Black Holes and Singularities in thermal CFT correlation functions

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Introduction

Topic:

Investigate the map between the singularity structure of certain thermal CFT two-point functions and their geometric counterparts via the holographic duality.

Why?

General objective: Investigate properties of gravity, including black holes.

A tool OR A modest start: Use the holographic principle.

Gravity in AdS and Conformal Field Theories (CFTs).

Introduction

Why and how is holography useful?

- Results for flat space can sometimes be extracted from AdS.
- Generic CFTs in d-dimensions admit an equivalent description in terms of Einstein gravity in AdS_{d+1} spacetimes.

[Heemskerk, Penedones, Polchinski, Sully, ...]

Holographic CFTs: minimal ingredients for a gravitational description.

Introduction

Given such an equivalence:

- CFT correlation functions $\leftrightarrow$ Scattering amplitudes
- Properties of finite temperature states $\leftrightarrow$ properties of black holes
- Thermal CFT correlation functions $\leftrightarrow$ Scattering amplitudes in Black Hole background
- Quantum Entanglement, Quantum Chaos
- $\cdots$

Introduction

Advantages of (holographic) CFTs

CFT "definition": $\{\Delta_i, c_i\}$

Consistency conditions (unitarity, causality, KMS condition, ...)

Crossing Equations & OPE:

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_1(x_2)\mathcal{O}_2(x_3)\mathcal{O}_2(x_4)\rangle$$

$$\mathcal{O}_1 \times \mathcal{O}_1 \leftrightarrow 1 + a_i\,\tilde{\mathcal{O}}_i + \cdots \leftrightarrow \mathcal{O}_2 \times \mathcal{O}_2, \ \text{direct-channel}$$

$$\mathcal{O}_1 \times \mathcal{O}_2 \leftrightarrow \mathcal{O}_1 \times \mathcal{O}_2, \qquad \text{cross-channel}$$

Holographic CFTs:

- Energy-momentum operator $T_{\mu\nu}$.

- Two parameters which can be taken large.

$$SU(N) \text{ gauge theory: } \{N, \lambda \equiv g^2_{YM} N\}.$$
$$\text{Abstract CFT: } \{c_T, \Delta_{gap}\}$$

Abstract CFT characteristic scales:

  - $\Delta_{gap} = \infty$: constrains the spectrum of operators; only a finite number of "primary" single-trace operators with spin $j \leq 2$.
  - Large c_T expansion, factorization.
    $\langle T_{\mu\nu} T_{\rho\sigma} \rangle \propto c_T$.

$$\langle \mathcal{O}_1 \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_2 \rangle = \langle \mathcal{O}_1 \mathcal{O}_1 \rangle \langle \mathcal{O}_2 \mathcal{O}_2 \rangle + \frac{1}{c_T} (\cdots)$$

- Spectrum building blocks: $\{\ \mathcal{O}_1, \mathcal{O}_2, \cdots, T^{\mu\nu}\}$.
- Composite operators (commonly, *double-traces*):

$$[\mathcal{O}_i\mathcal{O}_j]_{n,\ell} : \mathcal{O}_i\partial_{\mu_1 \dots}\partial_{\mu_\ell}(\partial^2)^n\mathcal{O}_j$$

- Higher composites, e.g., $[\mathcal{O}_{i_1}\mathcal{O}_{i_2}\cdots\mathcal{O}_{i_m}]_{n,\ell}$

$$\langle\mathcal{O}_1\mathcal{O}_1\rangle \sim 1+\cdots\,, \qquad \langle[\mathcal{O}_2\mathcal{O}_2][\mathcal{O}_2\mathcal{O}_2]\rangle \sim 1+\cdots$$

$$\langle\mathcal{O}_2\mathcal{O}_2[\mathcal{O}_2\mathcal{O}_2]_{n,\ell}\rangle \sim 1+\cdots\,, \qquad \langle\mathcal{O}_1\mathcal{O}_1[\mathcal{O}_2\mathcal{O}_2]_{n,\ell}\rangle \sim \frac{1}{c_T}+\cdots\,,$$

$$\langle\mathcal{O}_1\mathcal{O}_1\,T\rangle \sim \frac{1}{\sqrt{c_T}}+\cdots$$

Results:

- Crossing equation and locality/unitarity of the holographic description.
- Unitarity (causality) imply that Einstein's theory of general relativity is the only consistent description.
- Correlation functions in certain kinematic regimes.
- Higher order CFT-data (OPE coefficients, anomalous dimensions etc).
- Thermal CFT structure.
- $\cdots\cdots\cdots\cdots$

In this talk:

Thermal two-point functions in the regime where a gravity dual description is available, exhibits singularities when the insertions are at boundary points connected by null (real and complex) geodesics in the bulk.

$$G_+^{\beta}(t,x) \equiv \langle \mathcal{O}_\Delta(x,t)\mathcal{O}_\Delta(0)\rangle_\beta \propto (t_{BC}(x)-t)^{-f(\Delta,d)}, \quad f(\Delta,d)>0\,.$$

Work in collaboration with C. Esper, Y. Jia, and A.Parnachev.

