Dimitrios Kosmopoulos

Funded by
the European Union

Dynamical Love Numbers for Black Holes and Beyond from Shell Effective Field Theory

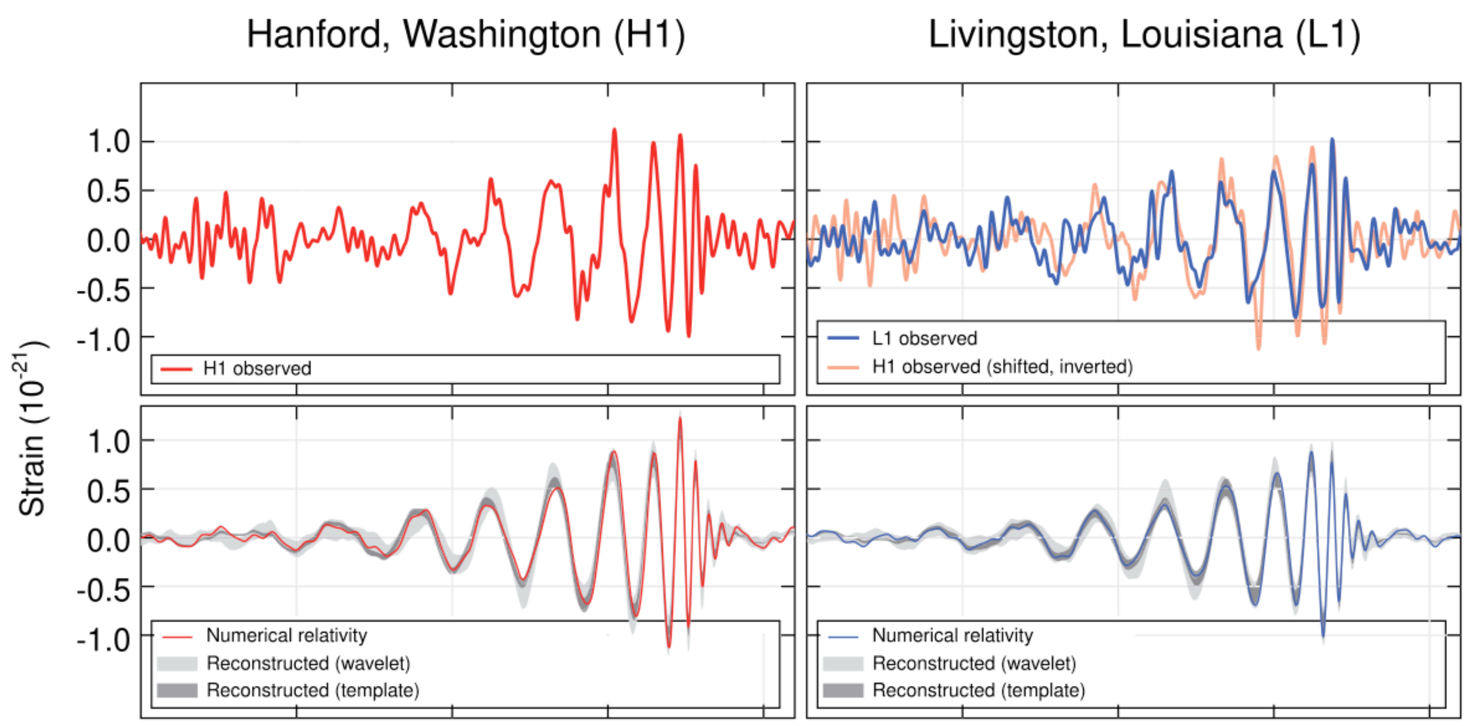
Based on PRL 136, 211401 (2026) [arXiv:2512.04002]
with **Davide Perrone** (Université de Genève) and **Mikhail Solon** (UCLA)

3rd INPP Demokritos-APCTP Workshop — June 29, 2026

Classical gravitational physics from Quantum Field Theory

Particle-physics tools

Effective Field Theory
+
Feynman integration
+
On-shell methods



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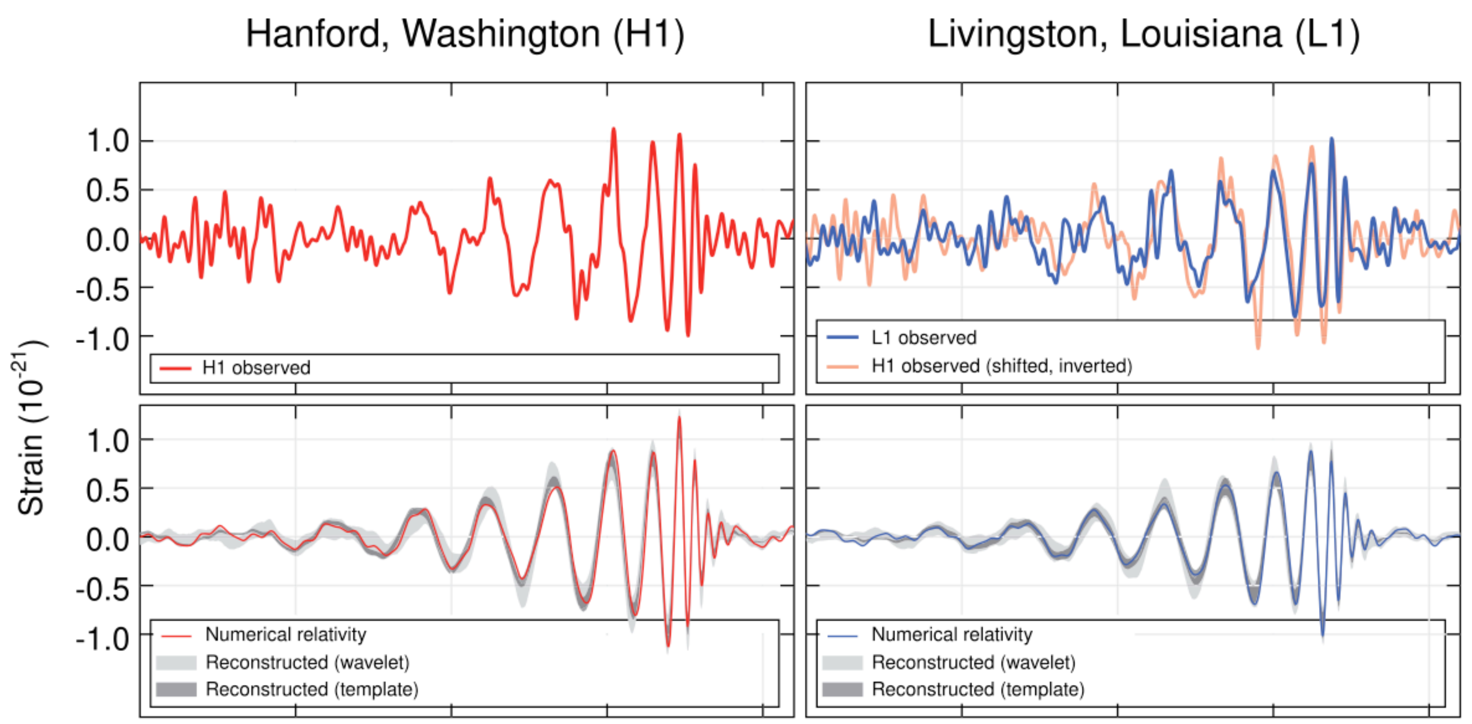

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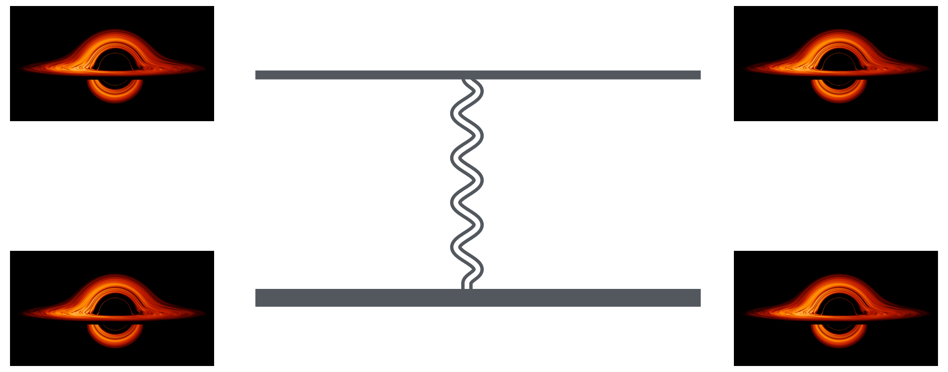

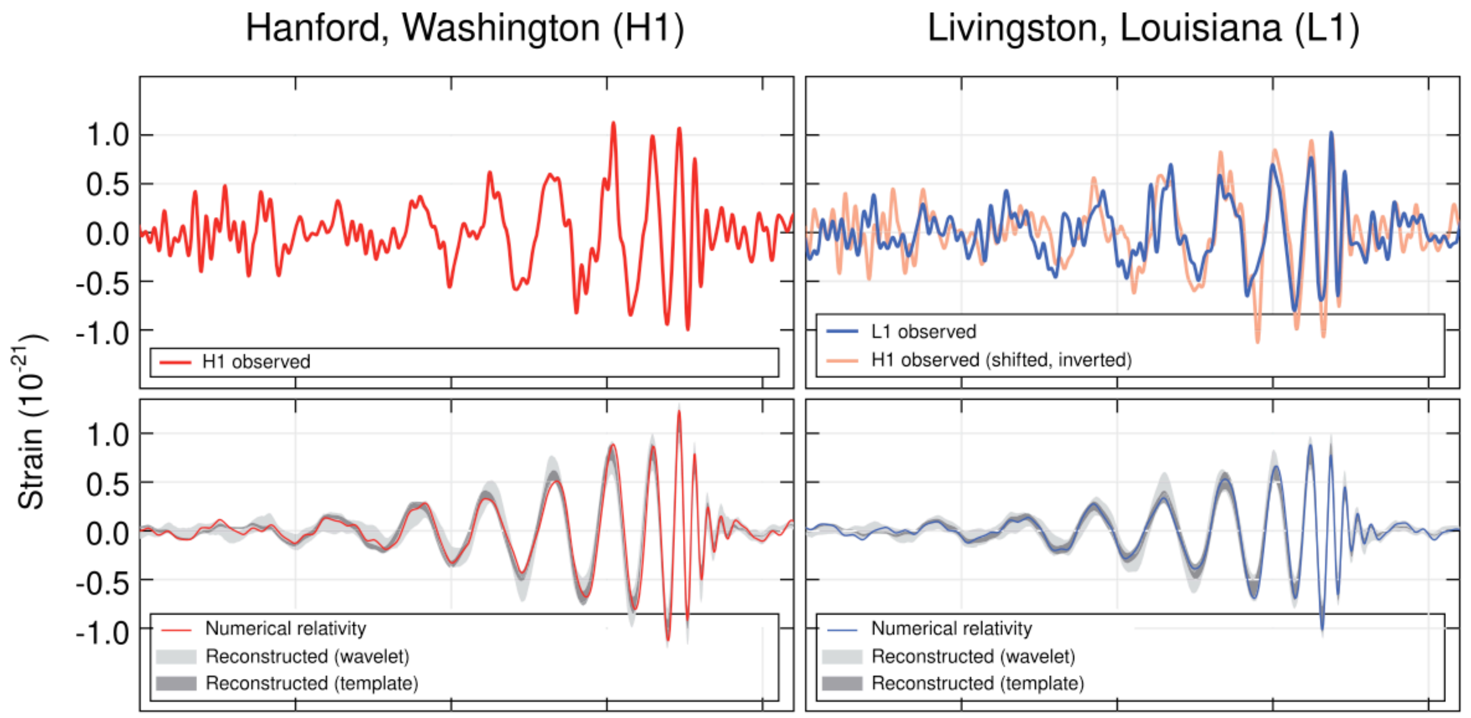

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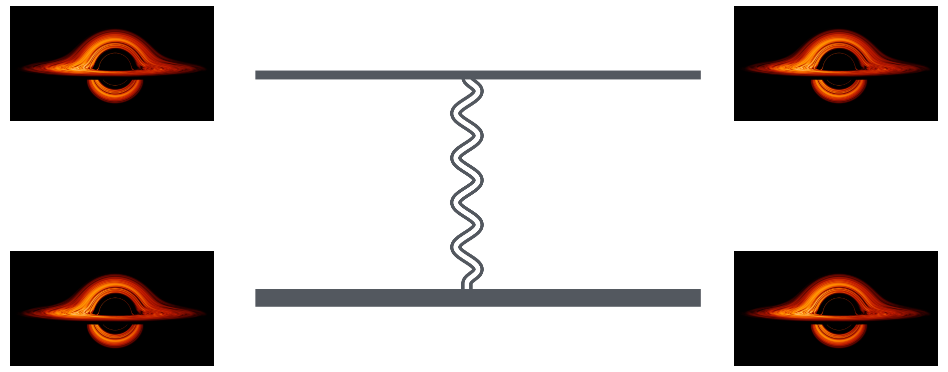
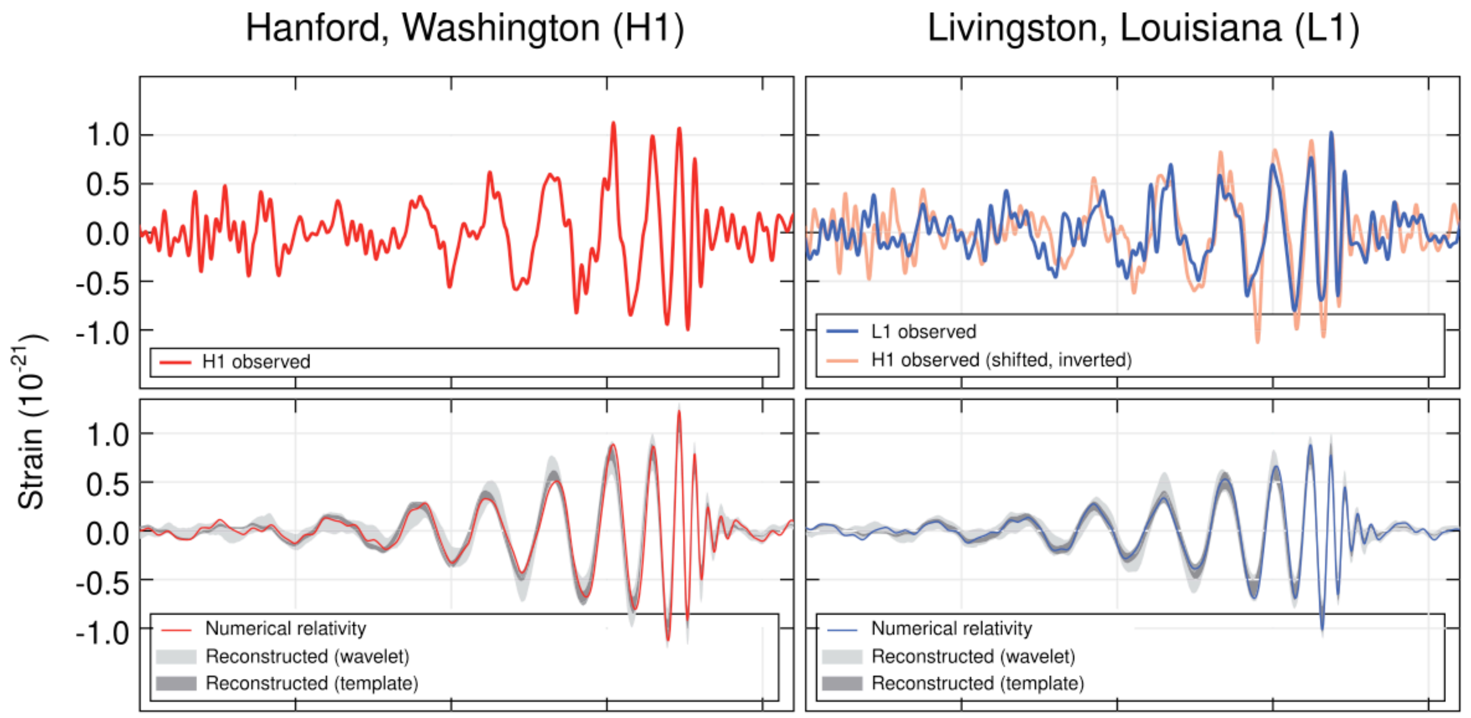


Corrections to the Newtonian potential

$$V = -\frac{Gm_1m_2}{r} + \left(\mathcal{O}(G^2), \text{spin, tidal response, } \ldots\right)$$



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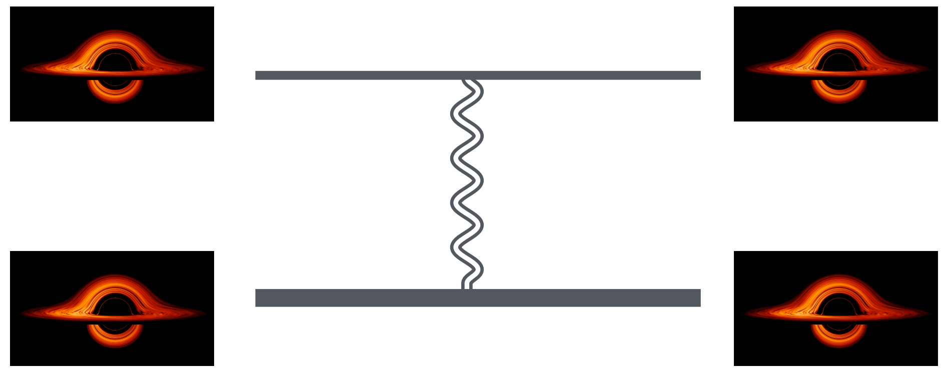

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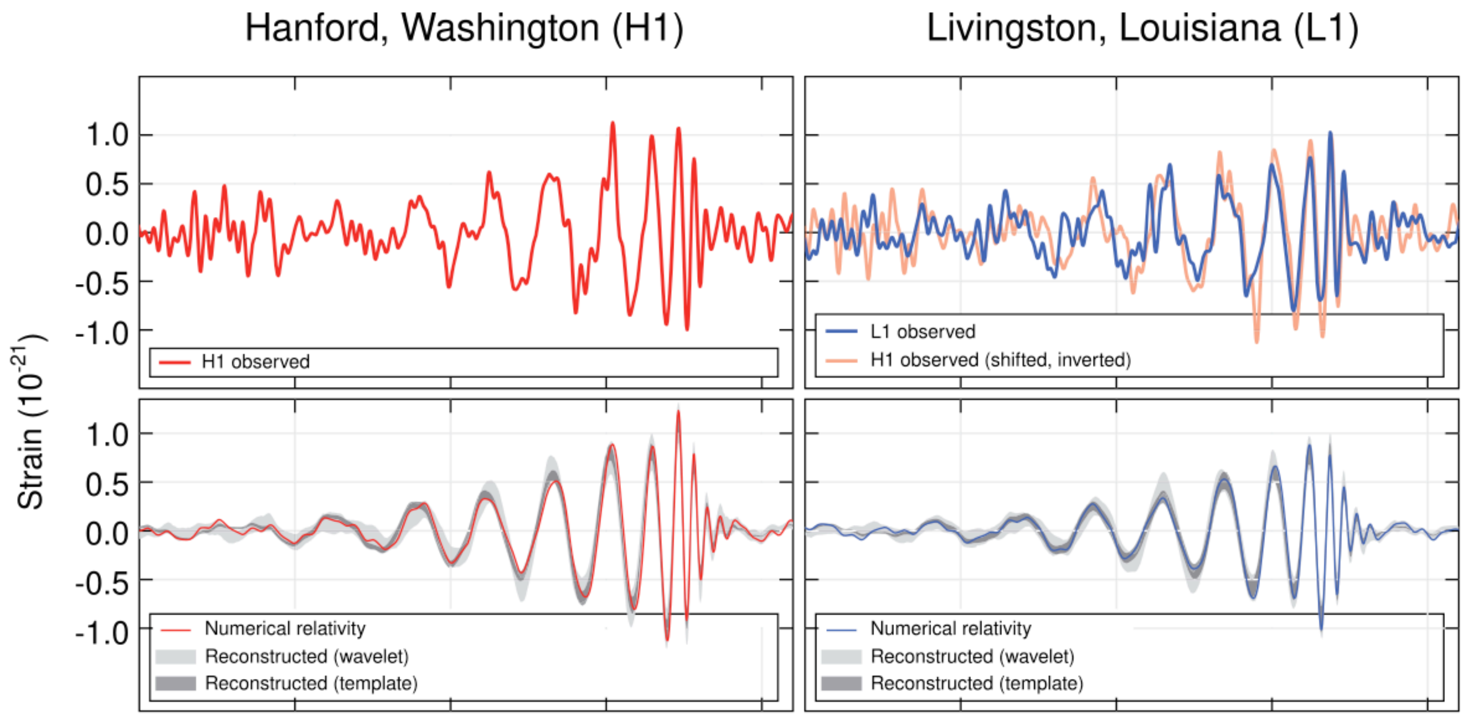
LIGO theorists



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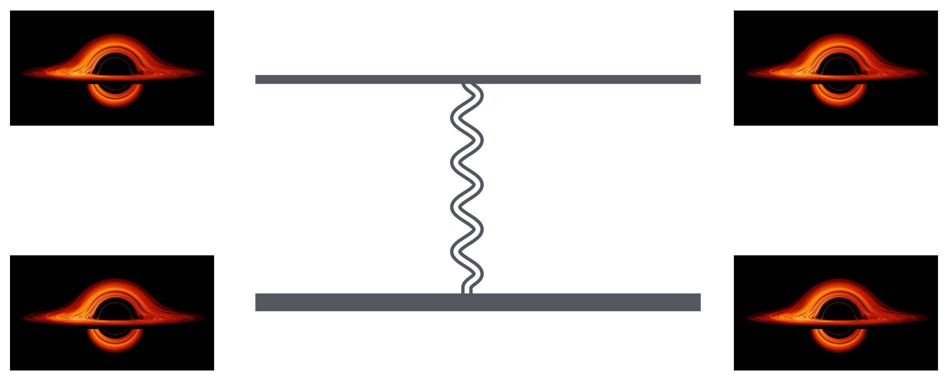

