane83'],

$$E_3 \; = \; \frac{1}{2\mu R^2} \Big(n(n+1) + s(2n+1) \Big) \tag{14}$$

The **thermodynamic limit** $R, s \to \infty$, with particle density $B = s/R^2$ fixed, is here an **IR limit**,

$$E_3 \; = \; \omega(R, \mu) \left(n + \frac{1}{2} \right) \tag{15}$$

Part II – Magnetic vacuum

In **global AdS$_4$ with magnetic flux** $F_{(2)} = Q_M \mathrm{vol}(\mathbb{S}^2)$, one may show that a massive fermion has spectrum,

$$E_A \;=\; \frac{1}{2mL^2}\Big(n(n+1) + Q_M(2n+1)\Big) \tag{16}$$

where we neglected radial energy, i.e. kept long-range modes reaching the boundary. The thermodynamic/IR limit $Q_M, L \to \infty$ gives

$$E_A \;=\; \omega(L,m)\left(n + \frac{1}{2}\right) \tag{17}$$

- Setting a universal scale, $\omega(m) = \omega(R,\mu) = \omega(L,m)$, implies

$$\boxed{R \;=\; L \;=\; \rho \;=\; \frac{1}{m}}$$

- In the **relativistic limit** $Bn \gg m$, the fermion Landau levels in $\mathbb{R}^{1,3}$, $E_4 = \sqrt{m^2 + 2Bn}$, reduce to

$$E_4 = \sqrt{2Bn} \tag{18}$$

- In this limit, fermions on the dual $\mathbb{S}^2 \times \mathbb{R}$ become **massless** again, obtaining the exact same spectrum [Arciniaga&Peterson16'], $E_3 = E_4$

- In this limit in AdS_4, we must consider the backreaction to the metric. It has been shown that the **magnetic extremal AdS black hole** also has the exact same spectrum [Basu.etal.10',Albash.etal.10'], $E_A = E_3 = E_4$.

Part II – Anomaly matching

In $\mathbb{R}^{1,3}$, the **chiral anomaly** in QED,

$$< \partial_\mu J_5^\mu > = \; N_f \frac{e^2}{16\pi^2} F \star F \tag{19}$$

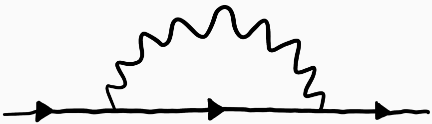
is realized as the **parity anomaly** on $\mathbb{S}^2 \times \mathbb{R}$,

$$\pm \frac{N_f}{4\pi} \int_{\mathbb{S}^2 \times \mathbb{R}} A \wedge dA \tag{20}$$

▶ Gauge invariance of the fermionic path-integral measure, implies $2N_f$ contributions of $\pm\frac{1}{2}\tilde{S}_{\text{CS}}$, with effective CS level $k_{eff} = k \pm N_f$.

We move on into interactions.

- The simplest in QED_4 is the **fermion self energy**,



- $\text{QED}_4/\text{CSM}_3$ duality should imply equal **energy shifts** due to quantum effects, $\delta E = \delta\tilde{E} \;\;\Rightarrow\;\; \delta m = (\delta\Delta_{\mathcal{O}})/R \;\;\Rightarrow\;\; \delta m/m = (\delta\Delta_{\mathcal{O}})$,

$$\boxed{\gamma_m \;=\; \gamma_{\mathcal{O}}}$$

where $\mathcal{O}$ is the operator dual to the 4d fermion.

Part II – Dual self energy

Considering that:

▹ In the free theory, the Landau-levels matching was reached by the **thermodynamic limit** (i.e. infinite degrees of freedom),

▹ CSM_3 contributes to anomalous dimensions from 2-loop order $1/k^2$. But at **leading order** $1/N_f$ at large N_f [Chen.etal.93'],

▶ We assume the $\mathrm{QED}_4/\mathrm{CSM}_3$ duality should hold at the **large-N_f limit**. At order $1/N_f$, QED_4 exhibits [Mestre&Pascual84'],

$$\gamma_m(\lambda) \;=\; -\frac{6\lambda}{\pi^2 N_f} \;+\; O(\lambda^2) \tag{21}$$

where $\lambda = e^2 N_f/16$ is kept fixed, and $\gamma(\lambda)$ converges for $\lambda < 15\pi^2/8$. At leading order in λ, this agrees with the usual QED_4 one-loop result.