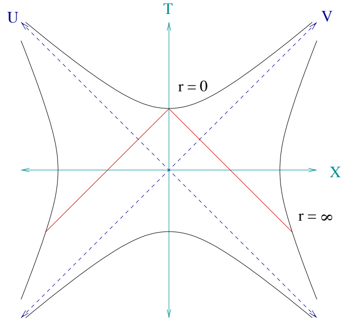
[Čeplak, Liu, Parnachev, Valach],[Burić, Gusev, Parnachev], [Afkhami-Jeddi, Caron-Huot, Chakravarty,Maloney], [Dodelson, Ioossa, Karlsson][Frederic Jia, Rangmani][Grozdanov, Valach, Vrbica]

Anticipated in:

[Hubeny, Liu, Rangamany][Fidkoesky, Hubeny, Kleban, Shener][Festuccia, Liu]

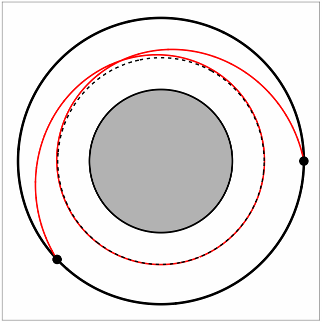
Real Null Geodesics

Two-sided Bouncing Geodesic



$t \in \mathbb{I}$

Geodesic anchored at the boundary



$t \in \mathbb{R}$

Momentum-space thermal correlator

$$G_+^\beta(\omega,k)=\int d^dx\, e^{ikx-i\omega t}\, G_+^\beta(t,\theta)\simeq e^{ik\,S(b)},$$
$$\omega,k\to\infty,\qquad b\equiv\frac{\omega}{k}=\text{fixed}$$

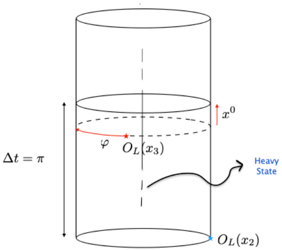
The description of the null geodesic lies in $\delta(b)\equiv kS(b)$.

(ω,k) are the conserved *energy* and *momentum* of a particle traveling along the respective null geodesic, according to Hamilton's equations of motion.

- More general complex geodesics can be understood as living in a different, analytically continued, spacetime.
- The singular behaviour of the correlator requires special analytic continuation.

High-energy scattering of light particle off black hole in AdS $\to$ **eikonal phase shift** (time delay + angular deflection).

$$\boxed{\begin{aligned} ds^2 &= -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega_{d-1}^2 \\ f(r) &= 1 + \frac{r^2}{R^2} - \frac{\mu}{r^{d-2}} \\ \delta(\omega, k) \equiv -p_\mu \Delta x^\mu &= k\,(b\,\Delta t - \Delta\theta) = kS(b) \end{aligned}}$$



Time delay and angular deflection:

$$\Delta t = 2\,b\int_{r_T}^{\infty}\frac{dr}{f\sqrt{b^2-\frac{f(r)}{r^2}}}, \qquad \Delta\theta = 2\int_{r_T}^{\infty}\frac{dr}{r^2\sqrt{b^2-\frac{f(r)}{r^2}}}\,.$$

$$r_T: \quad b^2-\frac{f(r)}{r^2}=0, \qquad \text{turning point}$$

Geodesics are labelled by the impact parameter

$$\frac{1}{\mathbf{b^2}} = b^2 - \frac{1}{R_{AdS}^2}$$

When $\mu=0$ reduces to the standard pure AdS impact parameter.
When $R_{AdS} \gg b^{-1}$, reduces to the flat space expression.

$$\delta(\omega,k) = k\,S(b) = 2k\int_{r_T}^{\infty}\frac{dr}{f}\sqrt{b^2-\frac{f}{r^2}}$$

The dual description:

Holographically, the correlator is obtained by studying fluctuations of a scalar field ϕ in the black hole background.

$$(\Box - m^2)\phi = 0, \qquad m^2 = \Delta(\Delta - d)$$

Fourier decomposition:

$$\phi(t, \Omega_{d-2}, r) = e^{-i\omega t} Y_{k,m}(\Omega_{d-1}) r^{-\frac{d-1}{2}} \psi_{\omega,k}(r)$$

with $Y_{k,m}$ the spherical harmonics on S^{d-1}.