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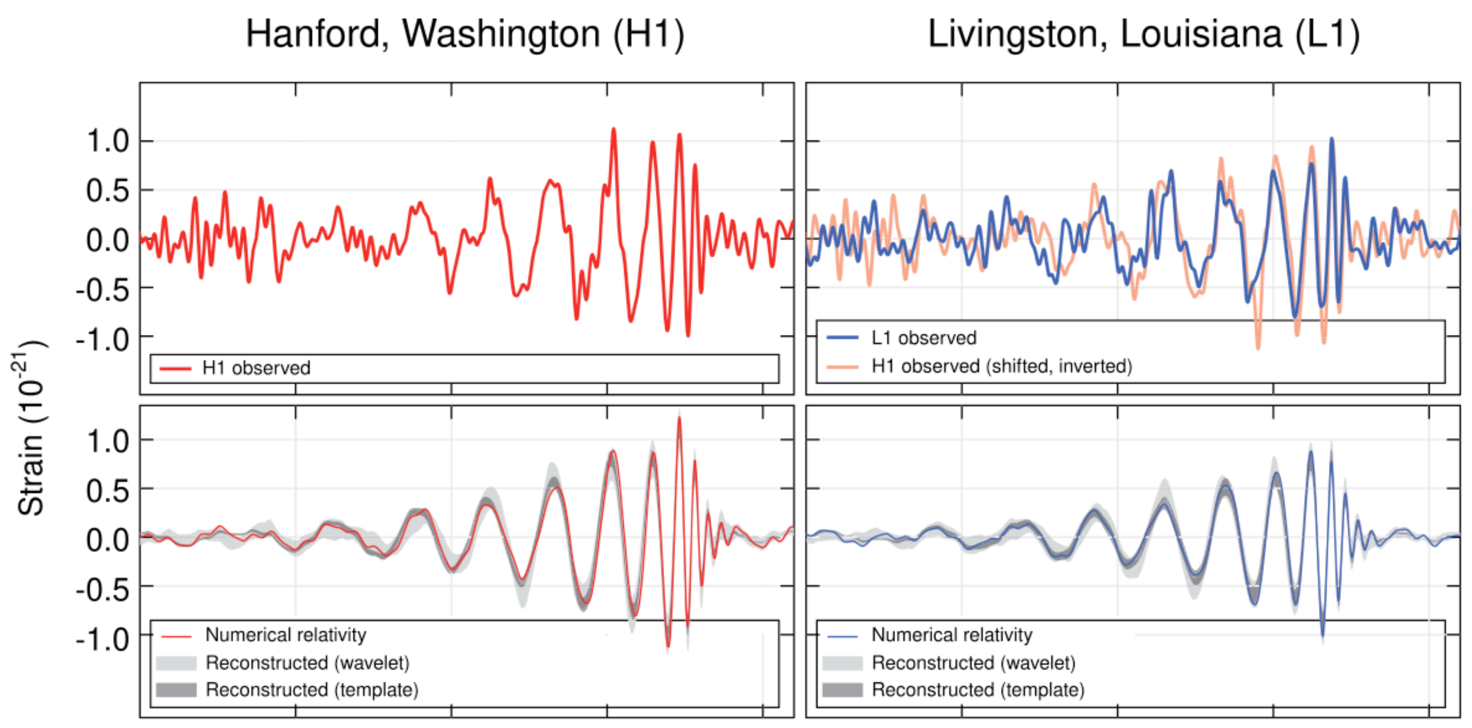

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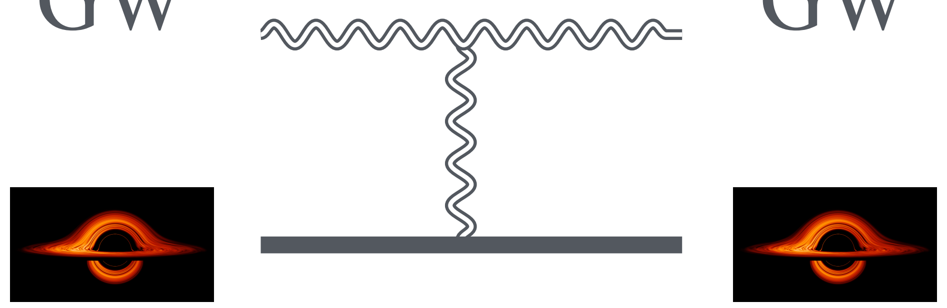

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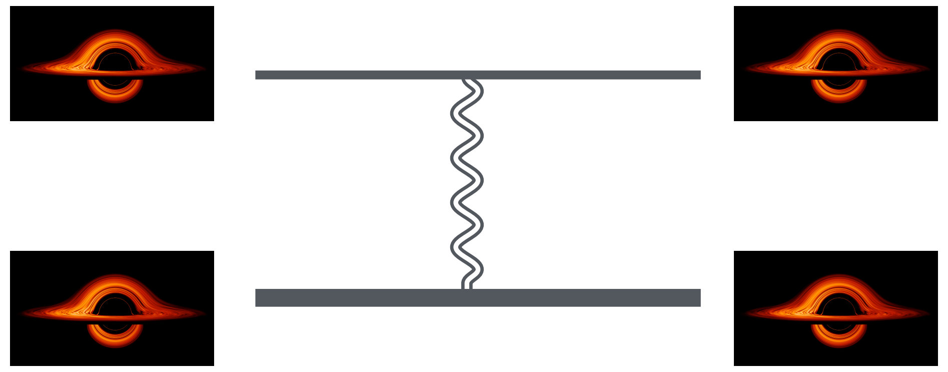

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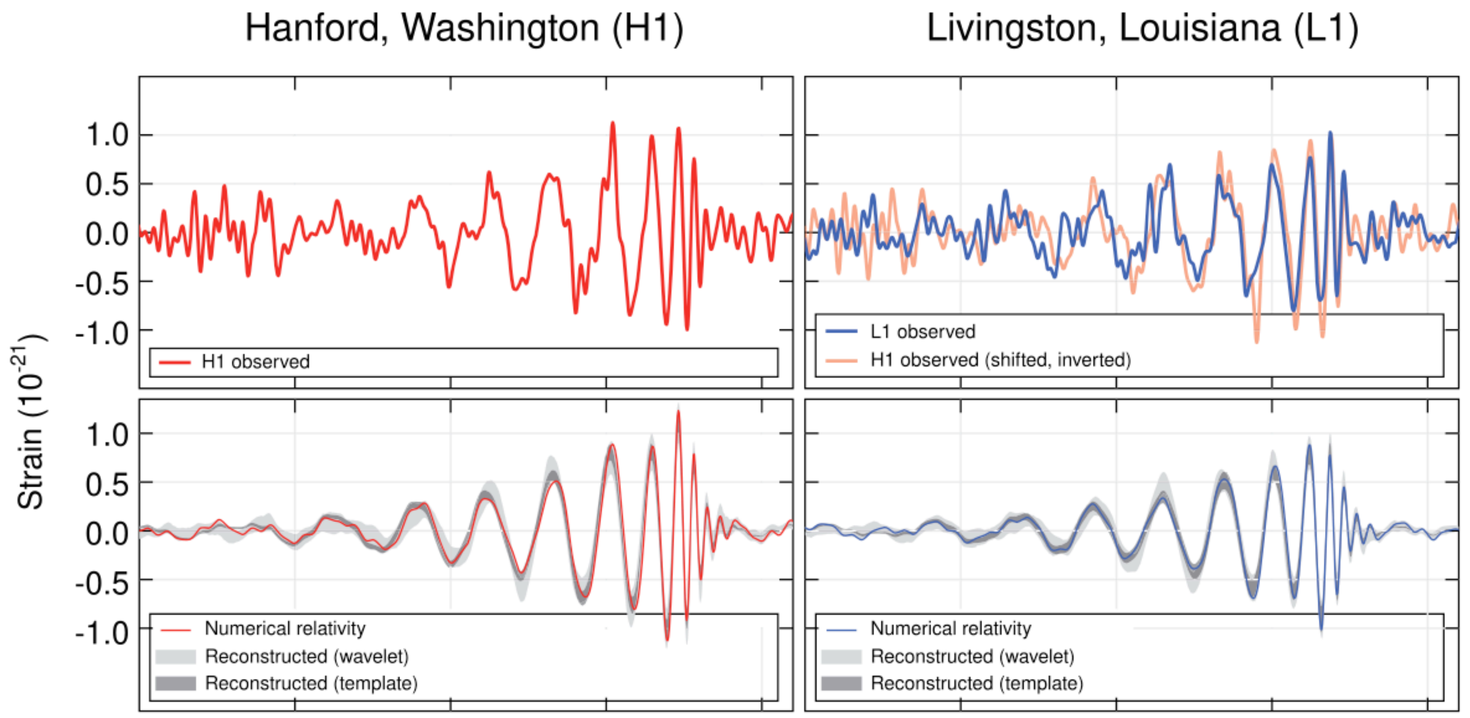

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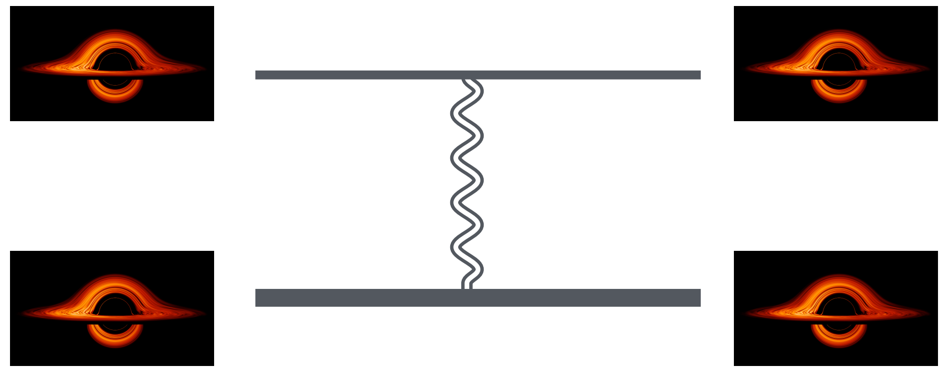

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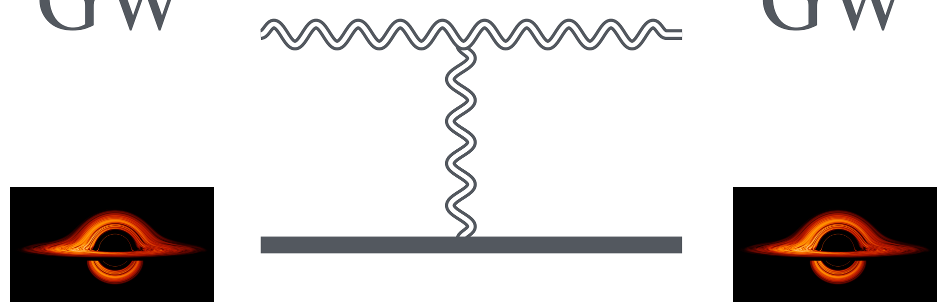

Match to Black Hole Perturbation Theory

Extract Wilson coefficients

↕

Love numbers

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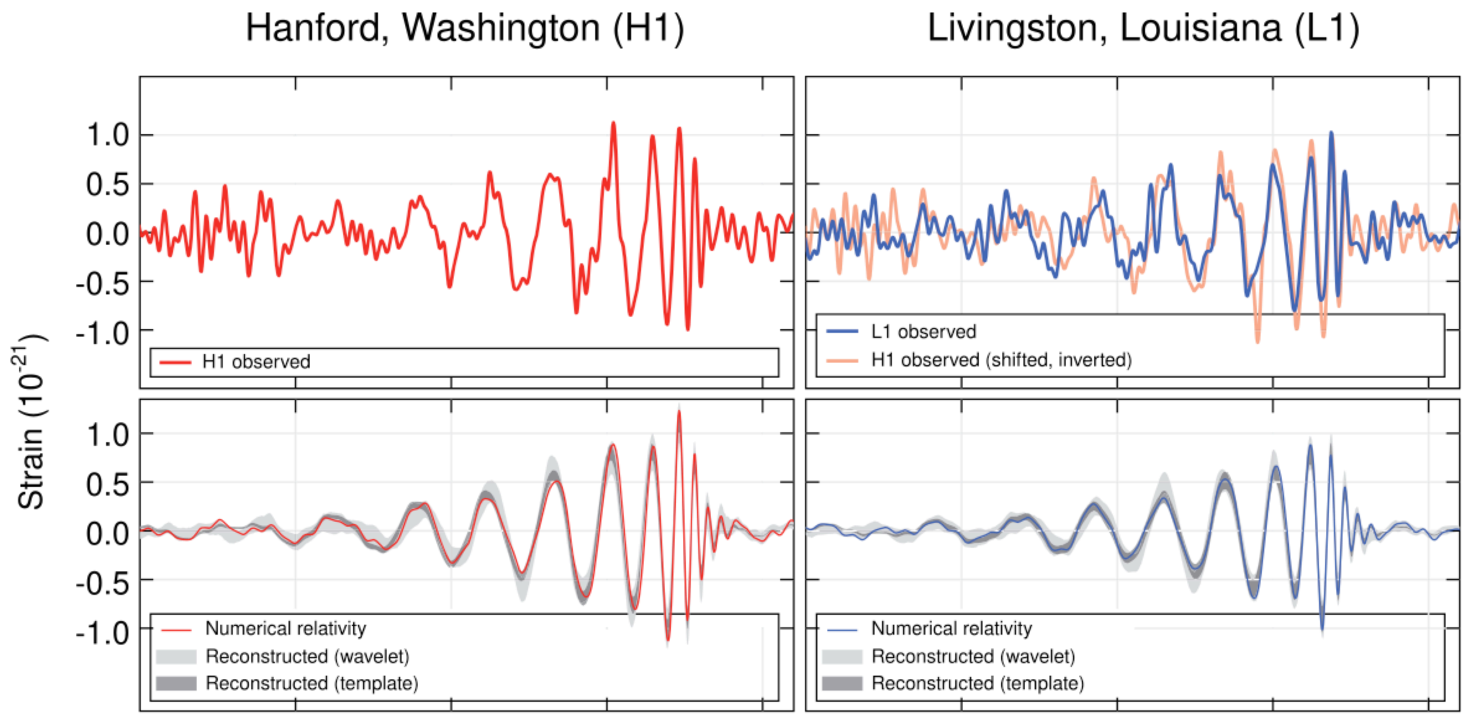

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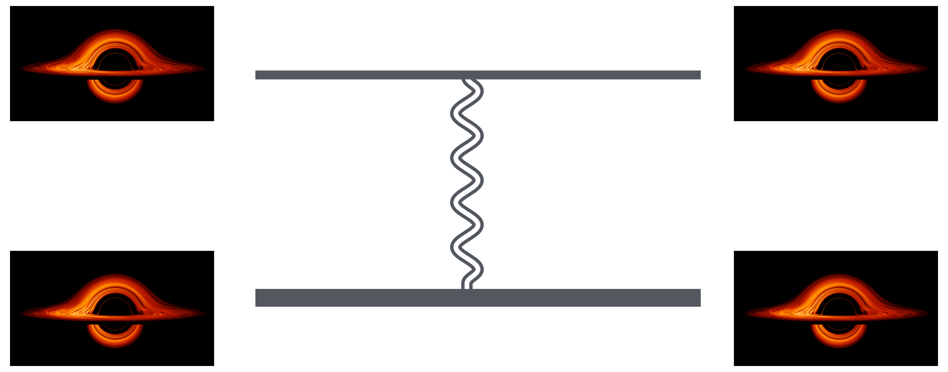

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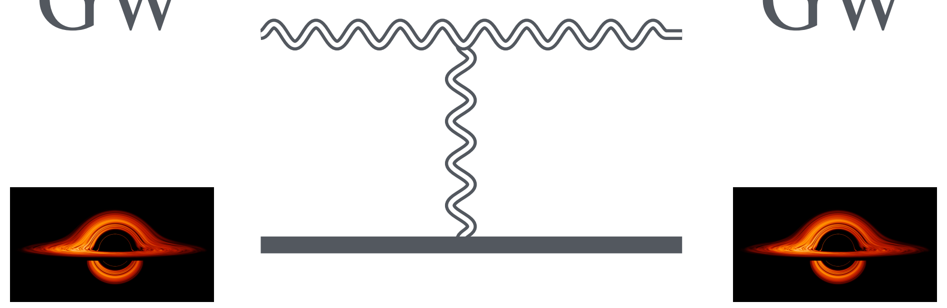

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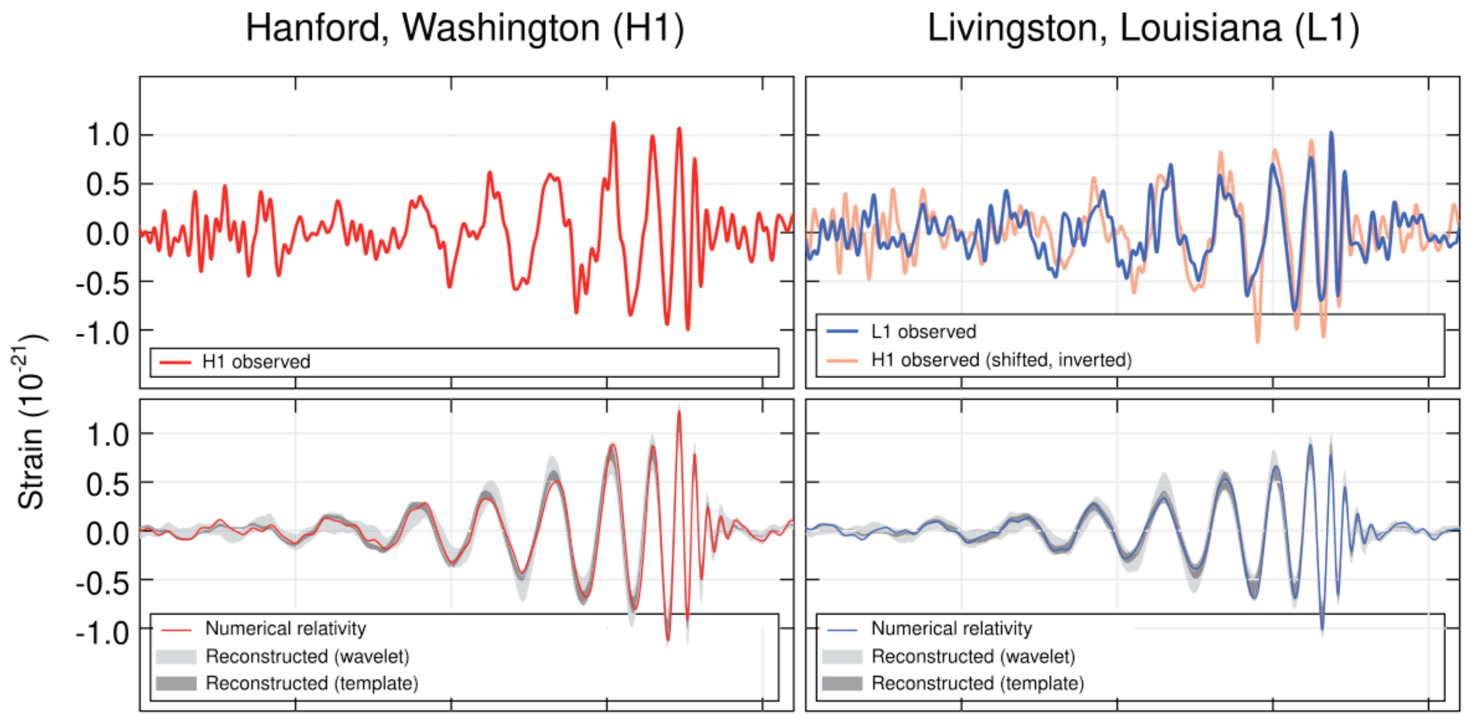

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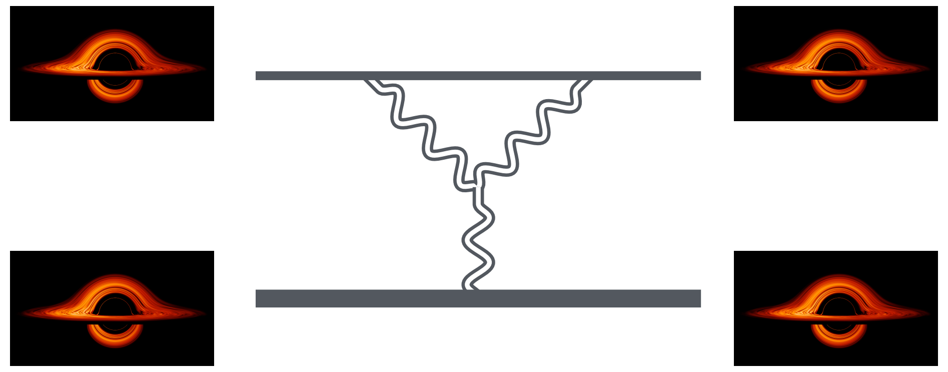

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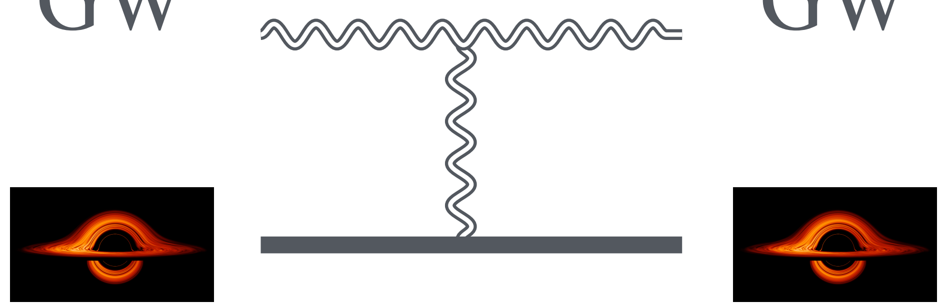