Part II – Dual self energy

For CSM_3, we consider:

- $\mathcal{O}$ should be $\text{SU}(2)_f$ rep. Since the **Hopf map** is $\Psi \mapsto \bar{\Psi}\sigma\Psi$, which is also the non-singlet $\text{SU}(2)_f$ irrep, we assume a $\mathbb{Z}_2$ flavor symmetric

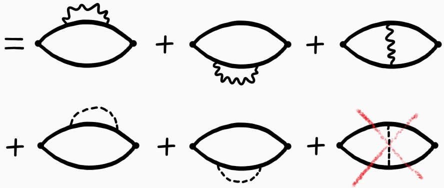
$$\mathcal{O} \;=\; c_0\bar{\Psi}\Psi \;+\; c_i\bar{\Psi}\sigma^i\Psi \;\;\rightarrow\;\; \bar{\psi}_1\psi_1 + \bar{\psi}_2\psi_2 + \bar{\psi}_1\psi_2 + \bar{\psi}_2\psi_1 \tag{22}$$

- UV divergences/renormalization are **background-independent**, so we may neglect the unit monopole to extract $\gamma_{\mathcal{O}}$.

- At $1/N_f$ expansion, ϕ becomes **dynamical** via fermion bubbles and, besides D_{CS}, we must consider also a $D_\phi = 1/\Pi$ propagator.

Part II – Dual self energy

Hence, $\langle\bar{\Psi}\Psi\rangle$ in $\mathbb{R}^4$ corresponds to $\langle\bar{\mathcal{O}}\mathcal{O}\rangle$ on $\mathbb{S}^2\times\mathbb{R}$, at $1/N_f$ order,



▷ Last diagram vanishes, since the Yukawa coupling $\phi(\bar{\psi}_1\psi_1-\bar{\psi}_2\psi_2)$, thus $\langle\bar{1}111\rangle+\langle\bar{2}222\rangle=-\langle\bar{1}212\rangle-\langle\bar{2}121\rangle$.

▶ For effective coupling $\tilde{\lambda}=(2\pi/k)\tilde{N}_f/16$, the total anomalous dimension is

$$\gamma(\tilde{\lambda}) \;=\; -\left(\frac{\tilde{\lambda}^2}{1+\tilde{\lambda}^2}\right)\frac{64}{3\pi^2\tilde{N}_f}-\frac{4}{3\pi^2\tilde{N}_f} \tag{23}$$

Part II – Dual self energy

Considering the duality between the (fixed) coupling constants,

$$\boxed{\left[\lambda = e^2 N_f\right] \quad \sim \quad \left[\tilde{\lambda} = \frac{2\pi \tilde{N}_f}{k}\right]^2}$$

▷ The expansion parameter is $1/N_f$, while $\lambda = O(1) = \tilde{\lambda}$ parametrize the resummed fermion-bubble chains in propagators.

▷ $e = e(\mu)$ runs while k does not; we understand that the duality makes sense only at some particular $\{\lambda, \tilde{\lambda}\}$.

▷ At $\tilde{\lambda} = 1$, (parity even) fluctuations perfectly balance (parity odd) topological phase attachment: the gauge field is **self-dual**.

▶ Indeed, given $\tilde{N}_f = 2N_f$, the duality **uniquely** enforces $\lambda = \tilde{\lambda} = 1$:

$$\boxed{\gamma_m \;=\; \gamma_{\mathcal{O}} \;=\; -\frac{6}{\pi^2 N_f}} \tag{24}$$

Part III
The conformal realms of QED_4

Part III – Holographic realms

The spin-orbit duality itself,

$$\boxed{\tilde{\rho} \;=\; \sqrt{R^2\,\frac{C_4 - Z^2}{\left(C_2 - J^2\right)^2}}\,\Bigg|_{\mathrm{SO}(2,3)} \;=\; \sqrt{\frac{W^2}{p^4}}\,\Bigg|_{\mathrm{ISO}(1,3)} \;=\; \frac{1}{m}}$$

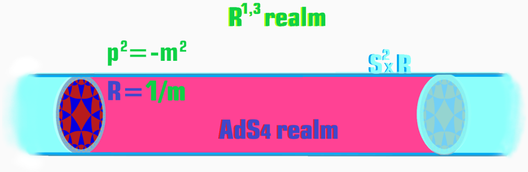
but also the cases of free fermions and the magnetic vacuum,

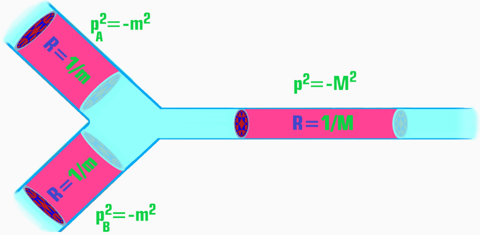
$$\boxed{\tilde{\rho} \;=\; R \;=\; L \;=\; \rho \;=\; \frac{1}{m}}$$

lead us to make the identification,

$$\boxed{\Big[\,\textbf{dual}\ \mathbb{S}^2 \times \mathbb{R}\,\Big]_{\mathrm{ISO}(1,3)} \quad = \quad \Big[\,\mathbb{S}^2 \times \mathbb{R}\ \textbf{worldtube}\ \text{in}\ \mathbb{R}^{1,3}\,\Big]}$$