The eikonal/Regge limit: $\omega, k \to \infty, \quad b \equiv \frac{\omega}{k} = \text{fixed}.$

WKB approximation with $\psi = e^{ikS}$.

$S(r)$ to leading order satisfies:

$$f(r)^2 \, (\partial_r S)^2 = b^2 - \frac{f(r)}{r^2}$$

Imposing appropriate boundary conditions:

$$\boxed{\begin{array}{c} G_+^{\beta}(\omega, k) \simeq e^{ik\, S(b)} \\ \textbf{Regge limit} : \omega, k \to \infty, \quad b \equiv \dfrac{\omega}{k} = \text{fixed} \end{array}}$$

Integrating the gravitational result back to position space, gives rise to bulk cone singularities (picture on the right).

Singularities are associated with real, null geodesic in the bulk. The singular structure is absent in the planar limit where real null geodesics anchored on the boundary do not exist.

Instead, of ad-hoc investigating complex geodesics:

- **Key question**: What happens when impact parameter b is less than critical value b_c (capture regime)?

In terms of geodesics, for values of the impact parameter beyond $\mathbf{b_c}$, the particle does not return to the boundary but rather falls into the black hole.

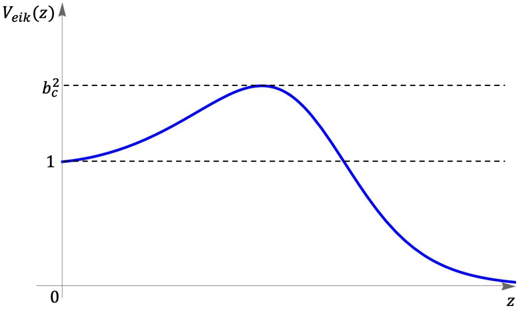
This is inelastic scattering. The phase shift develops an *imaginary* part related to the absorption cross section.

It is possible to compute the imaginary part by analytically continuing the exact result for the phase shift obtained. Technically, this analytic continuation corresponds to allowing for complex turning points in the bulk.

Phase shift acquires *imaginary* part $\rightarrow$ time delay/deflection become complex $\rightarrow$ physical interpretation unclear.

Alternative viewpoint:
Use the **position-space thermal correlator** and its **bulk-cone singularities** to give meaning to time delay and angular deflection.

$$\left[\partial_z^2 + k^2(b^2 - V_{eik})\right]\psi = 0$$
$$z : \frac{dz}{dr} = -\frac{1}{f}, \quad V_{eik} = \frac{f(r)}{r^2}$$



Evaluate the position space correlator with the analytically continued phase shift.

$$G_+(k,b) \sim \int_0^\infty dk \int_{b_c}^\infty db\, e^{ikbt - ik\theta}\, e^{ik(b\delta t(b) - \delta\theta(b))} =$$

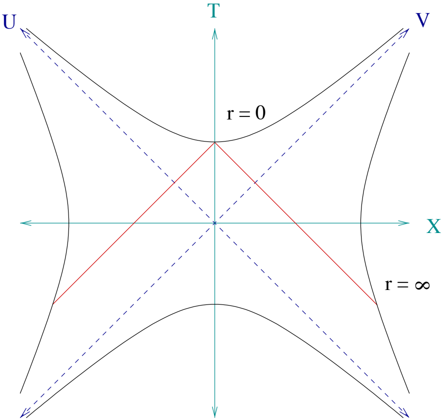
$$\int db \frac{(b^2-1)^{\delta-1}}{\{b(t-\Delta t(b)) - (\theta - \Delta\theta(b))\}^{2\Delta - d/2+1}}$$

Conditions for pinch singularity:

$$b_* : t = \Delta t(b_*), \quad \theta = \Delta\theta(b_*)$$

In general, $\Delta\theta$ is complex for $b > b_c$, but when $b \to \infty$:

$$b_* = \infty : \quad \Delta t = t_c + i\frac{\beta}{2}, \quad \Delta\theta = 0.$$



- Null geodesic bounces off singularity, emerges on opposite boundary.
- $G_+(t,\theta)$ acquires a singularity for times with fixed imaginary part. The imaginary part is understood from the entangled correlator (thermo-field double state)

Evaluated on the thermo-field double state

$$|\psi\rangle \equiv \frac{1}{\sqrt{\mathcal{Z}}}\sum_n e^{-\frac{1}{2}\beta E_n}\,|E_n\rangle_1\,|E_n\rangle_2$$