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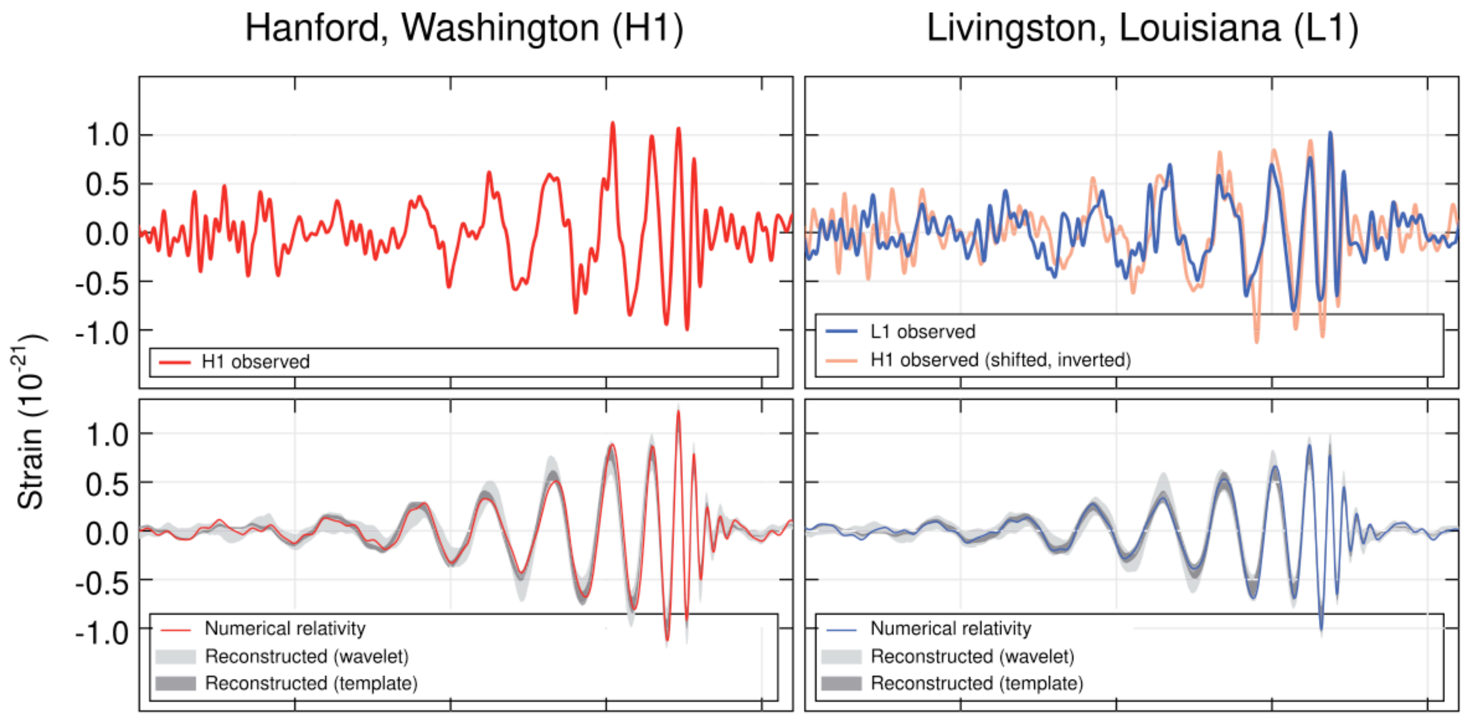

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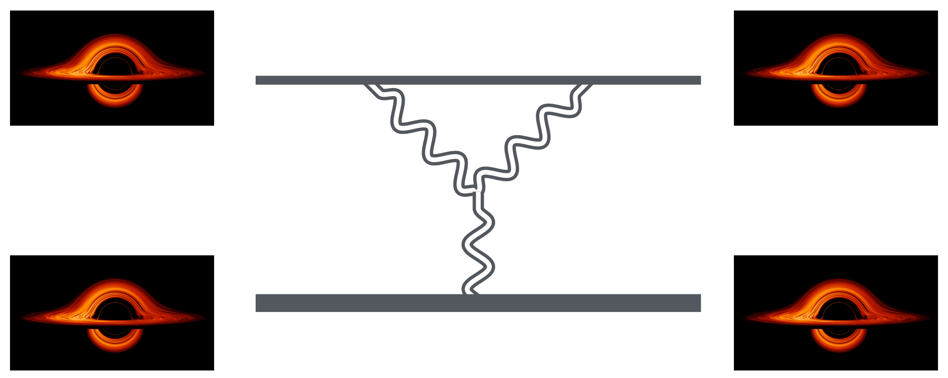

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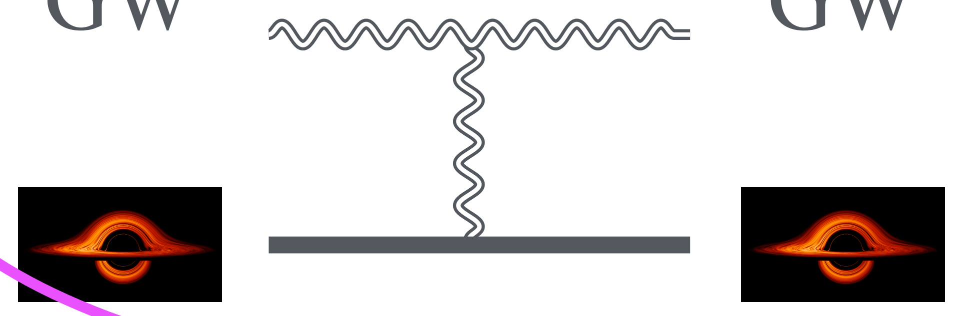

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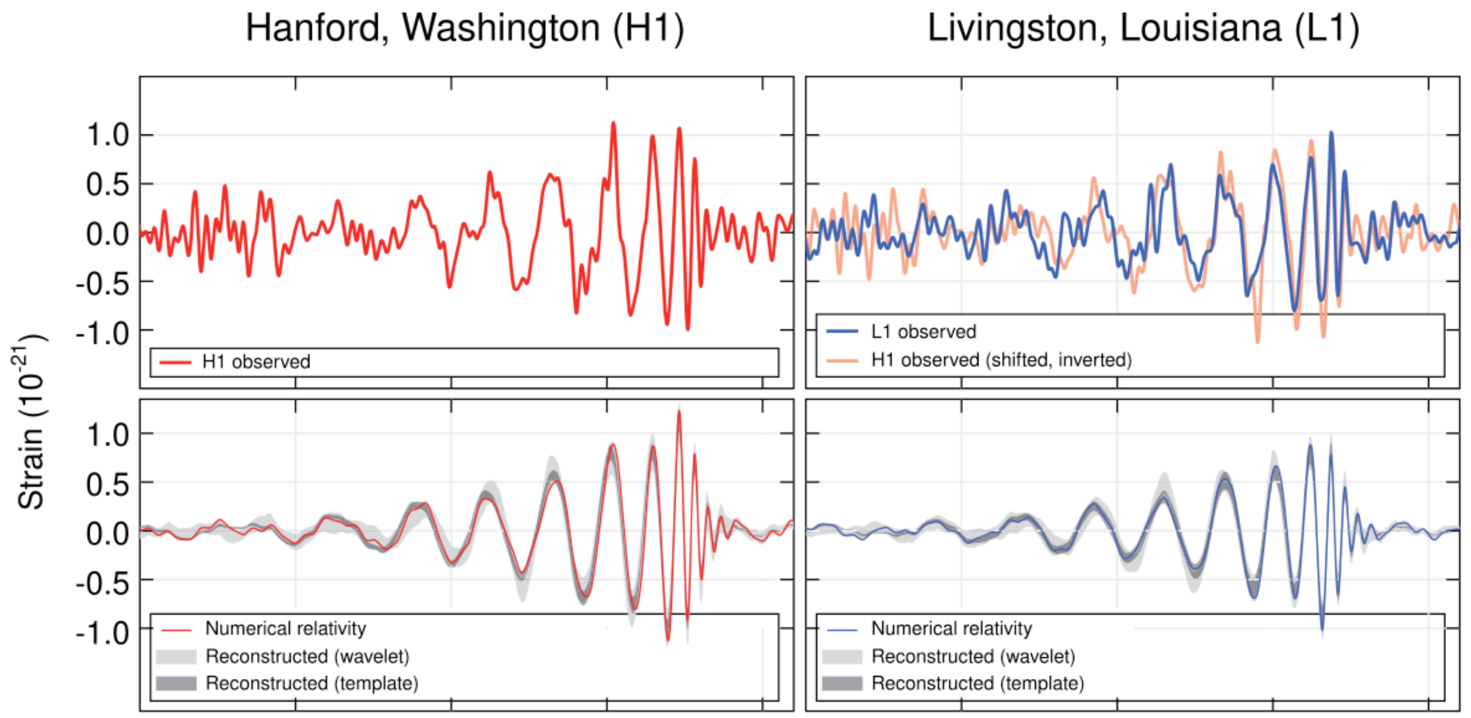
Love numbers

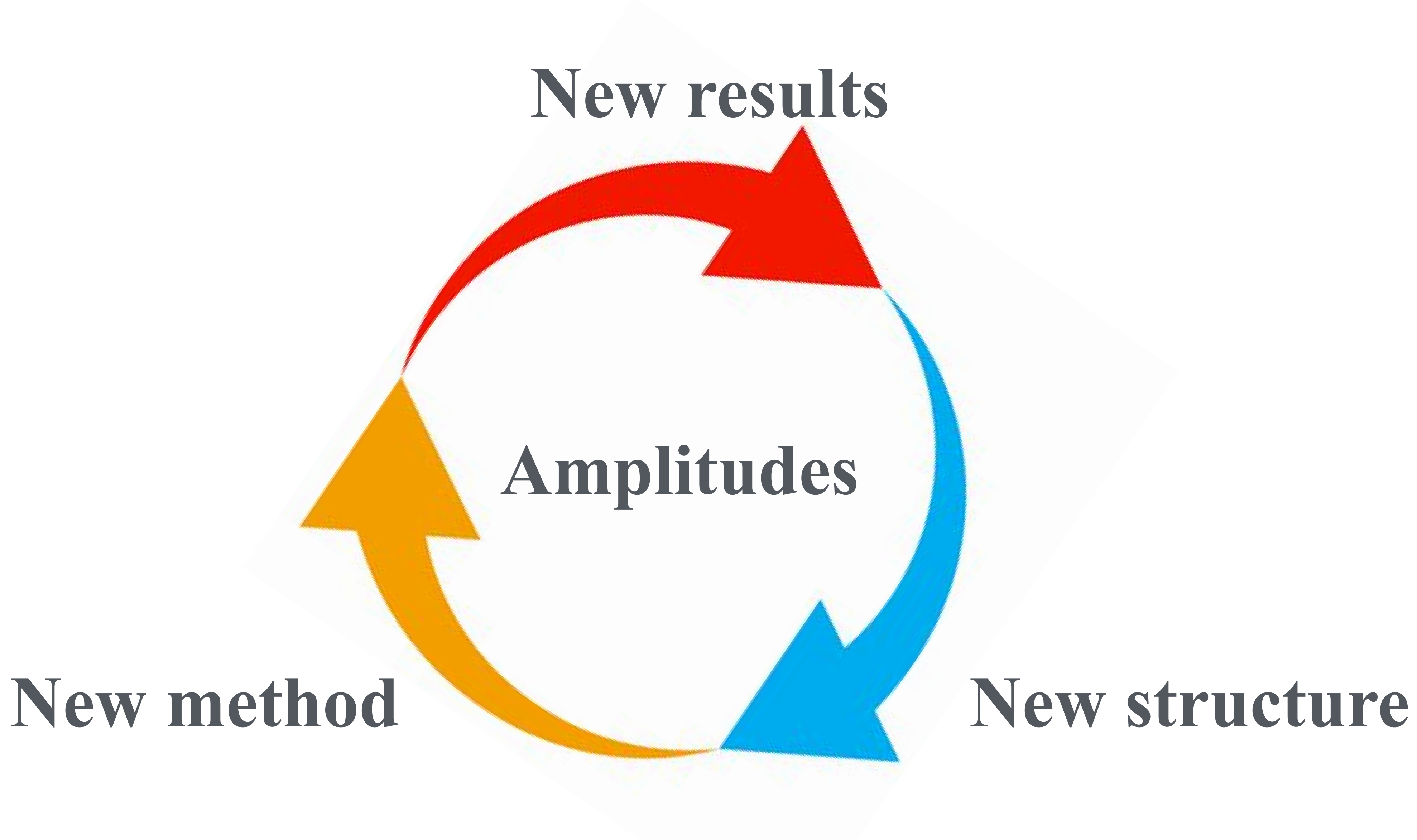
This talk

LIGO theorists



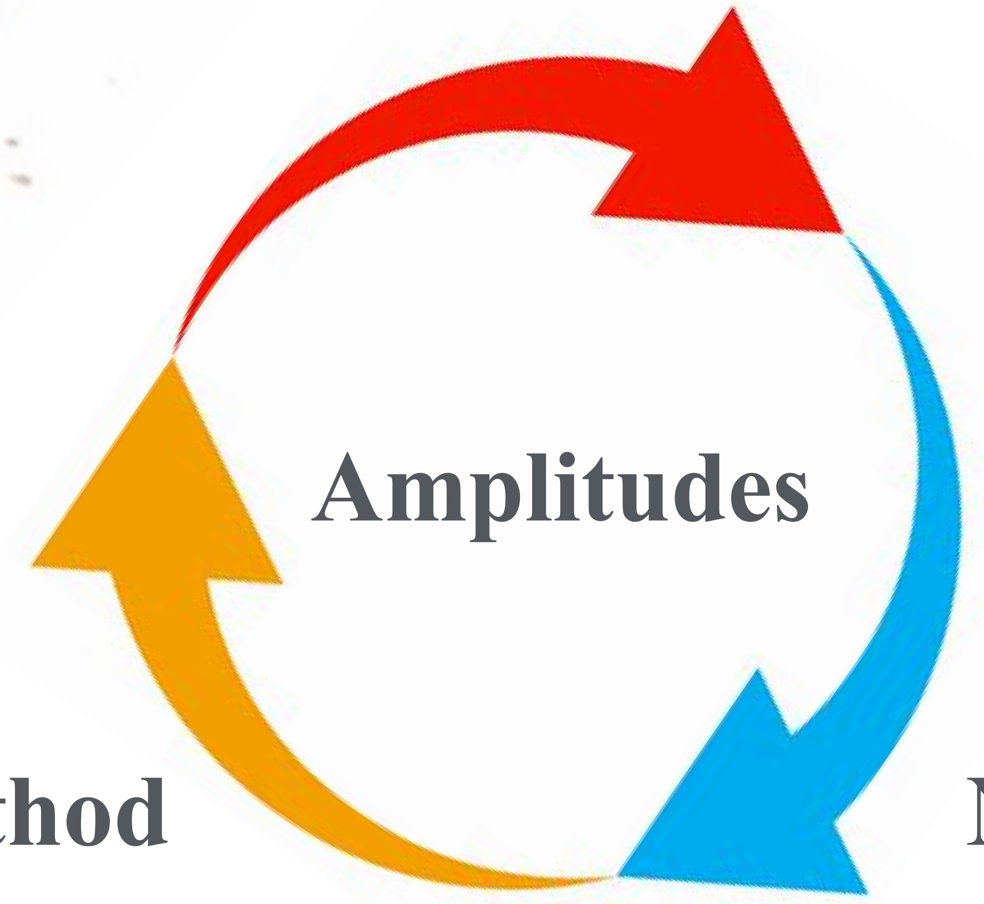
The amplitudes pipeline for classical gravitational physics

The virtuous cycle of Amplitudes



The virtuous cycle of Amplitudes

Shell EFT
4D Effective Field Theory
that incorporates solutions to
Black Hole Perturbation Theory



The virtuous cycle of Amplitudes

Scalar Love numbers
through G^9
(no integration!)

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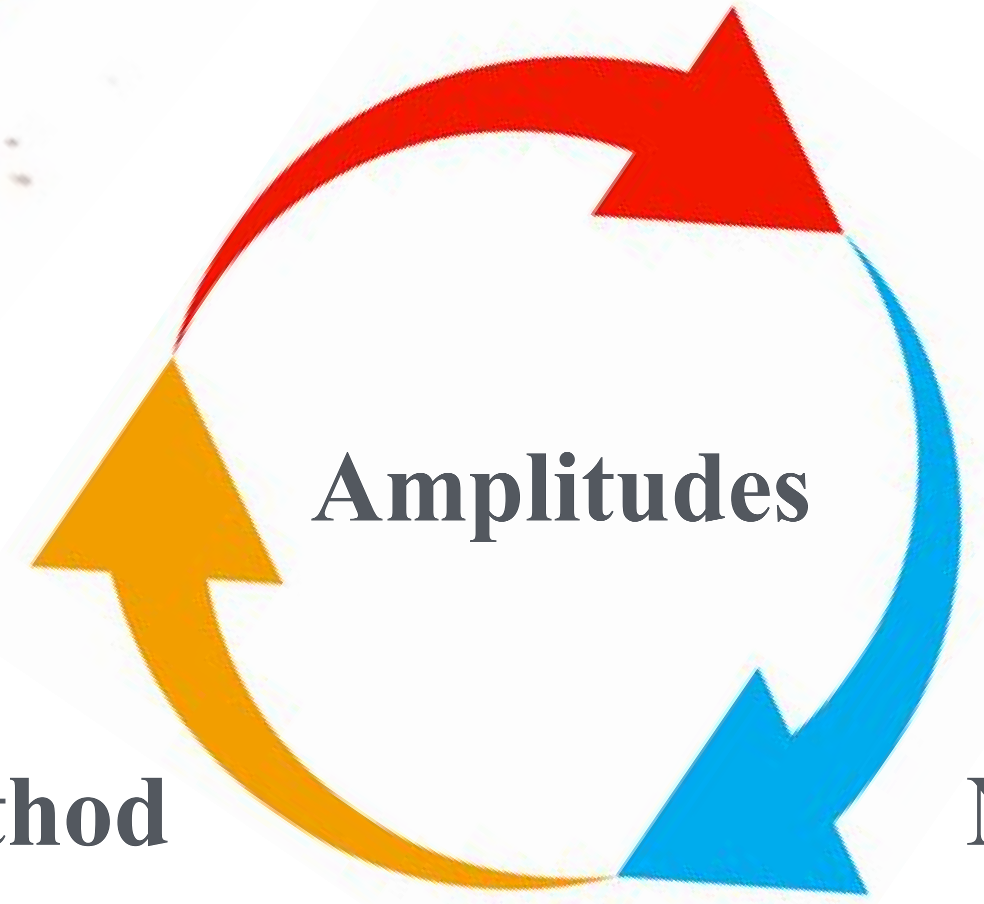

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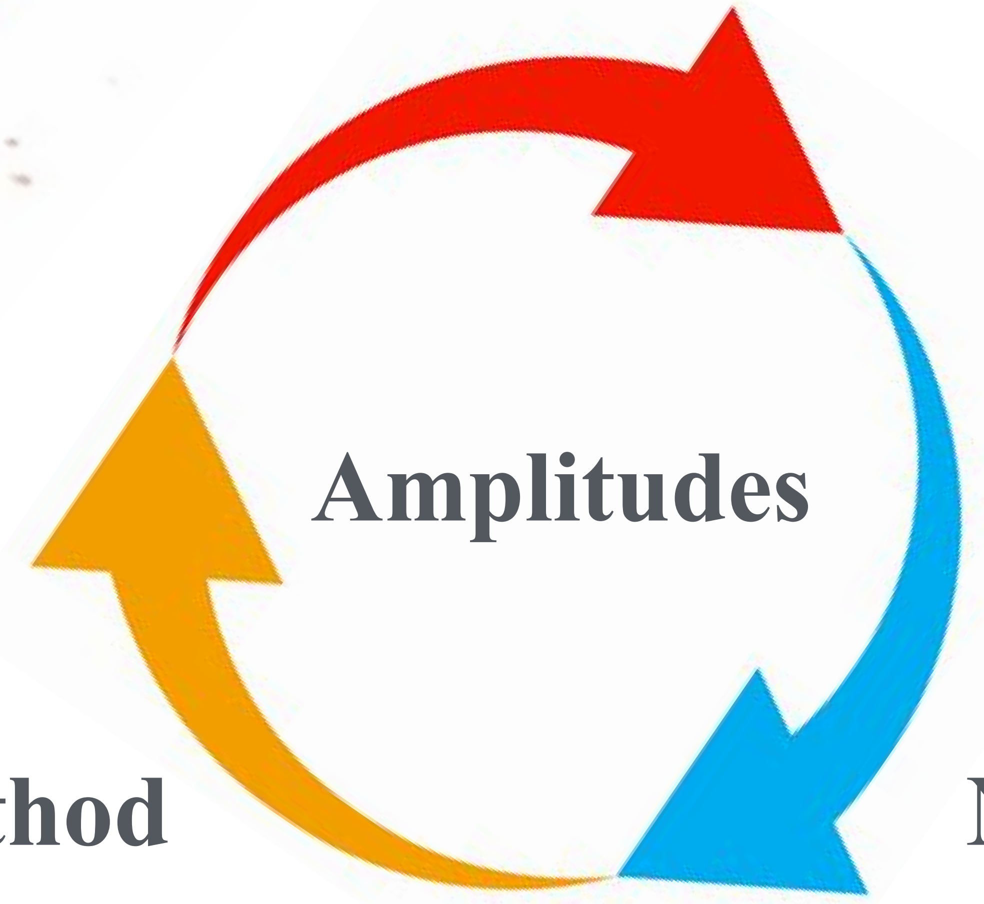
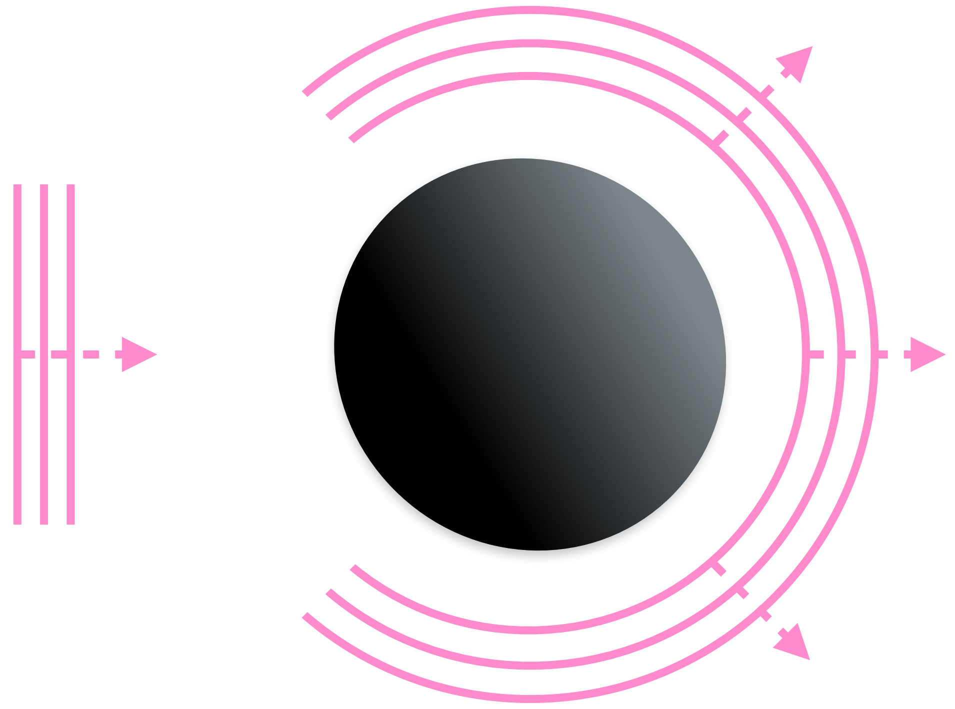
Conjecture
for the all-orders structure
of black-hole Love numbers



Introduction

What are Love numbers?