This superposes three particle theories across three domains, i.e. distinct **conformal realms**:

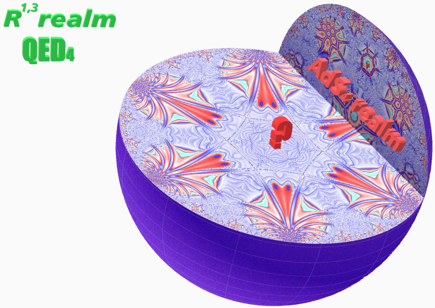




Part III – The AdS_4 realm of QED

For **large** N_f, what is understood as ordinary QED_4 in the $\mathbb{R}^{1,3}$ realm, in the **AdS_4 realm**, at least in its $SU(N_f)$-singlet sector, should be:

> A non-local, Abelian, higher-spin theory, on a $1/N_f$ expansion, with $\kappa = k/N_f$ kept fixed, similar to [Aharony&Chester&Urbach21']



- Consider **Minkowski vacuum** in $\mathbb{R}^{1,3}$ realm. A uniformly α-accelerated Rindler observer feels the **thermofield double (TFD) state**

$$|0\rangle_M = \frac{1}{Z(\beta,\mu_s)} \sum_n e^{-\frac{\beta(E_n-\mu_s s_n)}{2}} |E_n, s_n\rangle_L \otimes |E_n, s_n\rangle_R \tag{25}$$

An entangled pure state, where $|E_n, s_n\rangle_{L,R}$ are mixed states in thermal equilibrium at $T_U = \alpha/2\pi$. We also assume a spin chemical potential μ_s.

- We can trace one copy out of the total pure state to obtain the mixed thermal state of the other,

$$\rho = Z^{-1} e^{-\beta(H-\mu_s S)} \tag{26}$$

- Spin-orbit duality implies $s_n \longleftrightarrow l_n$. Each Rindler copy is dual to a mixed thermal density in the **AdS$_4$ realm**,

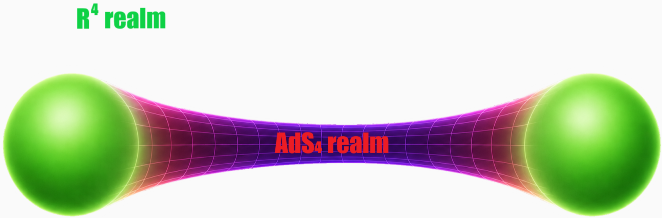
$$\rho = Z^{-1} e^{-\beta(H-\mu_l L)} \tag{27}$$

on boundary of an AdS$_4$ universe, the two universes being **distinct**.

The total vacuum state is a dual TFD state,

$$|0\rangle_A = \frac{1}{Z(\beta, \mu_I)} \sum_n e^{-\frac{\beta(E_n - \mu_I I_n)}{2}} |E_n, I_n\rangle_L \otimes |E_n, I_n\rangle_R \tag{28}$$

The two universes are **entangled**: it's AdS/CFT dual to Hartle-Hawking state of an eternal black hole in AdS, a spatial wormhole connecting those two AdS universes [Maldacena2001'].



Future studies

To be studied:

- QED_4/CSM_3 duality: Understand the coupling-constant duality. Do more computations. Look for an AdS/CFT dual.
- Study the non-Abelian case: More control is expected in large-N_c limit. Probably easier to find an AdS/CFT dual.
- Apply to quantum gravity: prove the duality holographic.

The end

thanks!

supplementary material

supplementary material

For **Poincarè generators J** and **P**, the possible compositions are

$$\mathbf{W} := \frac{\star(\mathbf{J} \wedge \mathbf{P})}{\mathbf{P}^2} \qquad \text{and} \qquad \mathbf{N} := \frac{\mathbf{J} \cdot \mathbf{P}}{\mathbf{P}^2} \tag{29}$$

whereas $\mathbf{L} = \mathbf{N} \wedge \mathbf{P}$. Then, $\mathbf{W}$ generates SO(3) and $\mathbf{N}$ boosts,

$$[\mathbf{W}, \mathbf{W}] = \frac{\mathbf{J}}{\mathbf{P}^2}\,, \qquad [\mathbf{W}, \mathbf{N}] = \frac{\star\mathbf{J}}{\mathbf{P}^2}\,, \qquad [\mathbf{N}, \mathbf{N}] = -\frac{\mathbf{J}}{\mathbf{P}^2} \tag{30}$$

The duality is

$$\begin{matrix} \mathbf{N} \mapsto \mathbf{W} \\ \mathbf{W} \mapsto -\mathbf{N} \end{matrix} \qquad \Leftrightarrow \qquad \mathbf{J} \mapsto \star\mathbf{J} \tag{31}$$

It leaves the $\mathbf{W}, \mathbf{N}$ algebra invariant $\quad\leftrightarrow\quad$ SO(1,3) is preserved.
It *does not* say anything (yet) about ISO(1,3).