The thermal correlator thus computed, corresponds to a two-sided one:

$$\left\langle \psi|\mathcal{O}_\Delta^{(1)}(0)\mathcal{O}_\Delta^{(2)}(t)|\psi\right\rangle = \left\langle \mathcal{O}_\Delta^{(1)}(0)\mathcal{O}_\Delta^{(1)}\left(-t-i\frac{\beta}{2}\right)\right\rangle_\beta$$

Near $\theta = 0$, expansion yields:

$$G_+ \sim \frac{1}{(t - t_c - i\beta/2)^{2\Delta - d/2}}$$

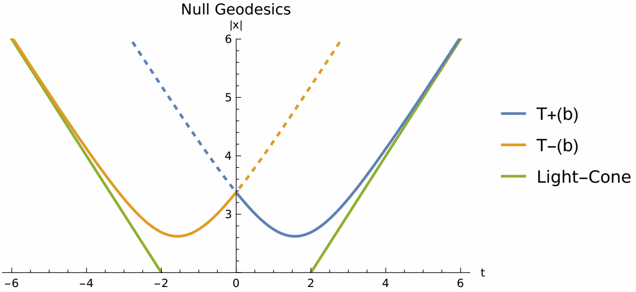
- Consistent with known stress-tensor OPE resummation.
- No singularity for $\theta \neq 0$ on real b-contour $\rightarrow$ only bouncing geodesic with zero angular deflection.
- For imaginary b, singularities with $\Delta\theta \neq 0$ and exponents of the bulk-cone type $2\Delta - \frac{d}{2} + 1$.

- Does the correlator exhibit further singularities? YES

Investigate the region $b < 1$.

The standard WKB phase provides a tunneling rate.

The complex turning point (associated to a complex geodesic) leads to further singularities, similar to the real bulk-cone ones.



Subtleties...

- Besides the bulk-cone singularities which correspond to real, null geodesics anchored on a single boundary, the others require analytic continuation of the correlator to a different sheet.

The correlator relevant for scattering in the eikonal regime:

$$G_+^\beta(t,\theta) \simeq_{x^\pm \ll 1} \langle \mathcal{O}^{pl}(x^+ + 2\pi, x^-)\mathcal{O}^{pl}(0)\rangle_\beta$$

where the 2π–shift corresponds to the zero temperature case.

The correlator singular for complex bulk geodesics, requires a further analytic continuation:

$$G_+^\beta(t,\theta) \simeq \langle \mathcal{O}^{pl}(x^+ e^{2i\pi} + 2\pi, x^-)\mathcal{O}^{pl}(0)\rangle_\beta$$

- Complex geodesics/turning points are generically related to singularities present in the stress-tensor sector of the correlator .

$$G_+^\beta = G_{+,T}^\beta + G_{+,[OO]}^\beta$$

The split distinguishes the operators contributing to the OPE:

$$\mathcal{O}_\Delta \times \mathcal{O}_\Delta \to 1 + a_1\, T_{\mu\nu} + a_2 T_{\mu\nu}^2 + \cdots$$
$$+ c_1[\mathcal{O}_\Delta\mathcal{O}_\Delta]_{n_1,2} + c_2[\mathcal{O}_\Delta\mathcal{O}_\Delta]_{n_2,\ell\leq 4} + \cdots$$

Analytic continuation enhances the stress-tensor contributions.

Summary

- Bulk cone singuglarities in thermal CFTs$\rightarrow$ identified with null geodesics returning to same boundary (elastic regime).
- Similarly, when $b < b_c$ (inelastic/capture) a singularity from bouncing (two-boundaries) null geodesic, and when $b > 1$, same boundary complex null geodesics, is identified.
- Singularities arise due to analytic continuation to a different sheet.
- Double-trace operators cancel singularity on principal sheet $\rightarrow$ present only on second sheet.
- The imaginary part of the inelastic process does not reveal further microscopic details. Likely, the analytic continuation probes a different saddle (a new complexified geometry) and not additional microscopic degrees of freedom.

Open Questions & Future Directions

- Investigate singularities for imaginary b to explore $\theta \neq 0$.
- Role of double-trace operators and KMS condition on second sheet.
- Corrections (stringy, gravitational).
- Applications to other backgrounds:
  - AdS-RN, Kerr.
  - Heavy states and differences.
  - Flat/dS spacetimes and observational physics.
- Thermal bootstrap and light-ray operator approaches.
- Causality analysis for impact parameters beyond the standard regime.

Thank you.