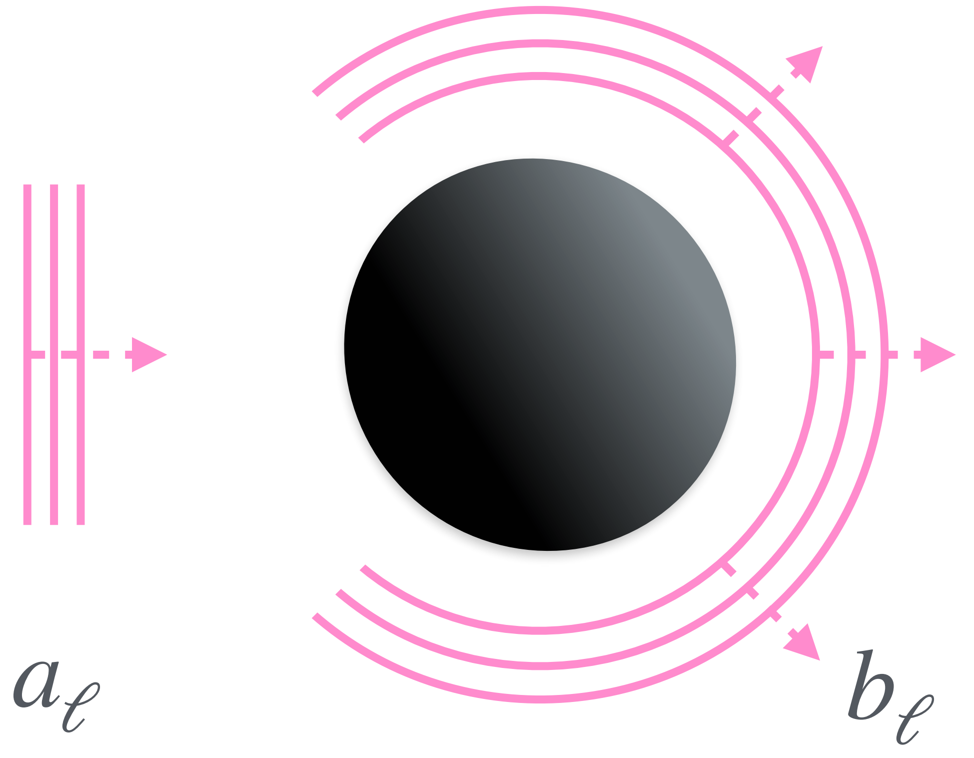


1. **Tidal response** of an object to a perturbation (e.g., Earth-Moon system)

2. Provide information about the object's **structure**

Love numbers are a principal goal of gravitational-wave astronomy

What are Love numbers?



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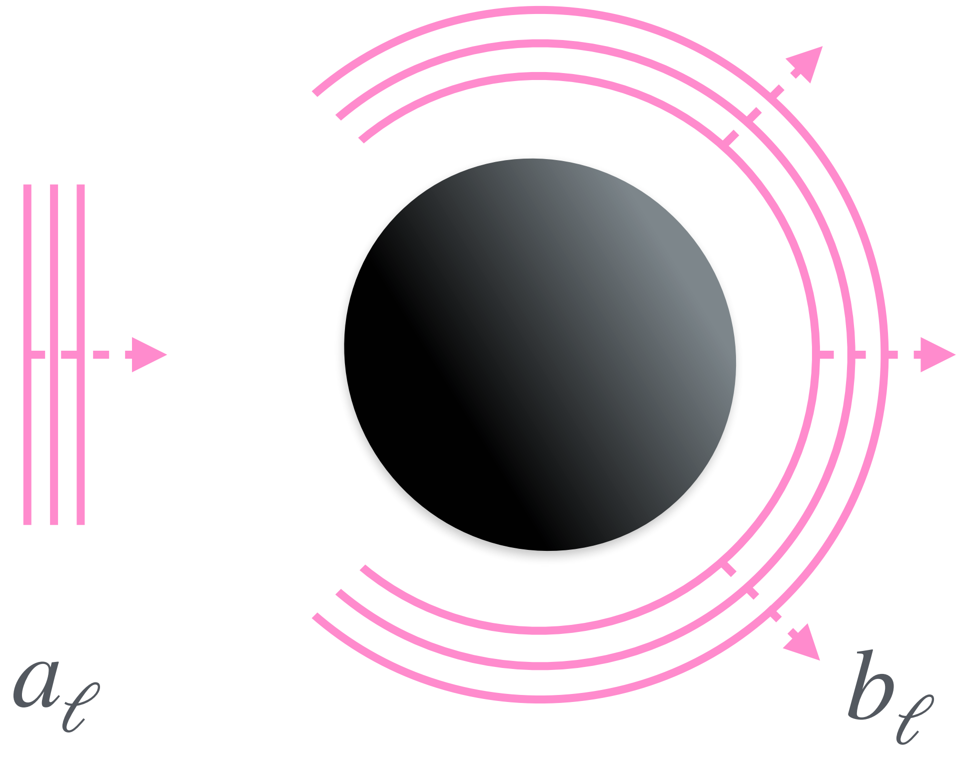

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Static Love numbers are **zero** for black holes in General Relativity in 4D in asymptotically flat space

→ Theoretically intriguing (e.g., symmetry, naturalness)

→ Good probe of beyond-GR effects

Love numbers are a principal goal of gravitational-wave astronomy

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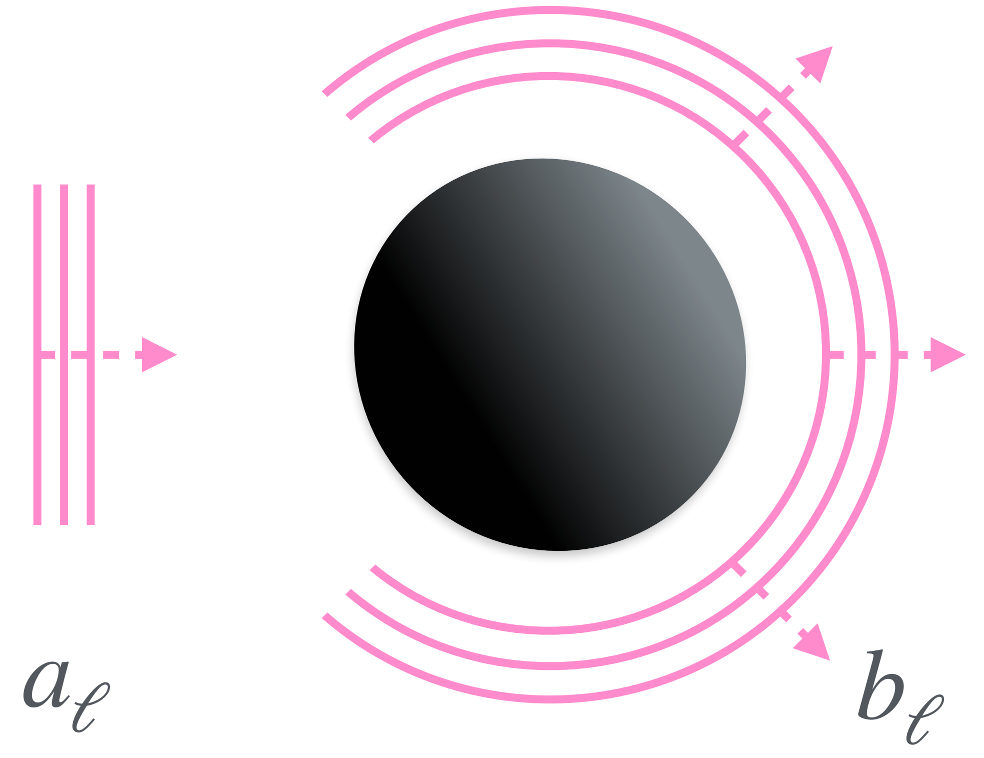

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Dynamical Love numbers for black holes are surprisingly simple [**DK**, Perrone, Solon (2025)]

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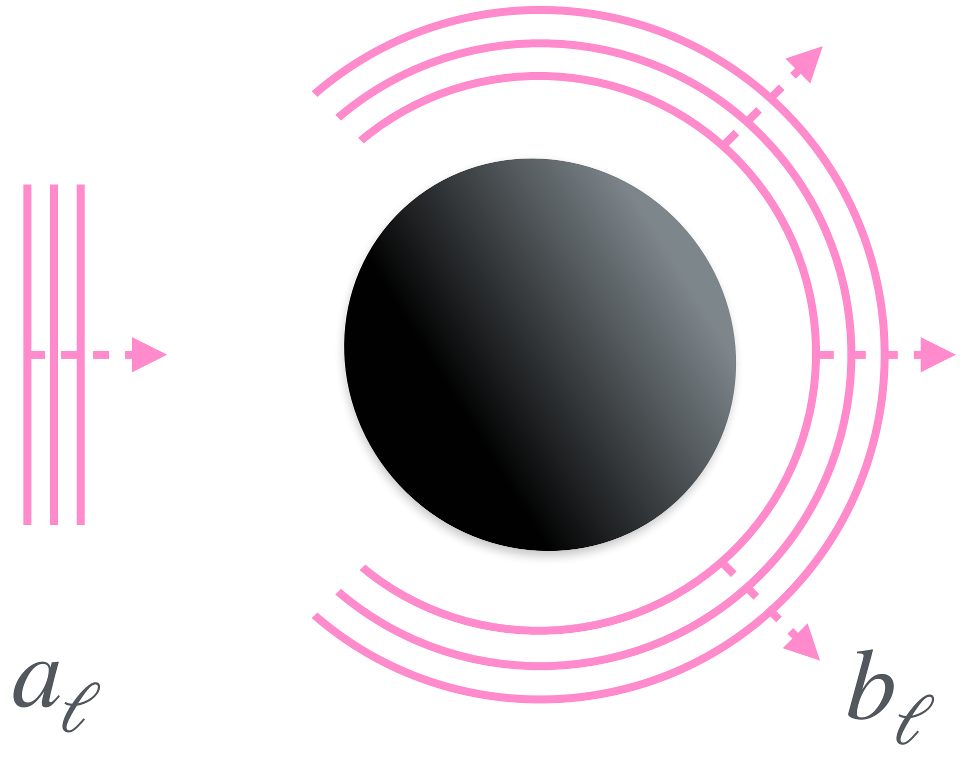

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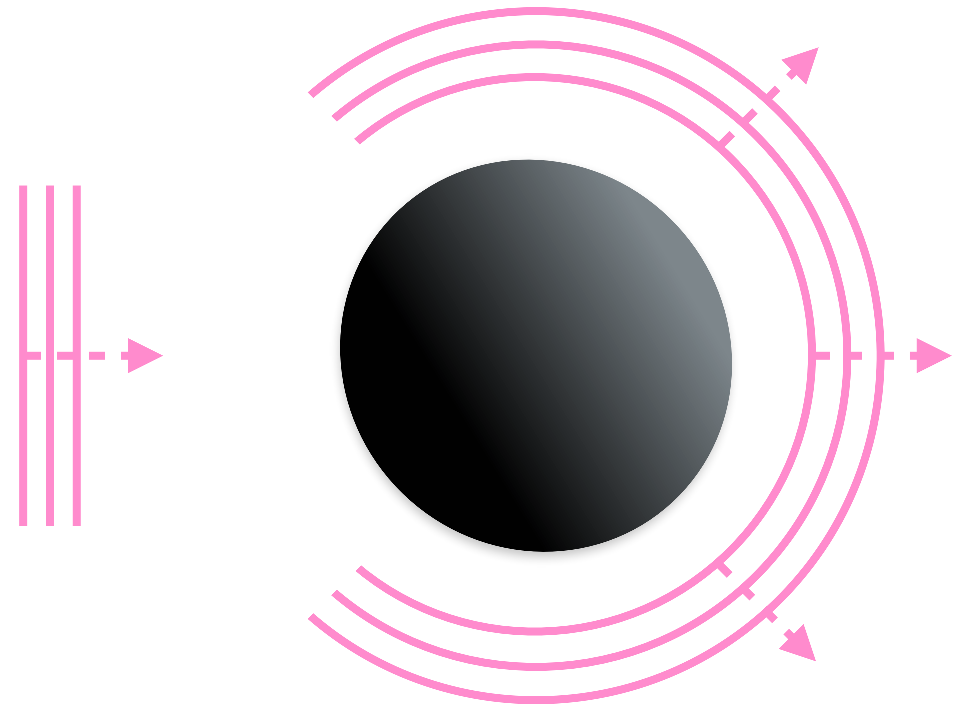
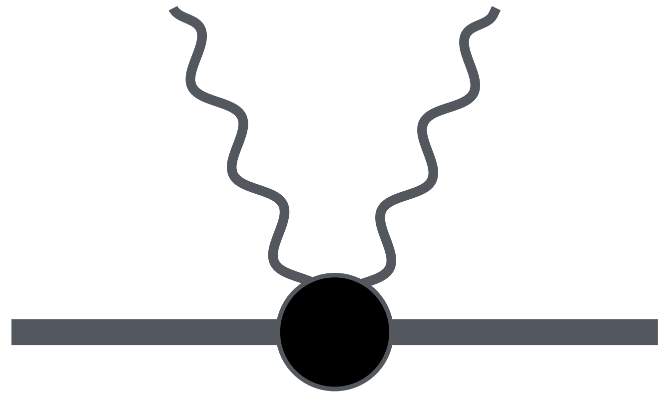
Love numbers for neutron stars ↔ **Equation of state**

Love numbers are a principal goal of gravitational-wave astronomy

Love numbers in Quantum Field Theory

Complete description
(e.g., black hole in GR)





Effective description
in terms of a
point particle

EFT tidal Wilson coefficients (c_ℓ) and Love numbers (k_ℓ)

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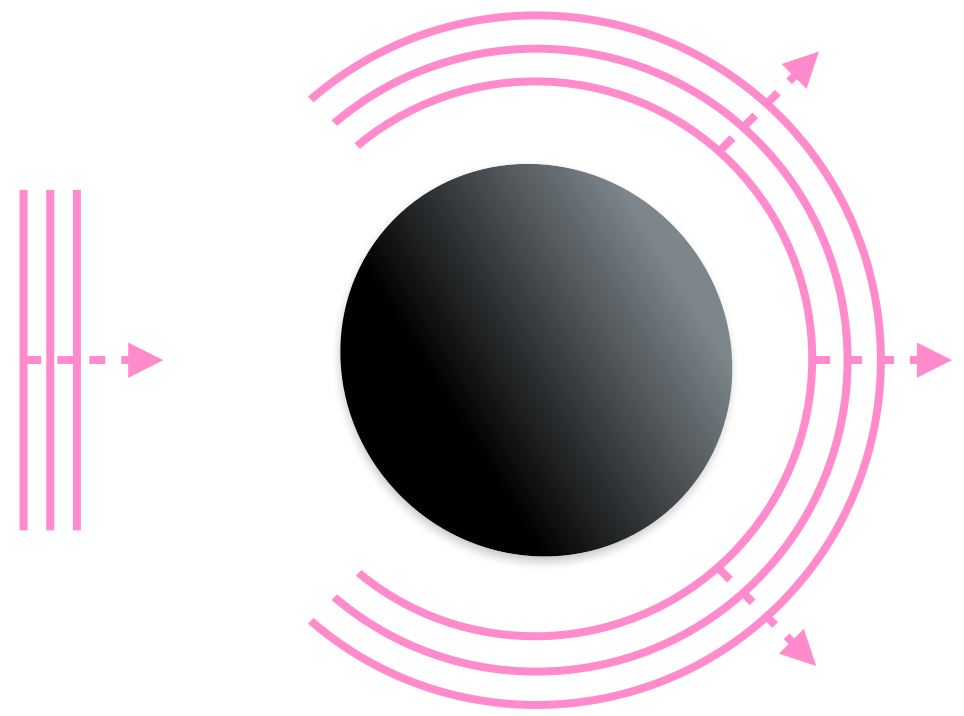

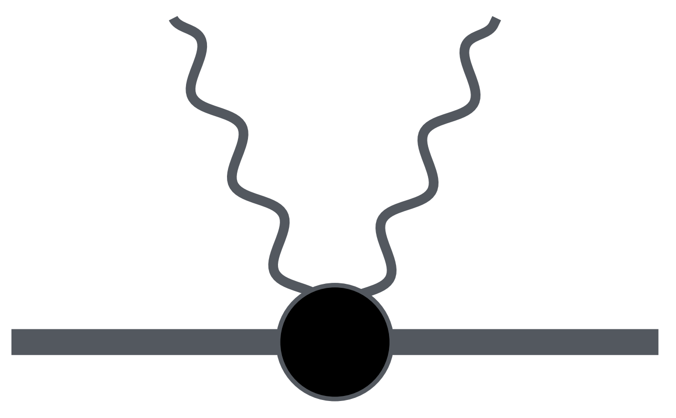

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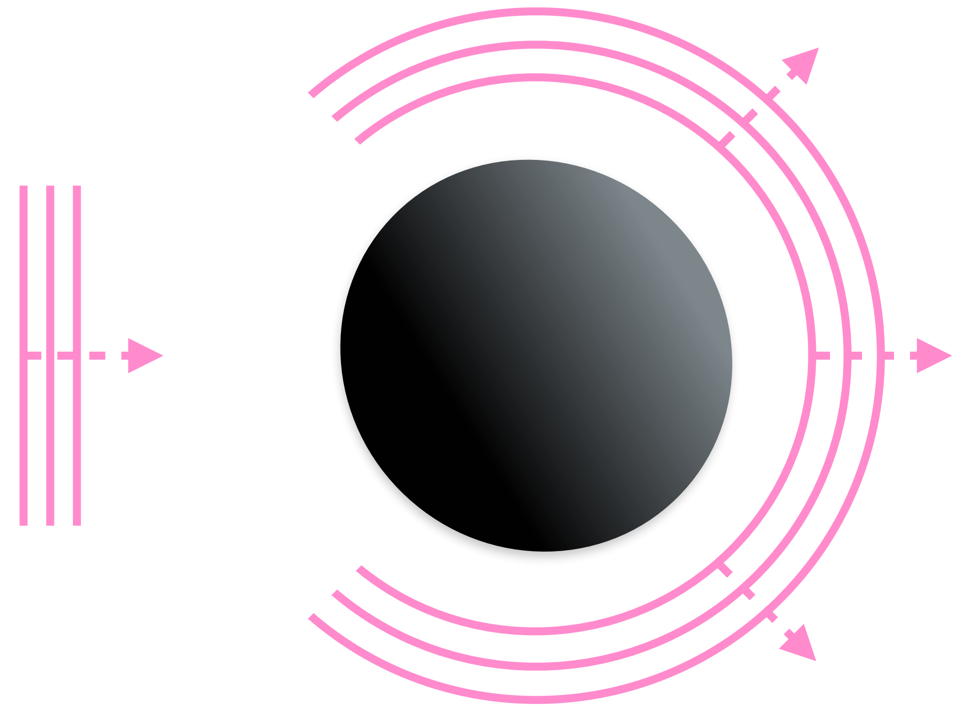
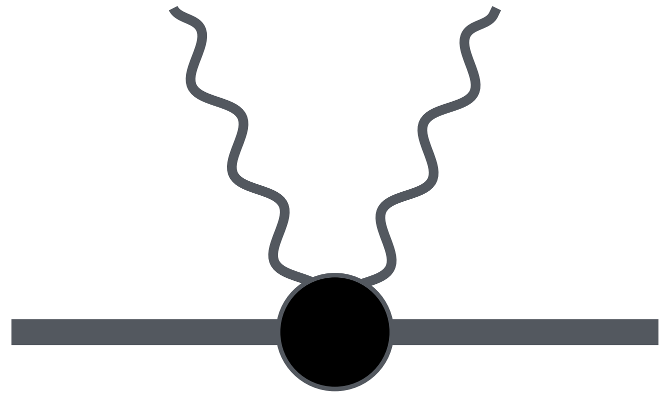


Toy graviton ϕ



Compact object

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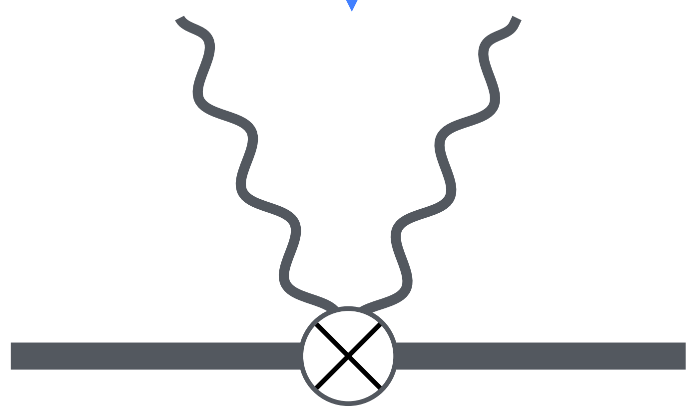

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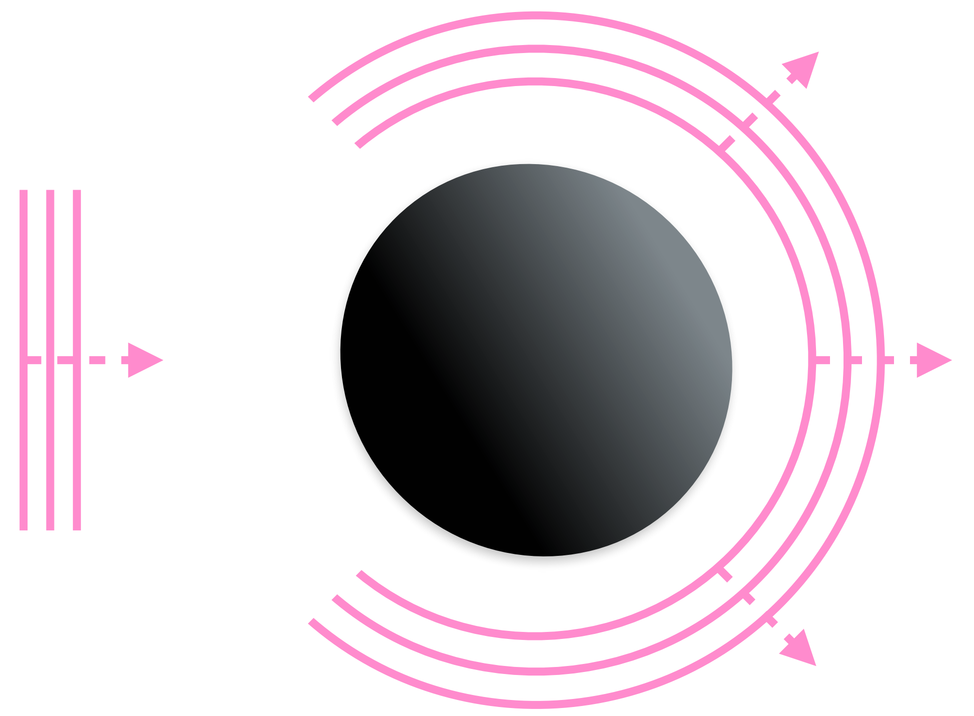