- In fact: defining the timelike position as $\mathbf{A} = \frac{\mathbf{DP}}{\mathbf{P}^2}$: $\mathbf{X} = \mathbf{A} + \mathbf{N}$,

$$[\mathbf{X}^\mu, \mathbf{X}^\nu] = -\frac{\mathbf{S}^{\mu\nu}}{\mathbf{P}^2} \tag{32}$$

Formally, this means a massive theory with spin is **noncommutative**. This was first seen in `relativistic mechanics` by [Pryce1948] and on the `superparticle` by [Casalbuoni1976] and [Brink&Schwarz1981].

▷ This sets the **fundamental scale** at $\hat{\rho} \sim \lambda_C$, exactly on $\mathbb{S}^2 \times \mathbb{R}$.

▷ It reaffirms $\hat{\rho}$ (where duality becomes trivial) as natural QM boundary.

QM on $\mathbb{S}^2 \times \mathbb{R}$ is **noncommutative**,

$$
\begin{aligned}
[\hat{X}^\mu, \hat{X}^\nu] &= \frac{i}{p^2}\left(\hat{X}^\mu \wedge p^\nu + \star(\hat{X}^\mu \wedge p^\nu)\right) \\
[\hat{X}^\mu, \hat{p}^\nu] &= i\frac{\hat{p}^\mu \hat{p}^\nu}{p^2}
\end{aligned}
\tag{33}
$$

Minus the 3rd term, it is a κ-**deformation** of the Poincarè-Hopf algebra, with $\kappa = m$. Also,

$$
[\hat{X}^i, \hat{p}^0] = -i\frac{\hat{p}^i}{\hat{p}^0} \qquad \rightarrow \qquad \textbf{Newton-Wigner localization} \tag{34}
$$

In the rest frame, or in the **low-energy** regime,

$$
\begin{aligned}
[\hat{X}^0, \hat{X}^i] &= -i\frac{\hat{X}^i}{m} && \rightarrow && \kappa\textbf{-Minkowski} \\
[\hat{X}^i, \hat{X}^j] &= -i\lambda_C \epsilon^{ijk}\hat{X}_k && \rightarrow && \textbf{fuzzy sphere}
\end{aligned}
\tag{35}
$$

Part II – Duality as the Hopf map

Realization: SU(2) spinor $\psi: \ \psi^\dagger\psi = \mathit{const}.$, a hypersurface $\mathbb{S}^3 \subset \mathbb{C}^2$. The Hopf map is $\mathbb{S}^3 \to \mathbb{S}^2 \subset \mathbb{R}^3$,

$$\psi \quad \to \quad x^i = \psi^\dagger \sigma^i \psi \tag{36}$$

where $x^2 = (\psi^\dagger\psi)^2 = \mathit{const}. \ \Rightarrow \ x^i \in \mathbb{S}^2$.

Example: the 4D CBS superparticle,

$$S_{CBS} = \int \mathrm{d}t \ e^{-1}(\dot{x}^\mu - i\dot{\theta}\sigma^\mu\bar{\theta} + i\theta\sigma^\mu\dot{\bar{\theta}})^2 - em^2 \tag{37}$$

feels the duality (i.e. true symmetry of S_{CBS}),

$$\begin{aligned} x^i \quad &\mapsto \quad W^i = \theta\sigma^i\bar{\theta} \\ W^i \quad &\mapsto \quad -x^i \end{aligned} \tag{38}$$

which realizes the Hopf map.

Part I – Duality and the BMS group

In fact, the duality is really the **general map**,

$$\begin{aligned} p^\mu &\mapsto \lambda p^\mu \\ N^\mu &\mapsto \tfrac{1}{\lambda} W^\mu \qquad \Leftrightarrow \qquad J \mapsto \star J \\ W^\mu &\mapsto -\tfrac{1}{\lambda} N^\mu \end{aligned} \tag{39}$$

iff $\lambda = \lambda(\mathbb{S}^2)$, while SO(1,3) still remains intact. Then, $\tilde{p}^t = \lambda p^t$ implies

$$t \to t + \lambda(\mathbb{S}^2) \tag{40}$$

and similar for $\tilde{p}^i = \lambda p^i$. These are the $\mathcal{T}_\lambda$ **super-translations** on $\mathbb{S}^2$.

- $\mathcal{T}_\lambda \rtimes$ SO(1,3) forms the **BMS group**,

$\hookrightarrow$ indeed the $\mathbb{R}^{1,3}$ asymptotic symmetries at **null infinity**.

- The $\lambda \in \mathbb{R}$ case picks SO(2,3) from the groupoid.