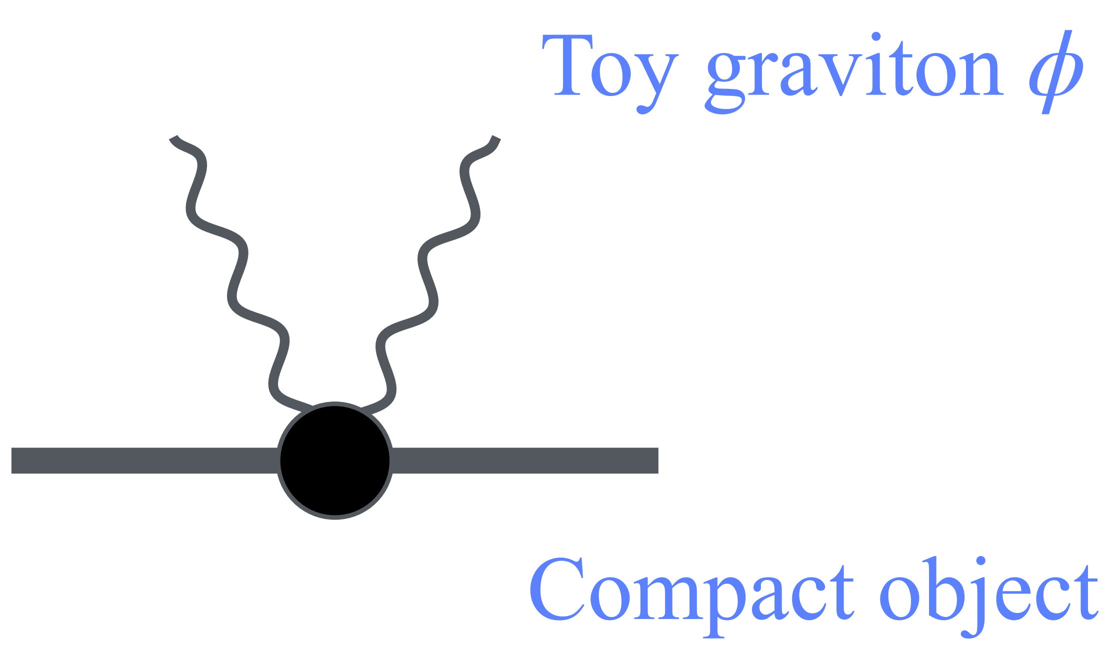

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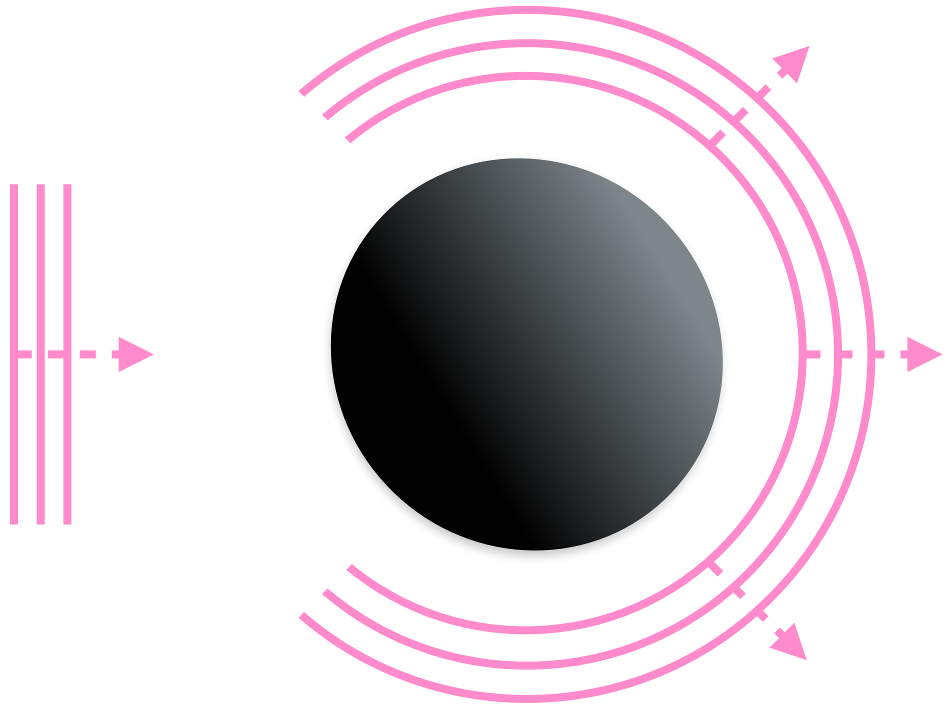

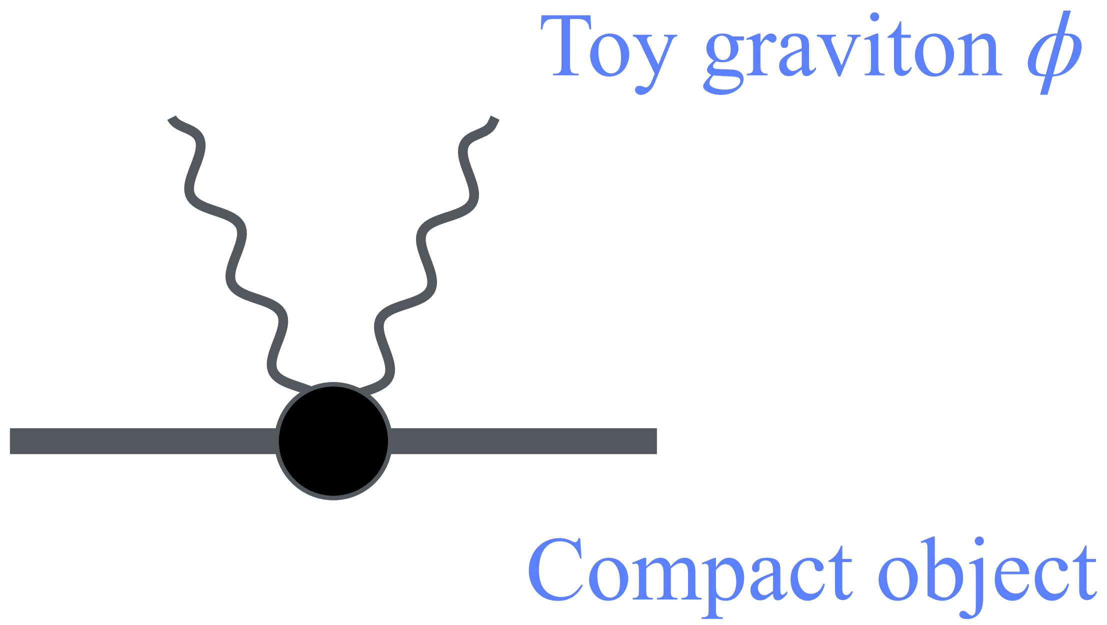

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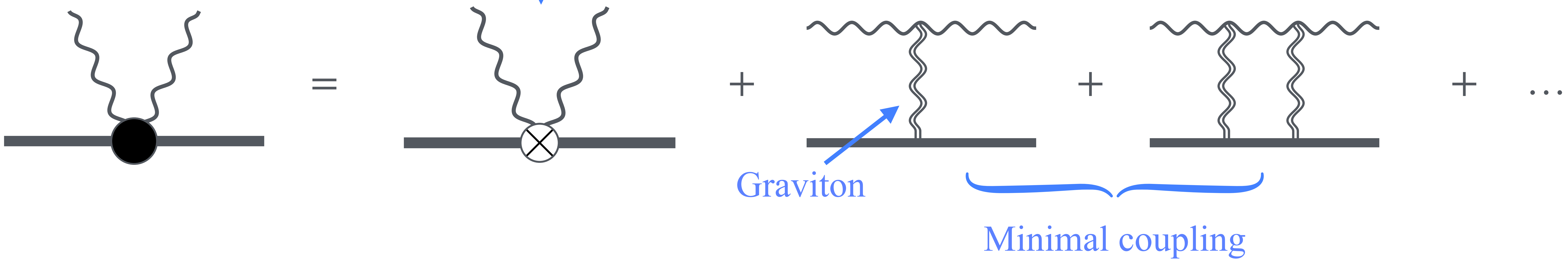

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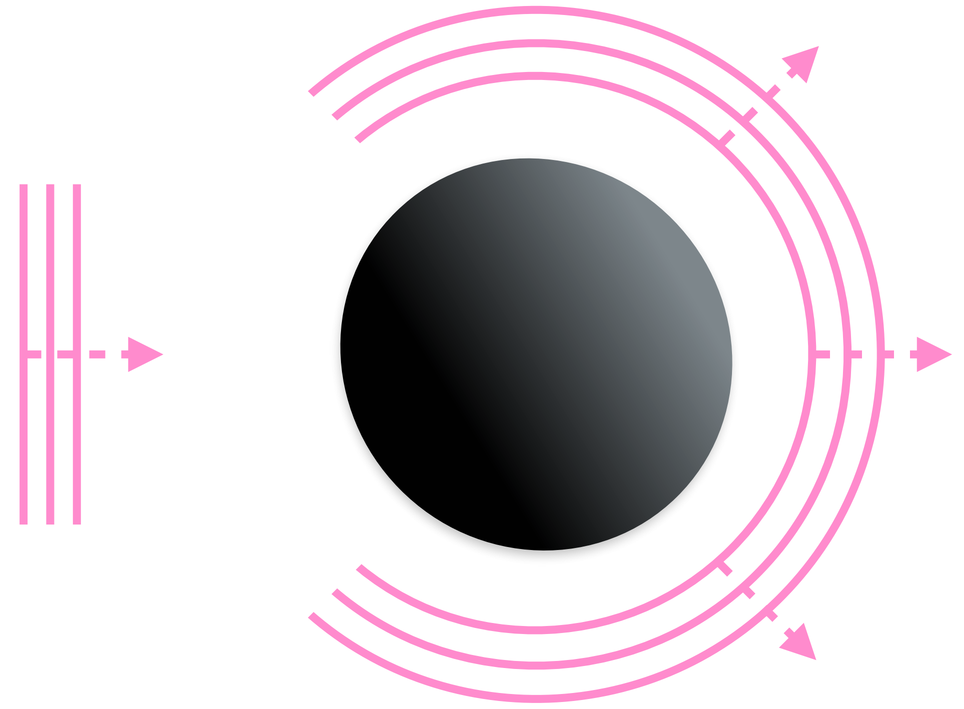

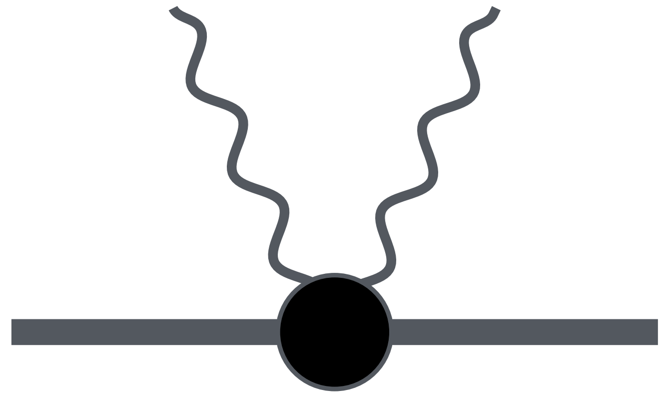

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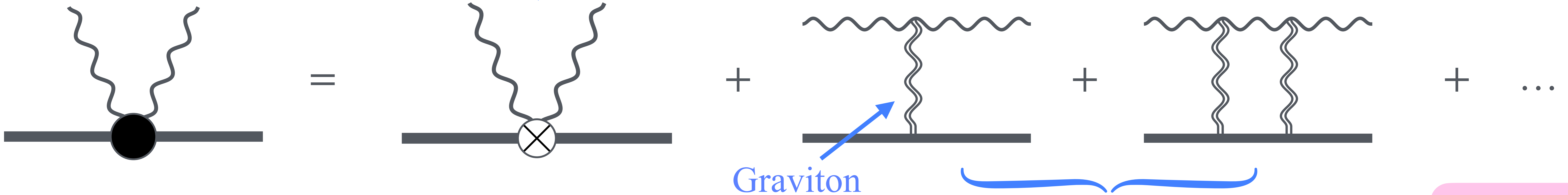

$$\boxed{c_\ell \sim k_\ell}$$

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The idea behind Shell Effective Field Theory

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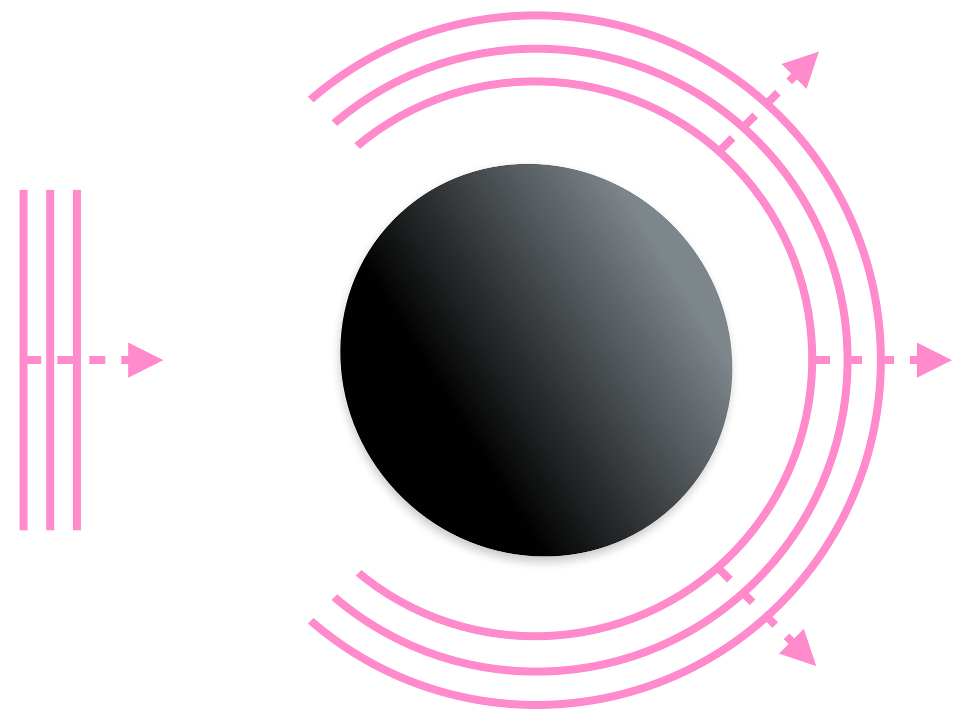

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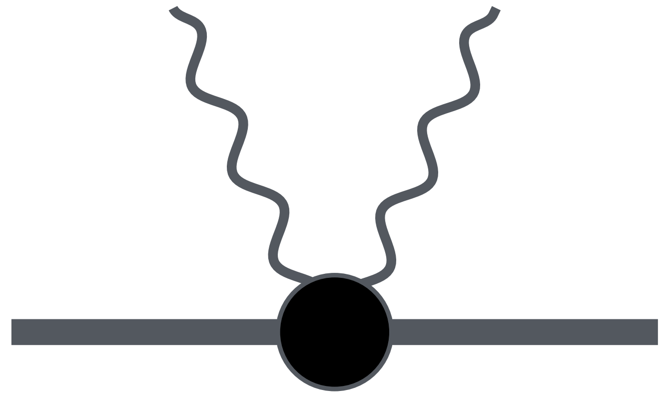


Compact object

Effective description
in terms of a
point particle

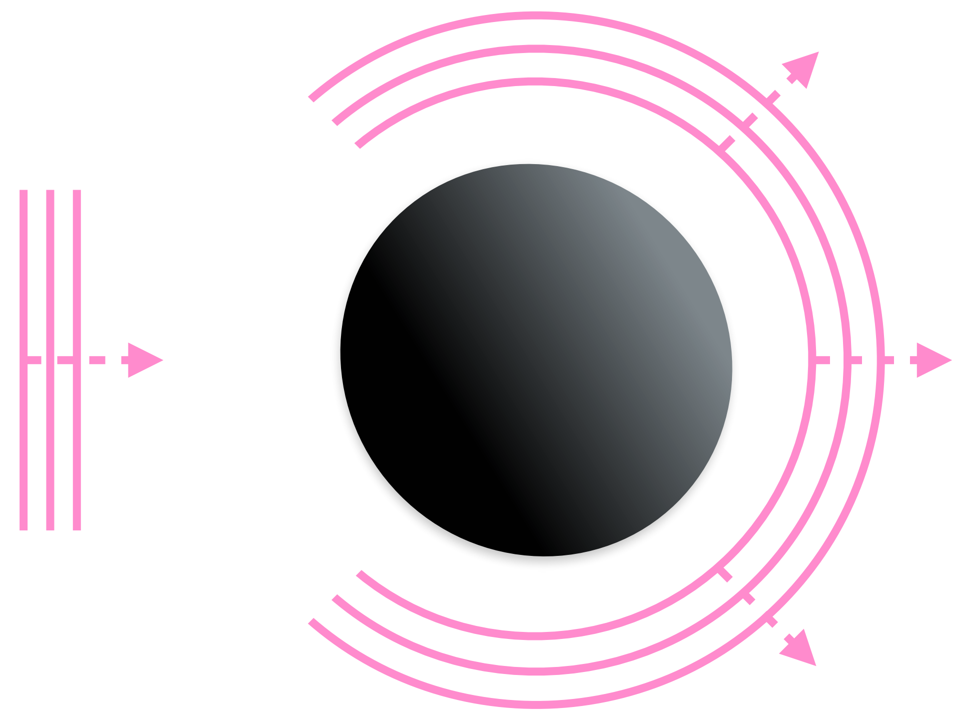
Solved to any order in G
in Black Hole Perturbation Theory

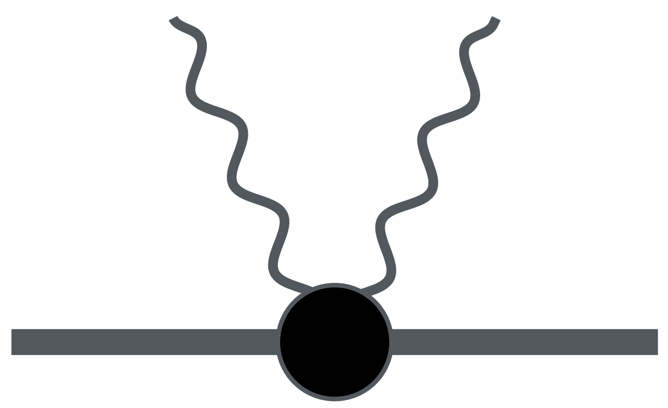
Higher orders in G require
hard loop integrals

Designing an EFT framework that employs techniques from BHPT

The idea behind Shell Effective Field Theory

Complete description
(e.g., black hole in GR)





Effective description
in terms of a
point particle

Solved to any order in G
in Black Hole Perturbation Theory

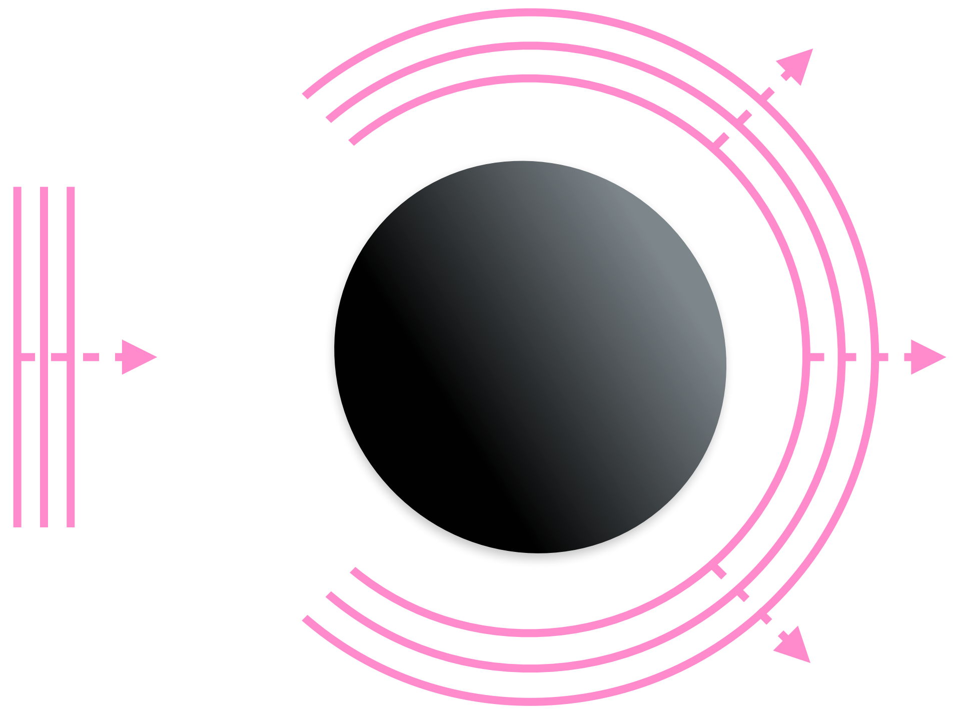
Higher orders in G require
hard loop integrals

Can we bypass loop integration
by incorporating BHPT solutions?

Designing an EFT framework that employs techniques from BHPT

Shell Effective Field Theory

Preliminary: EFT description of compact objects



EFT paradigm: From far away, any compact object looks like a point particle

The shell is a 4D regularization of the point-particle description

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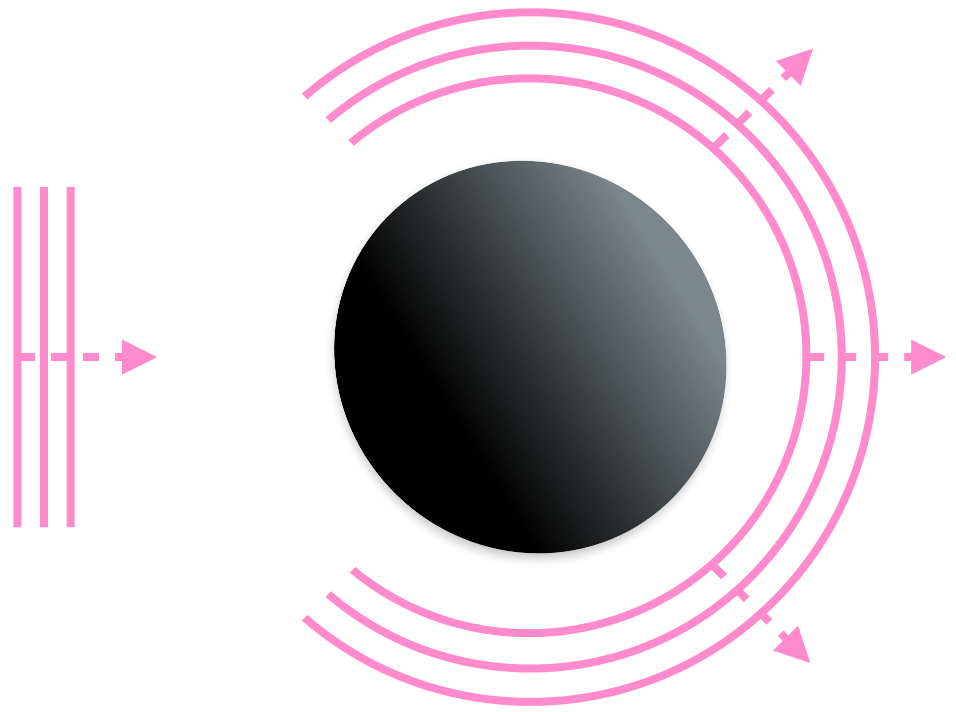

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A black hole, a neutron star, a cat, …

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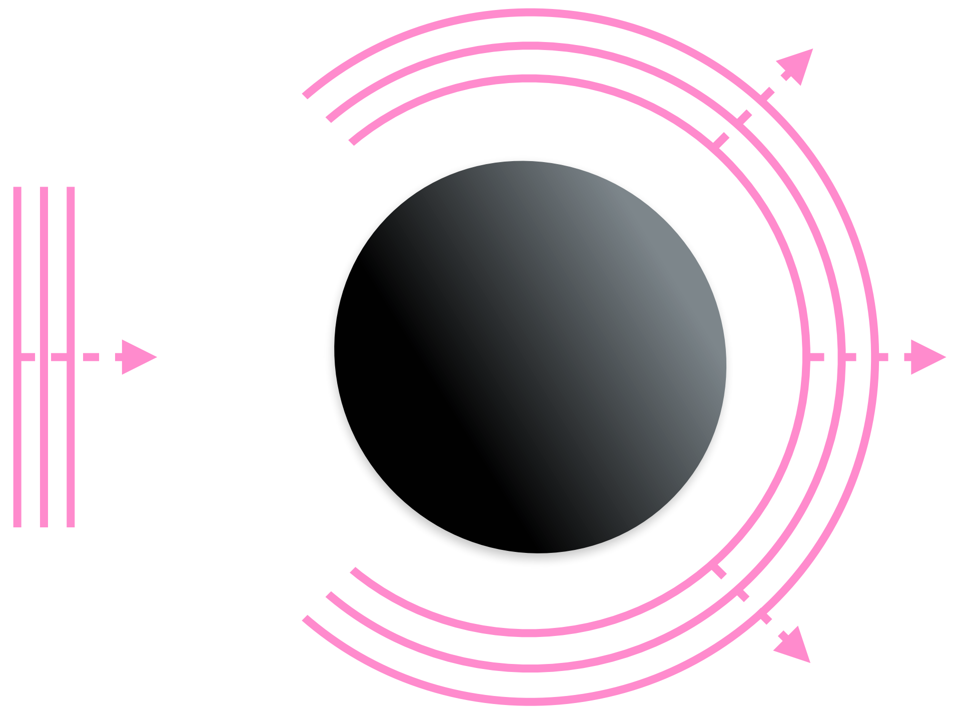

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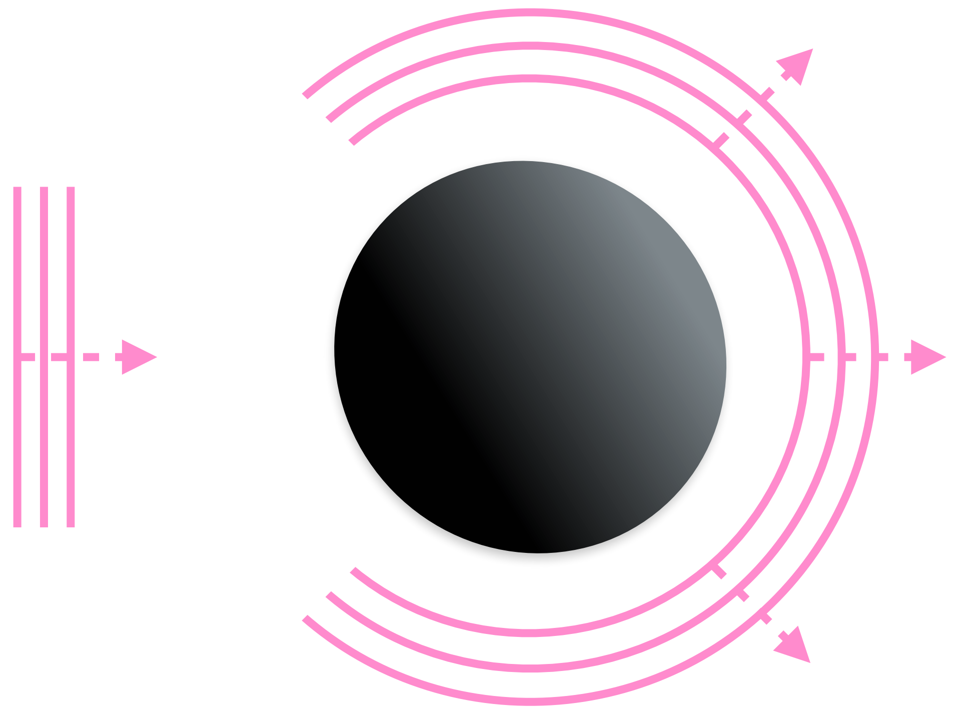

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Dimensional regularization? **Black-Hole-Perturbation-Theory solutions don't exist in** $4-\epsilon$ **dimensions**

(their use in the EFT is our main innovation)

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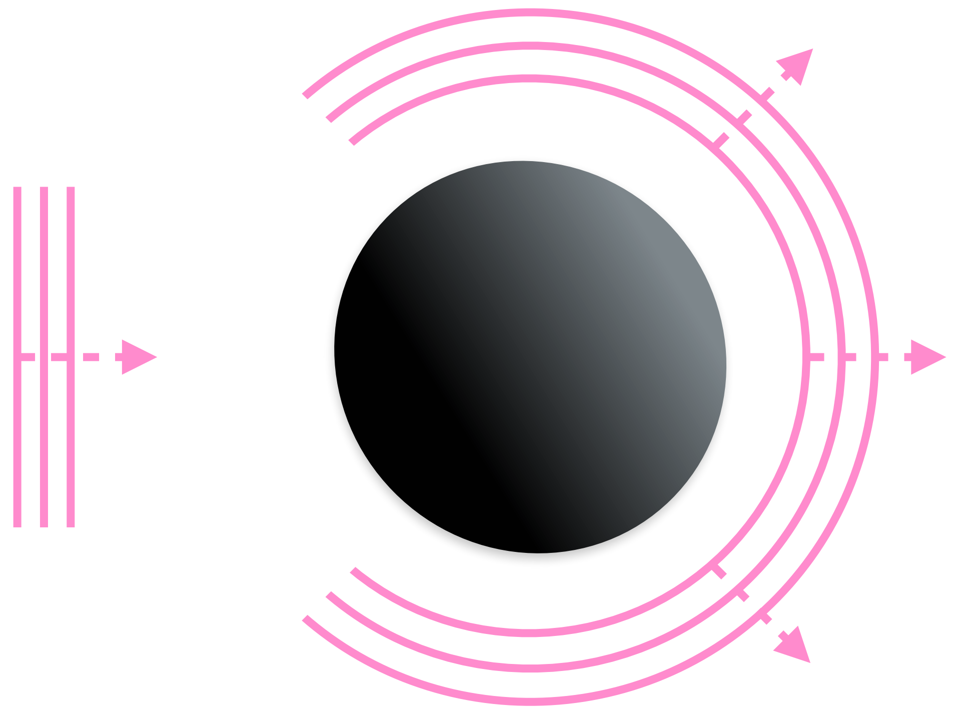

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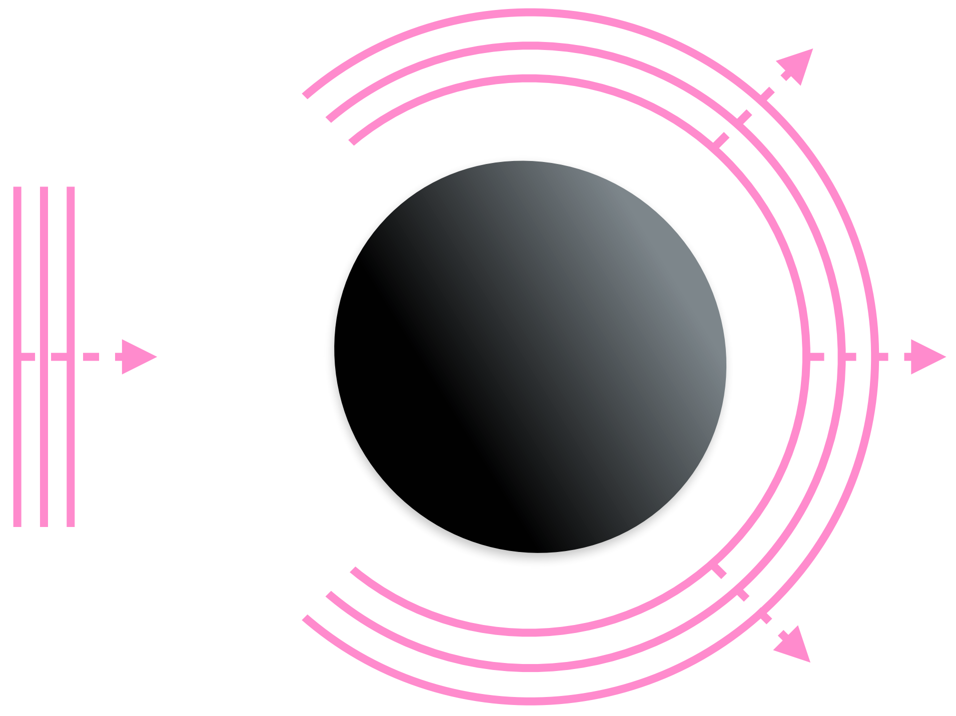
Effective description with 4D cutoff-style regulator: **Shell EFT**

The shell is a 4D regularization of the point-particle description

The setup of Shell EFT



Black-hole exterior

$$ds_+^2 = f(r)dt^2 - \frac{dr^2}{f(r)} - r^2 d\Omega^2, \quad r > R > R_S$$

Flat-space interior

$$ds_-^2 = d\tilde{t}^2 - d\tilde{r}^2 - \tilde{r}^2 d\Omega^2, \quad \tilde{r} < R$$

Modeling any compact object in terms of a thin shell

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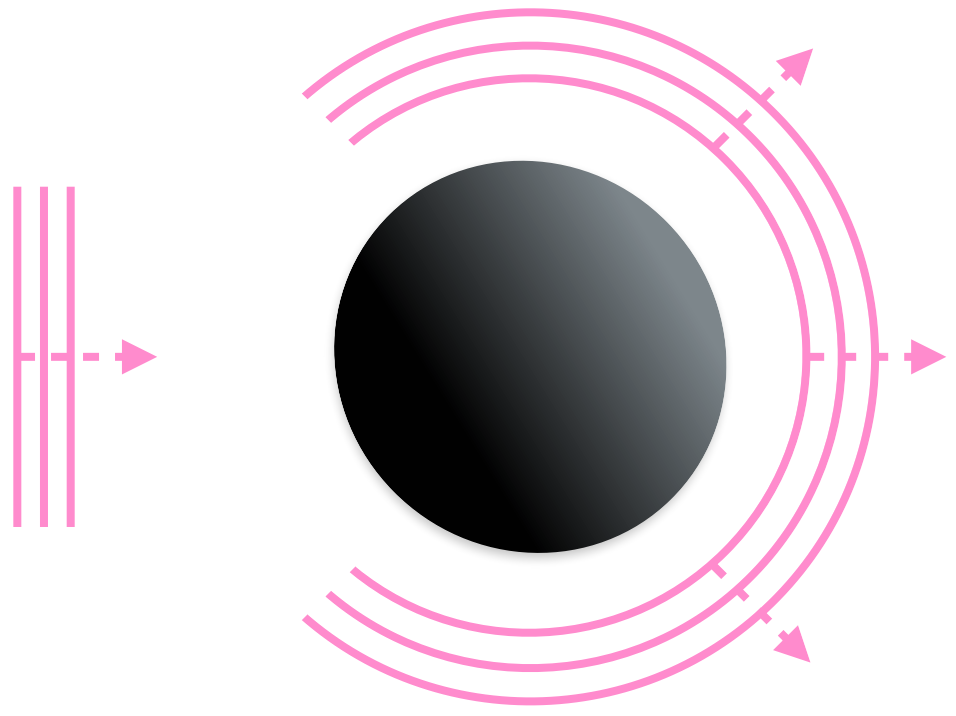

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Keldysh effective action for dissipation

Doubling of fields: $\phi \to (\phi, \varphi)$

Modeling any compact object in terms of a thin shell

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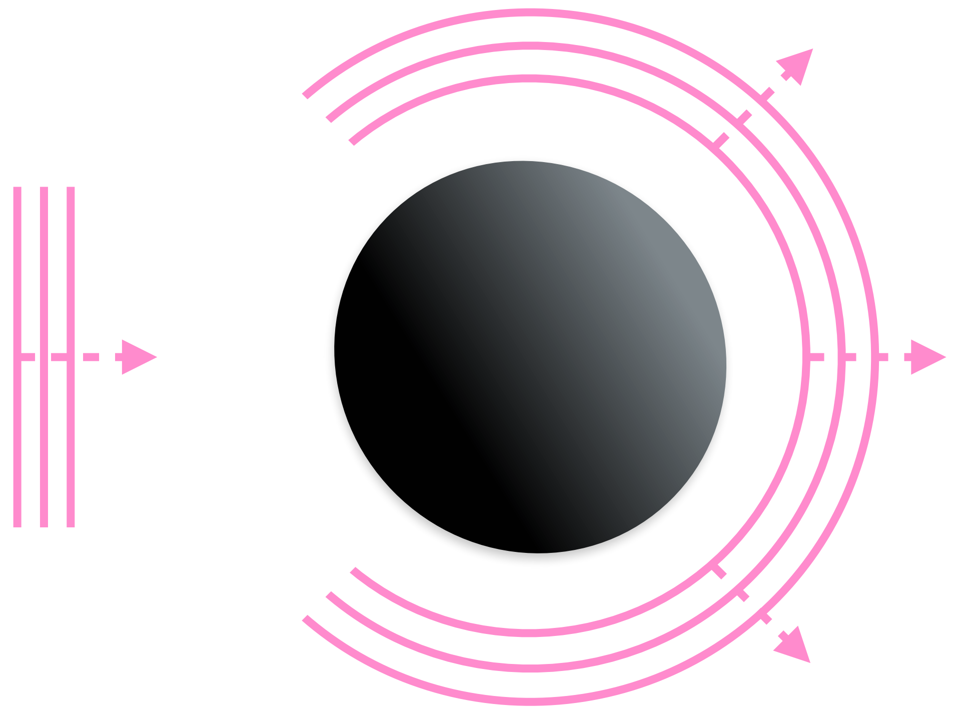

$$S_{\text{Bulk}} = \int d^4x\sqrt{-g}\; g^{\mu\nu}\partial_\mu\varphi\partial_\nu\phi$$

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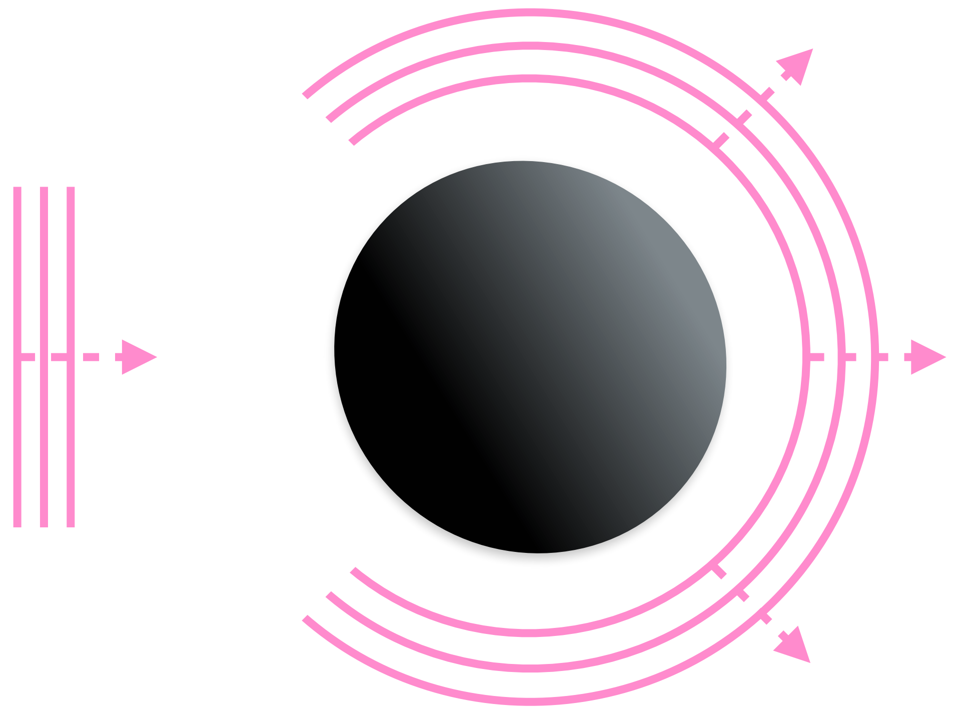

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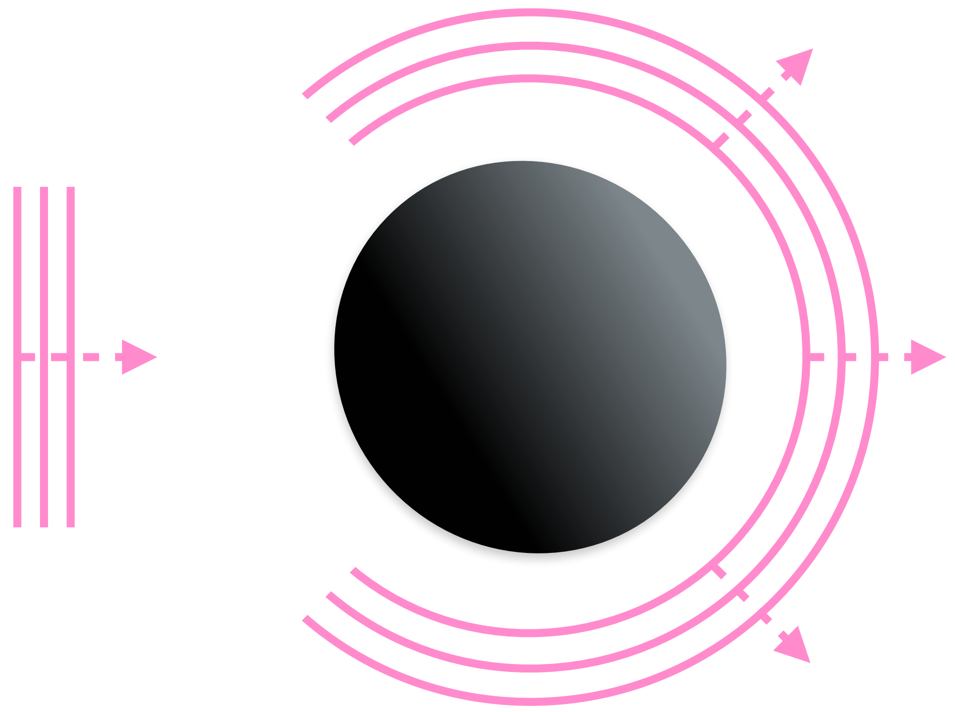

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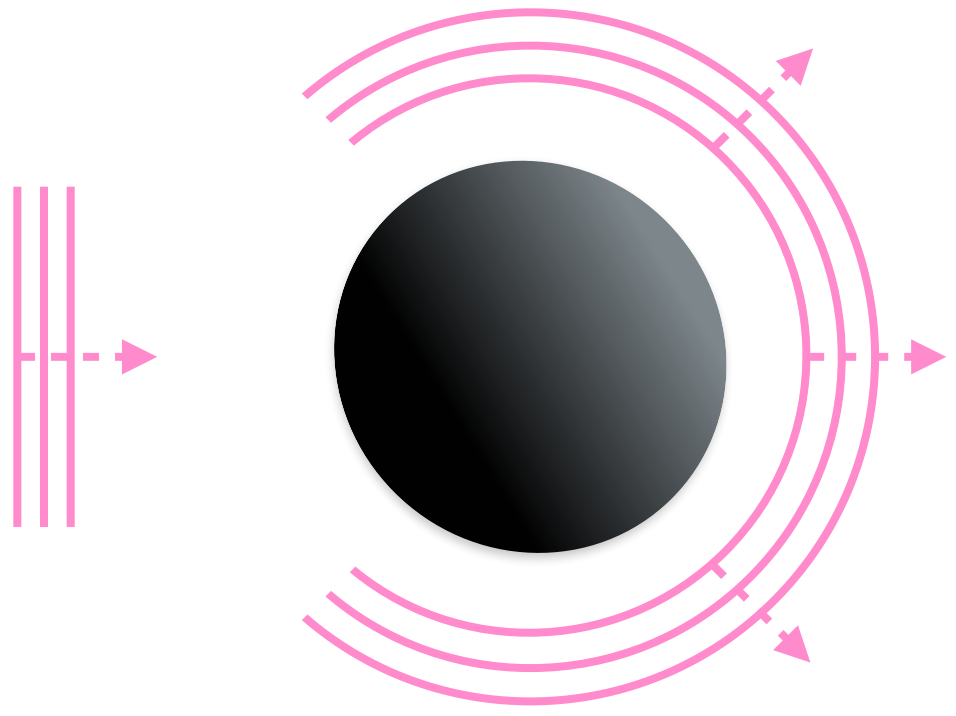

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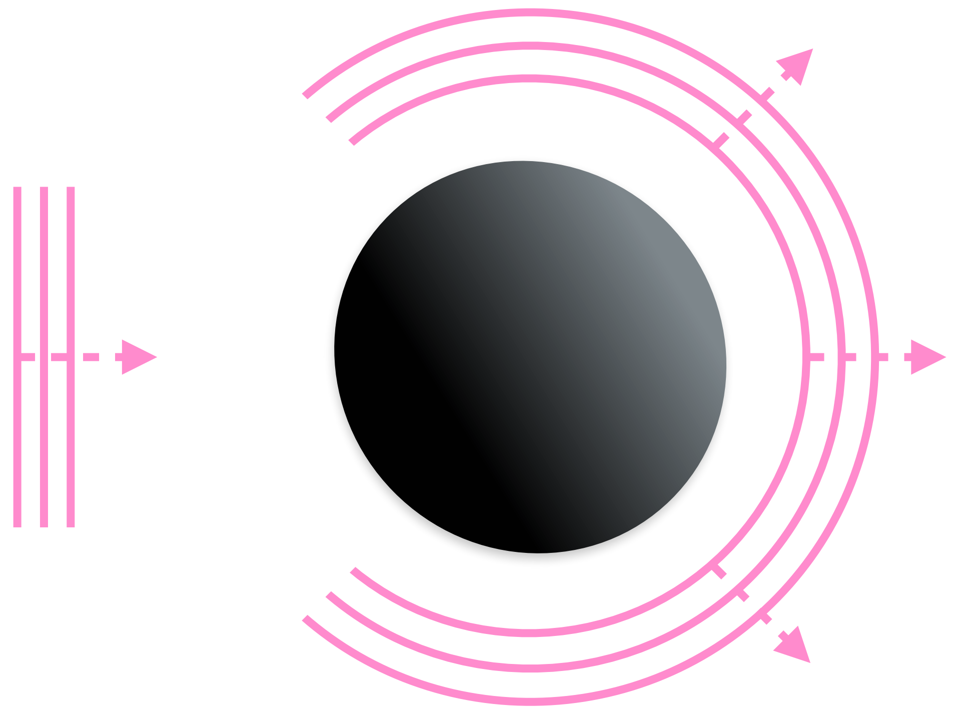

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Time derivatives

$D_\ell \sim \left(\nabla^2_\Omega\right)^\ell$ ← Angular derivatives

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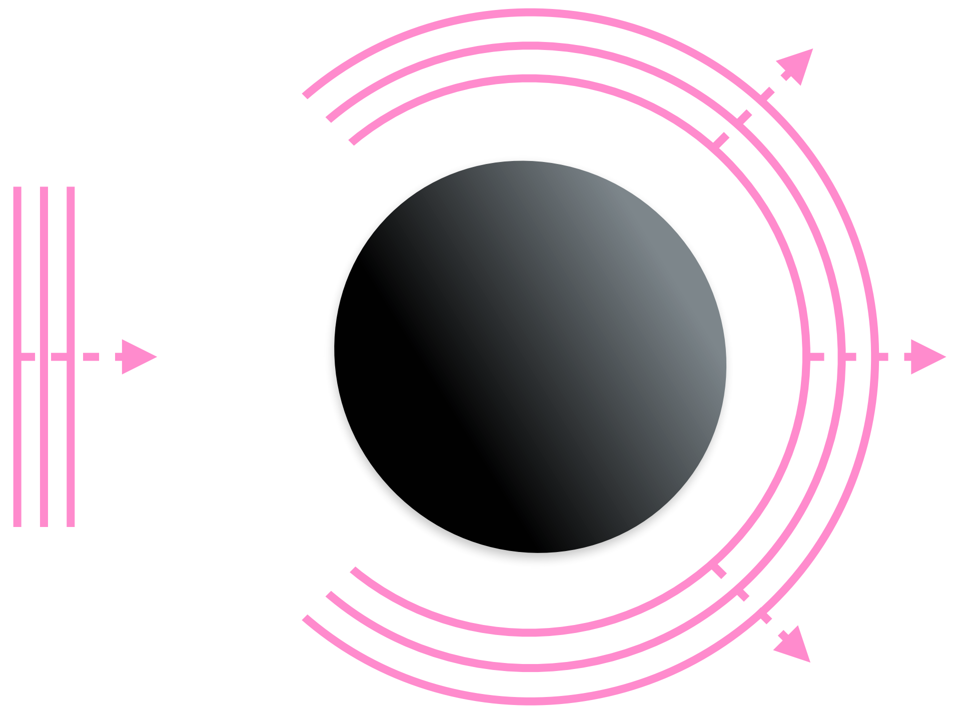

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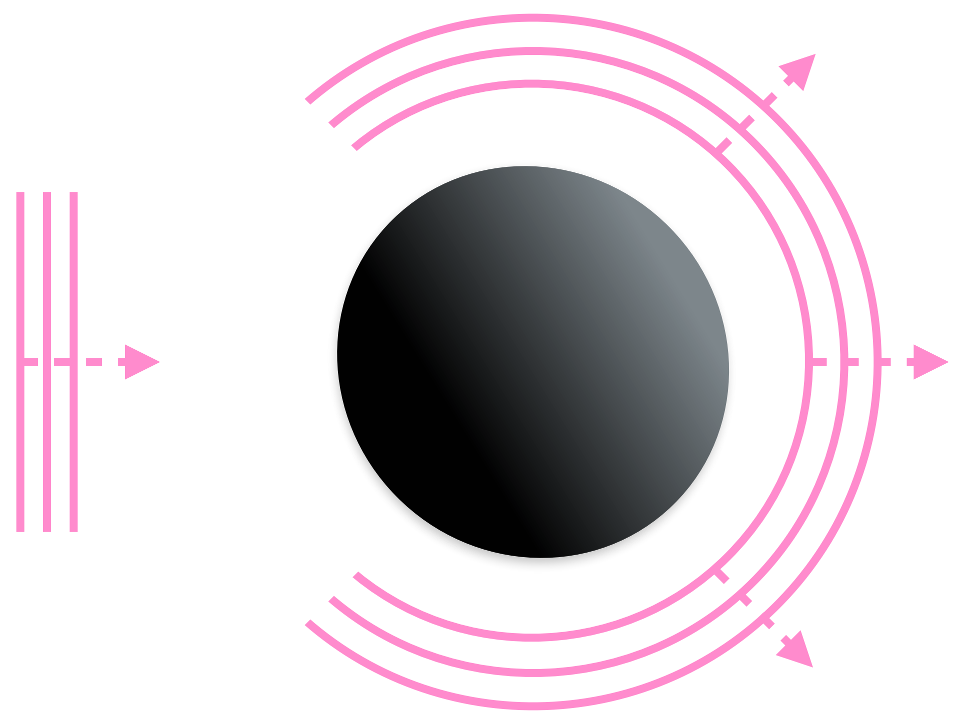
Keldysh effective action for dissipation

Goal: Solve for $\hat{c}_{\ell,n}$ and relate them to point-particle definition

Modeling any compact object in terms of a thin shell

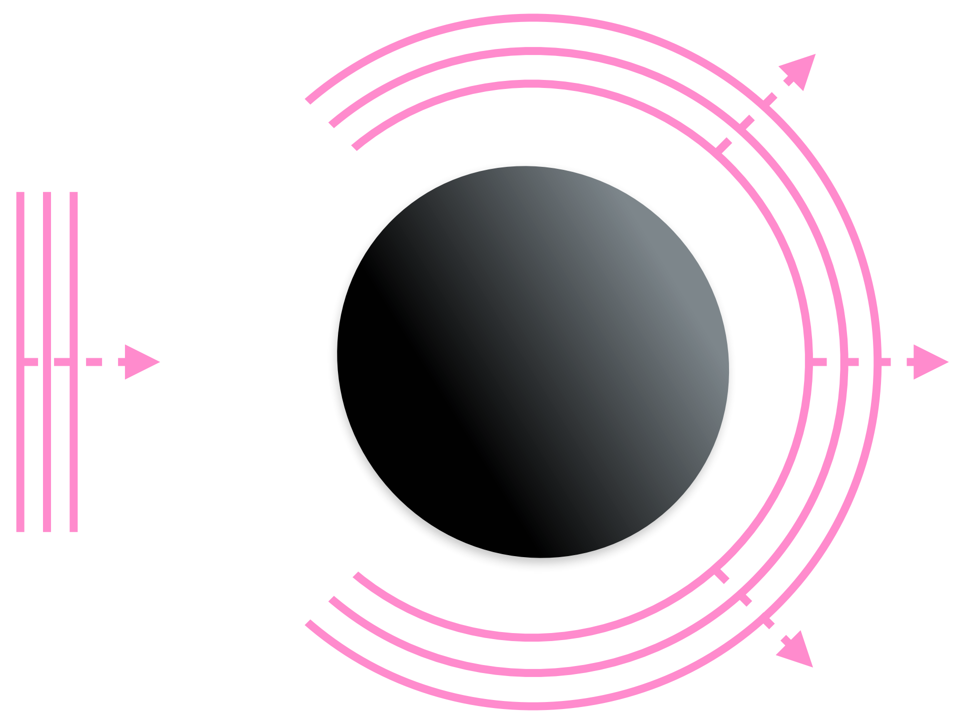
The bulk equations of motion



Equations of motion outside the shell: $\Box\phi^+ = 0$ $\qquad \phi = e^{-i\omega t}\sum_\ell \psi_\ell^+(r)P_\ell(\cos\theta)$

Leaver/MST solutions reproduce the results of loop integration

The bulk equations of motion



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[Leaver (1986), Mano, Suzuki, Takasugi (MST) (1996)]

Coulomb wavefunctions
(They carry the r dependence)

Leaver/MST solutions reproduce the results of loop integration

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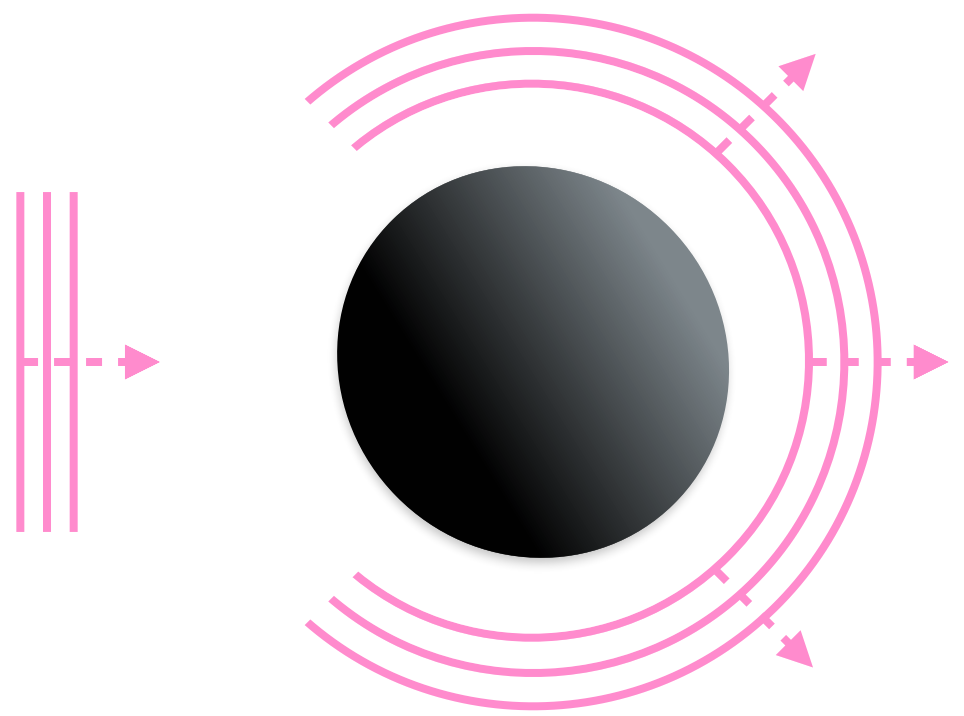

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Renormalized angular momentum: $\nu_\ell - \ell \sim \left(R_S \omega\right)^2$

$b_n^\ell \sim \left(R_S \omega\right)^{|n|-1}$

Found recursively **without integration**

Leaver/MST solutions reproduce the results of loop integration

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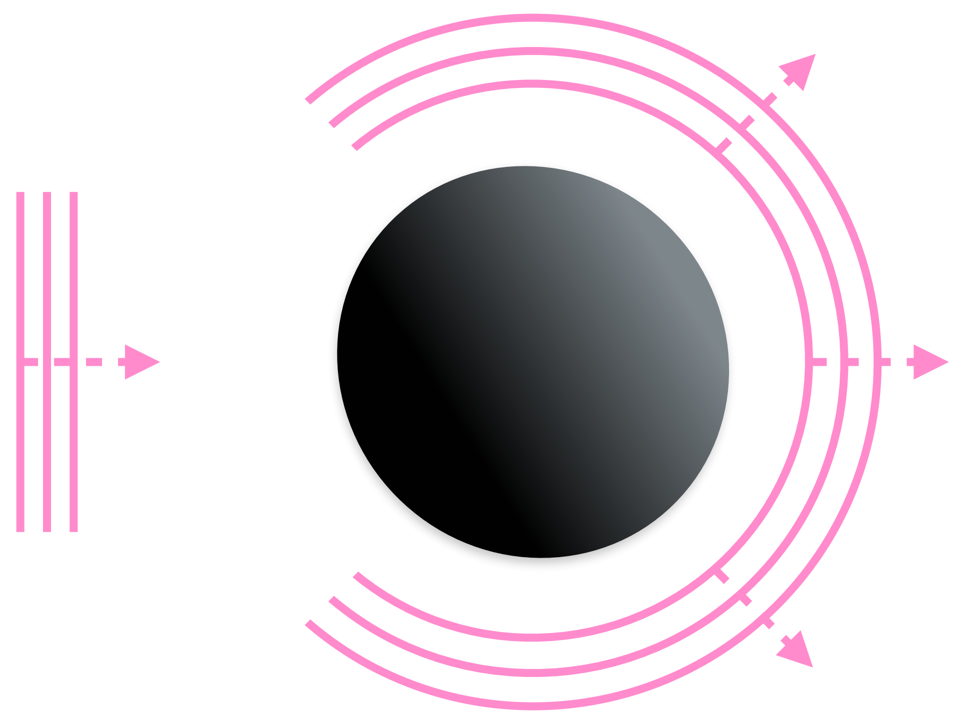

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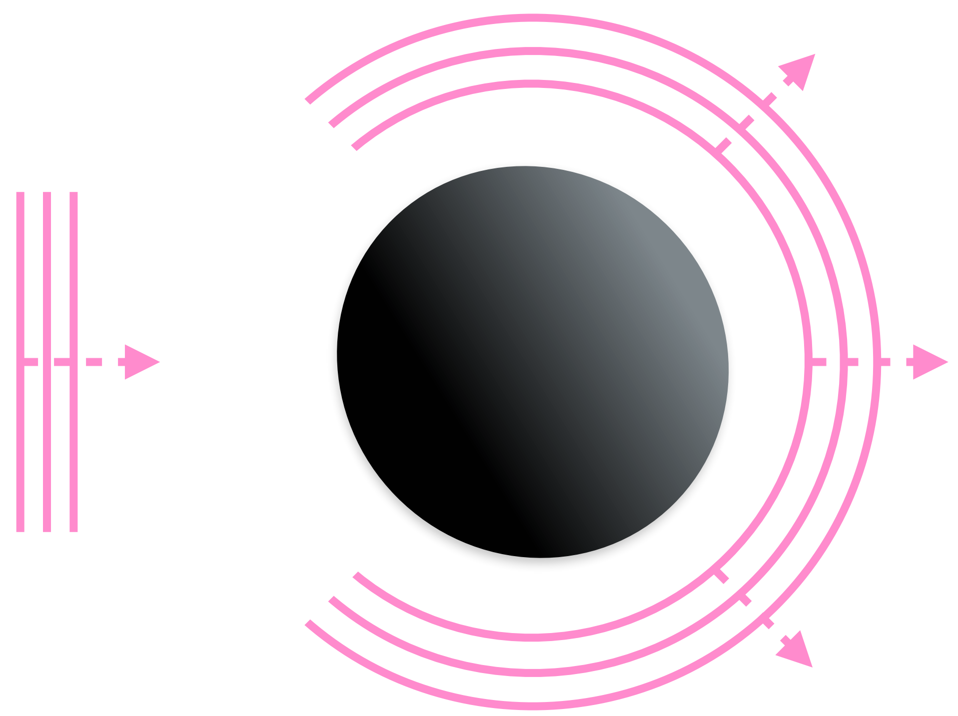
$\left(A_\ell, B_\ell\right)$: Characterize the compact object

Leaver/MST solutions reproduce the results of loop integration

The bulk equations of motion



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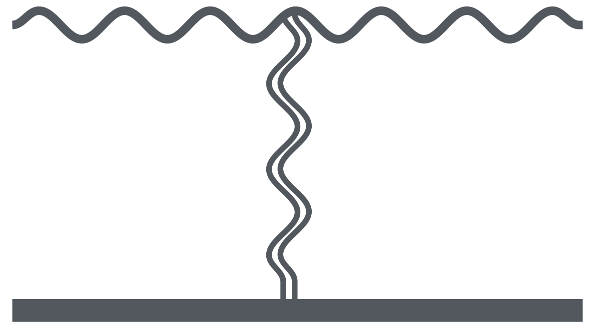
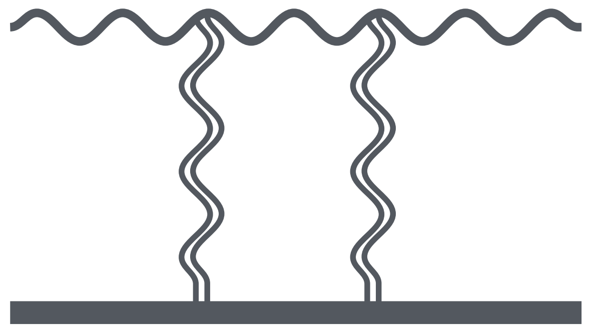
Coulomb wavefunctions
(They carry the r dependence)

$$\psi_\ell^+(r) \sim j_\ell(\omega r) + \mathcal{O}\left(R_S\,\omega\right) + \mathcal{O}\left((R_S\,\omega)^2\right) + \dots$$

Free propagation



Tree-level interaction
with matter



One-loop-level interaction
with matter

Leaver/MST solutions reproduce the results of loop integration

Perturbations around the shell

Equations of motion: $\Box \phi = 0\,, \quad r \neq R$ $\qquad g_{\mu\nu} = g^{+}_{\mu\nu}\,\theta(r-R) + g^{-}_{\mu\nu}\,\theta(R-r)$ $\qquad \phi = \phi^{+}\theta(r-R) + \phi^{-}\theta(R-r)$

Solution:

Black-hole exterior $\quad \phi^{+} = e^{-i\omega t}\sum_{\ell}\psi^{+}_{\ell}(r)P_{\ell}(\cos\theta) \qquad \psi^{+}_{\ell}(r) = \dfrac{1}{\omega\left(r-R_S\right)}\left(A_{\ell}\sum_{n=-\infty}^{\infty} b^{\ell}_{n}\,F_{n+\nu_{\ell}} + B_{\ell}\sum_{n=-\infty}^{\infty} b^{\ell}_{n}\,\bar{F}_{n+\nu_{\ell}}\right)$

Flat-space interior $\quad \phi^{-} = e^{-i\tilde{\omega}\tilde{t}}\sum_{\ell}\psi^{-}_{\ell}(r)P_{\ell}(\cos\theta) \qquad \psi^{-}_{\ell}(r) = \tilde{A}_{\ell}\,j_{\ell}(\tilde{\omega}\tilde{r})$

The contribution on the shell from the discontinuity of the bulk solutions

Perturbations around the shell

Equations of motion: $\Box\phi = 0\,,\quad r \neq R$ $\qquad g_{\mu\nu} = g^+_{\mu\nu}\,\theta(r-R) + g^-_{\mu\nu}\,\theta(R-r)$ $\qquad \phi = \phi^+\theta(r-R) + \phi^-\theta(R-r)$

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Flat-space interior $\quad \phi^- = e^{-i\tilde{\omega}\tilde{t}}\sum_\ell \psi^-_\ell(r)P_\ell(\cos\theta) \qquad \psi^-_\ell(r) = \tilde{A}_\ell j_\ell(\tilde{\omega}\tilde{r})$

Continuity of ϕ: Finite scalar charge on the shell $\qquad \tilde{\omega}^2 = \omega^2/f(R) \qquad \tilde{A}_\ell j_\ell(\tilde{\omega}R) = \psi^+_\ell(R) \equiv \psi_\ell(R)$

The contribution on the shell from the discontinuity of the bulk solutions

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Discontinuous derivative: Scalar charge distributed on the shell

$$\Box\phi = \delta(r-R)\,f(R)e^{-i\omega t}\sum_{\ell} P_{\ell}(\cos\theta)\left(\partial_r\psi^{+}_{\ell} - \partial_r\psi^{-}_{\ell}\right)$$

The contribution on the shell from the discontinuity of the bulk solutions

The complete equations of motion

(Shell contribution)

$$\frac{\delta S_{\text{Shell}}}{\delta \varphi} = \delta(r-R) f(R) e^{-i\omega t} \sum_{\ell,n} P_\ell(\cos\theta)\psi_\ell(R) \frac{1}{4\pi\sqrt{f(R)}R^2} \frac{(2\ell+1)!}{\ell!2^\ell f(R)^\ell R^{2\ell}} (-i\omega)^n c_{\ell,n}$$

$$S_{\text{Shell}} = \int \frac{d^3y\sqrt{\gamma}}{4\pi R^2}\left(-\hat{M} + \sum_{\ell,n} \frac{\hat{c}_{\ell,n}}{\ell!}\varphi\,(u\cdot\nabla)^n D_\ell\,\phi\right)$$

The tidal Wilson coefficients in terms of the solutions to the wave equation

The complete equations of motion

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The complete equations of motion

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(Wilson coefficients)

$$\boxed{\sum_n (-i\omega)^n c_{\ell,n} = 4\pi\sqrt{f(R)}R^2 \frac{\ell!2^\ell f(R)^\ell R^{2\ell}}{(2\ell+1)!} \frac{\partial_r \psi_\ell^- - \partial_r \psi_\ell^+}{\psi_\ell}\Big|_{r=R}}$$

(Solutions to the wave equation)

The tidal Wilson coefficients in terms of the solutions to the wave equation

Results

Dynamical Love numbers for Schwarzschild black holes

$$\mathcal{F}_\ell \equiv \sum_n (-i\omega)^n c_{\ell,n} = 4\pi\sqrt{f(R)}R^2 \frac{\ell!2^\ell f(R)^\ell R^{2\ell}}{(2\ell+1)!} \frac{\partial_r \psi_\ell^- - \partial_r \psi_\ell^+}{\psi_\ell}\Big|_{r=R}$$

$$\psi_\ell^+ \leftrightarrow \mathrm{MST}$$

$$\psi_\ell^- = \tilde{A}_\ell j_\ell(\tilde{\omega}\tilde{r})$$

Results for dynamical Love numbers and comparison to dim. reg.

Dynamical Love numbers for Schwarzschild black holes

$$\mathcal{F}_\ell \equiv \sum_n (-i\omega)^n c_{\ell,n} = 4\pi\sqrt{f(R)}R^2 \frac{\ell\,!2^\ell f(R)^\ell R^{2\ell}}{(2\ell+1)\,!} \frac{\partial_r\psi_\ell^- - \partial_r\psi_\ell^+}{\psi_\ell}\Big|_{r=R}$$

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Scheme dependent

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Scheme independent (Match dim. reg.)

Scheme dependent

Scheme-dependent power-law divergencies (Absent in dim. reg.)

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Results for dynamical Love numbers and comparison to dim. reg.

Sample results

The dynamical Love number for a black hole for $\ell = 0$ through $\mathcal{O}\left(G^9\right)$

Power-law divergencies removed

$$
\begin{aligned}
\bar{\mathcal{F}}_0 = {} & 4i\pi\omega R_{\mathrm{S}}^2 - \frac{1}{2}\pi\omega^2 R_{\mathrm{S}}^3\left(8\log\left(\frac{R_{\mathrm{S}}}{R}\right)+11\right) + \frac{1}{6}i\pi\omega^3 R_{\mathrm{S}}^4\left(-88\log\left(\frac{R_{\mathrm{S}}}{R}\right)+8\pi^2-25\right) \\
& - \frac{\pi\omega^4 R_{\mathrm{S}}^5}{8640}\left(69120\zeta_3 - 63360\log^2\left(\frac{R_{\mathrm{S}}}{R}\right) - 221280\log\left(\frac{R_{\mathrm{S}}}{R}\right) + 42240\pi^2 - 7583\right) \\
& - \frac{i\pi\omega^5 R_{\mathrm{S}}^6}{4320}\left(384\pi^4 - 150720\log^2\left(\frac{R_{\mathrm{S}}}{R}\right) + 21120\pi^2\log\left(\frac{R_{\mathrm{S}}}{R}\right) - 165232\log\left(\frac{R_{\mathrm{S}}}{R}\right) + 126720\zeta_3\right. \\
& \qquad \left. + 36880\pi^2 + 71417\right) \\
& - \frac{\pi\omega^6 R_{\mathrm{S}}^7}{21772800}\left(-174182400\zeta_5 + 10644480\pi^4 + 253209600\log^3\left(\frac{R_{\mathrm{S}}}{R}\right) + 1911047040\log^2\left(\frac{R_{\mathrm{S}}}{R}\right)\right. \\
& \qquad - 506419200\pi^2\log\left(\frac{R_{\mathrm{S}}}{R}\right) + 1101382800\log\left(\frac{R_{\mathrm{S}}}{R}\right) - 878169600\zeta_3 \\
& \qquad \left. -638668800\zeta_3\log\left(\frac{R_{\mathrm{S}}}{R}\right) - 277589760\pi^2 - 936497323\right) \\
& + \frac{i\pi\omega^7 R_{\mathrm{S}}^8}{1360800}\left(11520\pi^6 + 39916800\zeta_5 - 97944000\log^3\left(\frac{R_{\mathrm{S}}}{R}\right) + 15825600\pi^2\log^2\left(\frac{R_{\mathrm{S}}}{R}\right)\right. \\
& \qquad - 4336080\pi^4 - 264464760\log^2\left(\frac{R_{\mathrm{S}}}{R}\right) + 443520\pi^4\log\left(\frac{R_{\mathrm{S}}}{R}\right) + 79626960\pi^2\log\left(\frac{R_{\mathrm{S}}}{R}\right) \\
& \qquad - 13305600\pi^2\zeta_3 + 49764960\zeta_3 + 189907200\zeta_3\log\left(\frac{R_{\mathrm{S}}}{R}\right) \\
& \qquad \left. +25605367\log\left(\frac{R_{\mathrm{S}}}{R}\right) + 22945475\pi^2 + 58628869\right) \\
& + \frac{\pi\omega^8 R_{\mathrm{S}}^9}{438939648000}\left(-3511517184000\zeta_7 - 30628233216000\zeta_3\log^2\left(\frac{R_{\mathrm{S}}}{R}\right) - 12875563008000\zeta_5\log\left(\frac{R_{\mathrm{S}}}{R}\right)\right. \\
& \qquad - 136581761433600\zeta_3\log\left(\frac{R_{\mathrm{S}}}{R}\right) + 7898204160000\log^4\left(\frac{R_{\mathrm{S}}}{R}\right) + 107694565171200\log^3\left(\frac{R_{\mathrm{S}}}{R}\right) \\
& \qquad - 31592816640000\pi^2\log^2\left(\frac{R_{\mathrm{S}}}{R}\right) + 164490996579840\log^2\left(\frac{R_{\mathrm{S}}}{R}\right) \\
& \qquad + 1020941107200\pi^4\log\left(\frac{R_{\mathrm{S}}}{R}\right) - 56870501990400\pi^2\log\left(\frac{R_{\mathrm{S}}}{R}\right) + 3712941711360\pi^4 \\
& \qquad - 32042594304000\zeta_5 + 12875563008000\zeta_3^2 + 20418822144000\pi^2\zeta_3 \\
& \qquad - 24046102886400\zeta_3 + 34062336000\pi^6 + 2753089059840\pi^2 \\
& \qquad \left. -123148321363200\log\left(\frac{R_{\mathrm{S}}}{R}\right) + 2165341072007\right) + \mathcal{O}(R_{\mathrm{S}}^{10})
\end{aligned}
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Sample results

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Power-law divergencies removed

Can you spot the structure?

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Hidden simplicity in black-hole Love numbers

$$\frac{(-1)^n}{4\pi R_S^{2\ell+n+1}}\,\lambda_{\ell,n} = \mathbf{L}_\ell\,\zeta_{n-1} + \sum_{\mathbf{a}} \left(\mathbf{L}^{(2)}_{\ell,n}\right)_{a_1,a_2}\zeta_{a_1}\,\zeta_{a_2} + \sum_{\mathbf{a}} \left(\mathbf{L}^{(3)}_{\ell,n}\right)_{a_1,a_2,a_3}\zeta_{a_1}\,\zeta_{a_2}\,\zeta_{a_3} + \ldots$$

$$\lambda_{\ell,n} = c_{\ell,n} - \text{(power-law div.)}$$

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Conjecture for the all-orders structure of black-hole Love numbers

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All $\mathbf{L}$: Rational coefficients

Universality for scheme-independent coefficients:

(e.g., shell or dim. reg.)

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All $\mathbf{L}$: Rational coefficients

Universality for scheme-independent coefficients:

(e.g., shell or dim. reg.)

$$\left(\mathbf{L}^{(N)}_{\ell,n}\right)_{\mathbf{a}} = \mathbf{L}^{(N)}_\ell$$

n counts the order in G

	$\mathbf{L}_\ell$	$\mathbf{L}_\ell^{(2)}$	$\mathbf{L}_\ell^{(3)}$	$\mathbf{L}_\ell^{(4)}$
$l=0$	-2	$22/3$	$-628/27$	$5830/81$
$l=1$	$-1/6$	$19/90$	$-361/2025$	–
$l=2$	$-1/270$	$79/28350$	–	–
$l=3$	$-1/21000$	–	–	–

Table I. Table for the scheme-independent entries of $\mathbf{L}$.

Conjecture for the all-orders structure of black-hole Love numbers

Sample results

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Outlook

Future directions

Adding spin: Graviton Love numbers for a Schwarzschild/Kerr black hole

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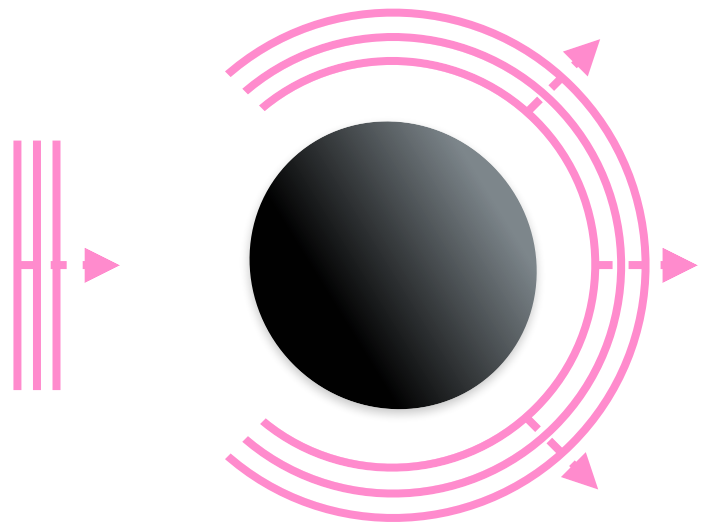
gives toy-graviton **propagator** in Schwarzschild background



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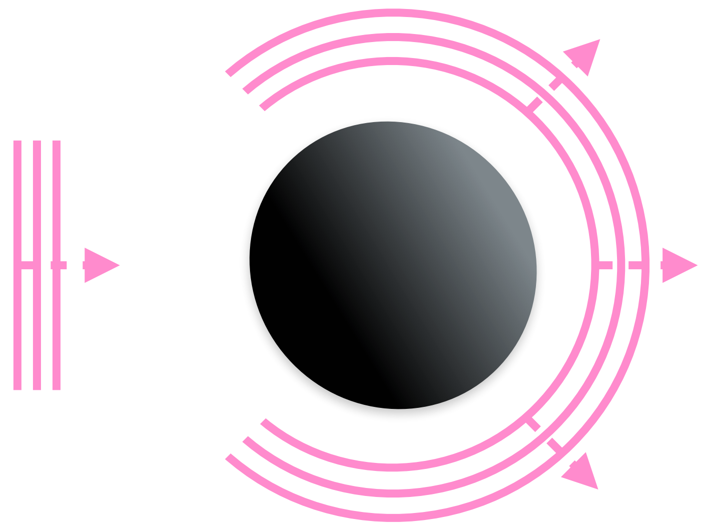

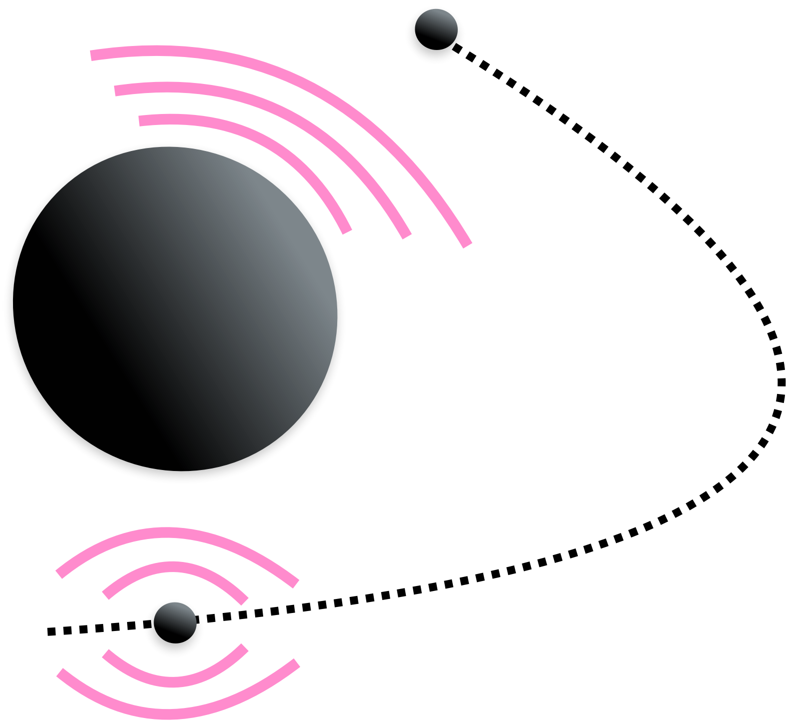

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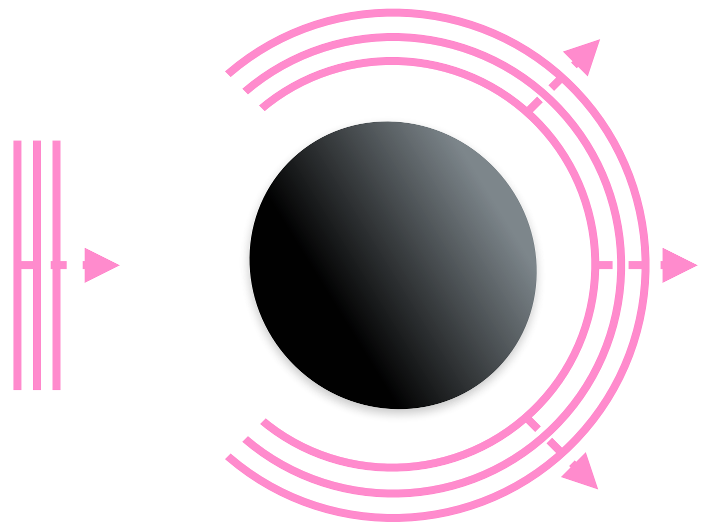

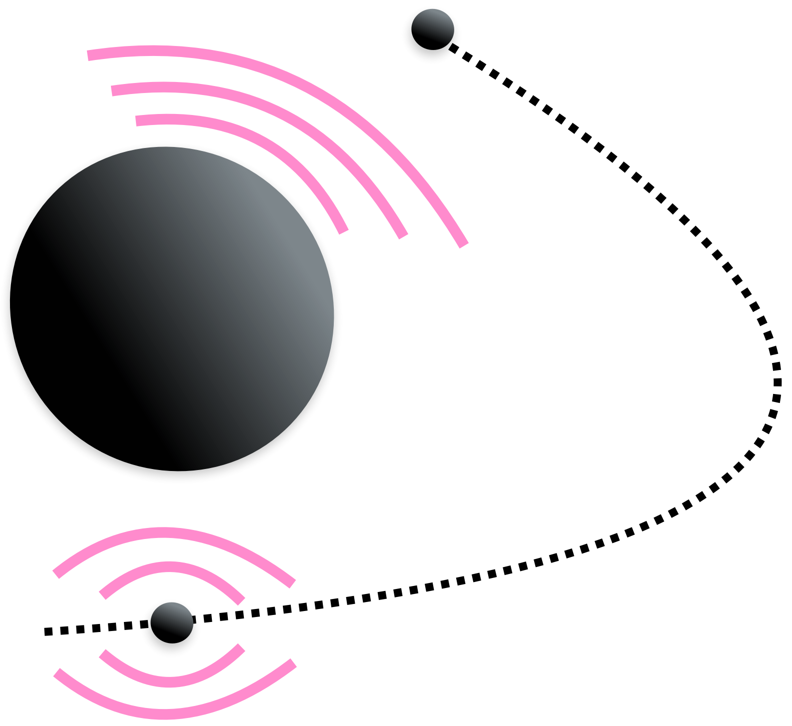

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[Alrashed, **DK**, Solon (in preparation)]

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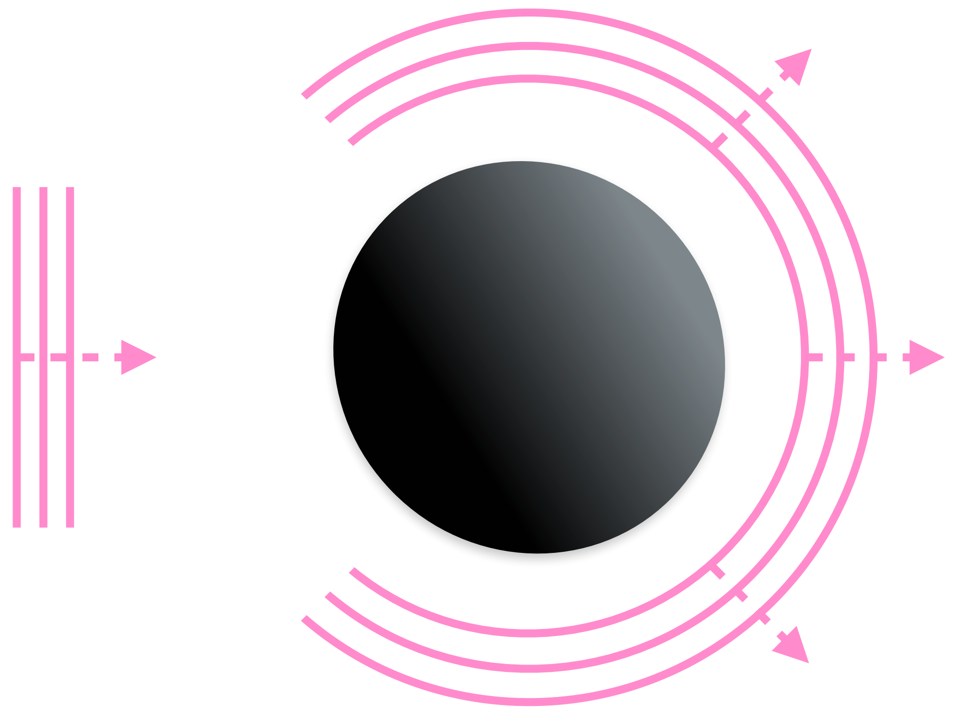

New method (Shell EFT) that employs
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New results for dynamical Love numbers

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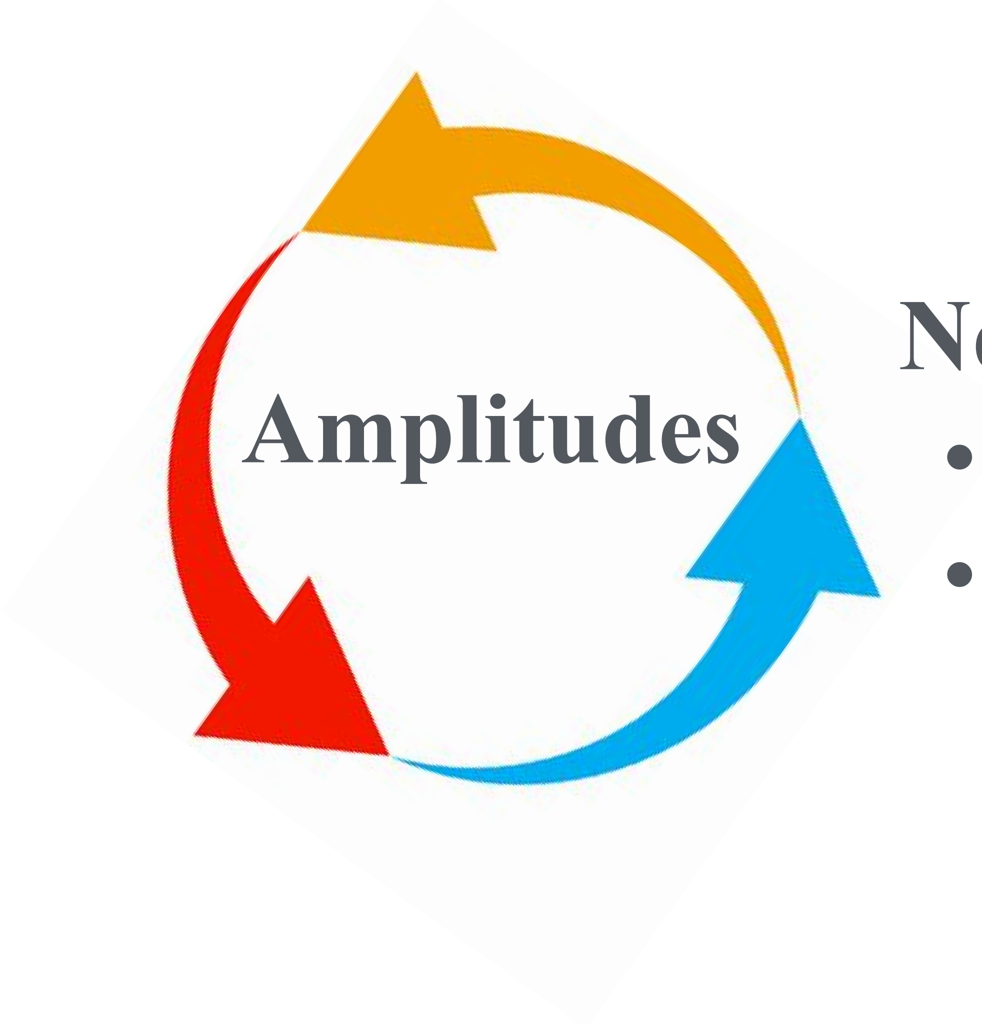

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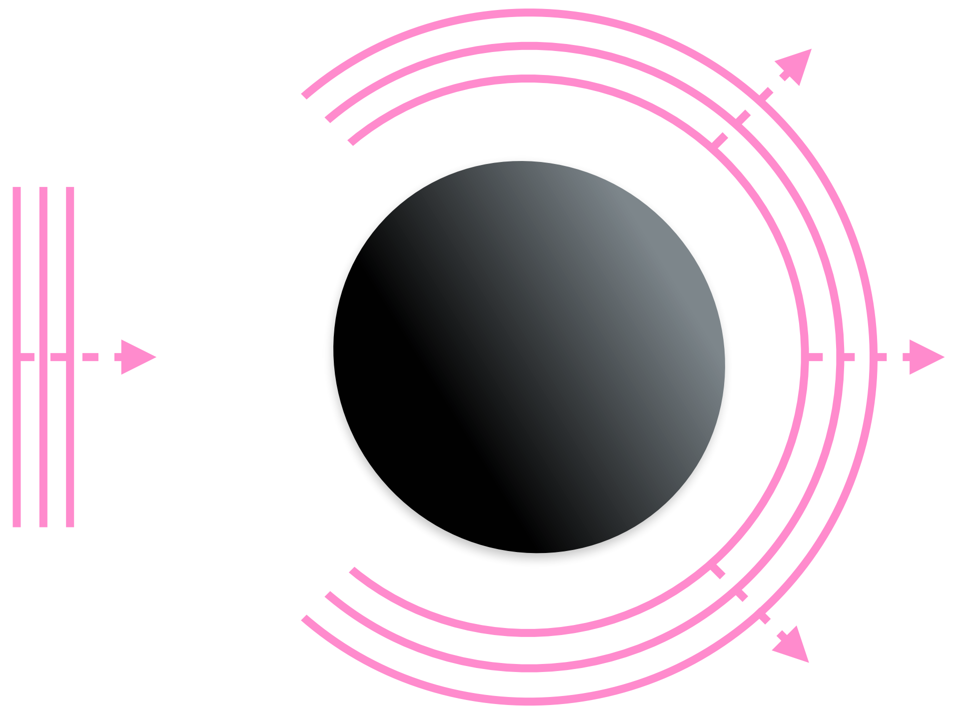

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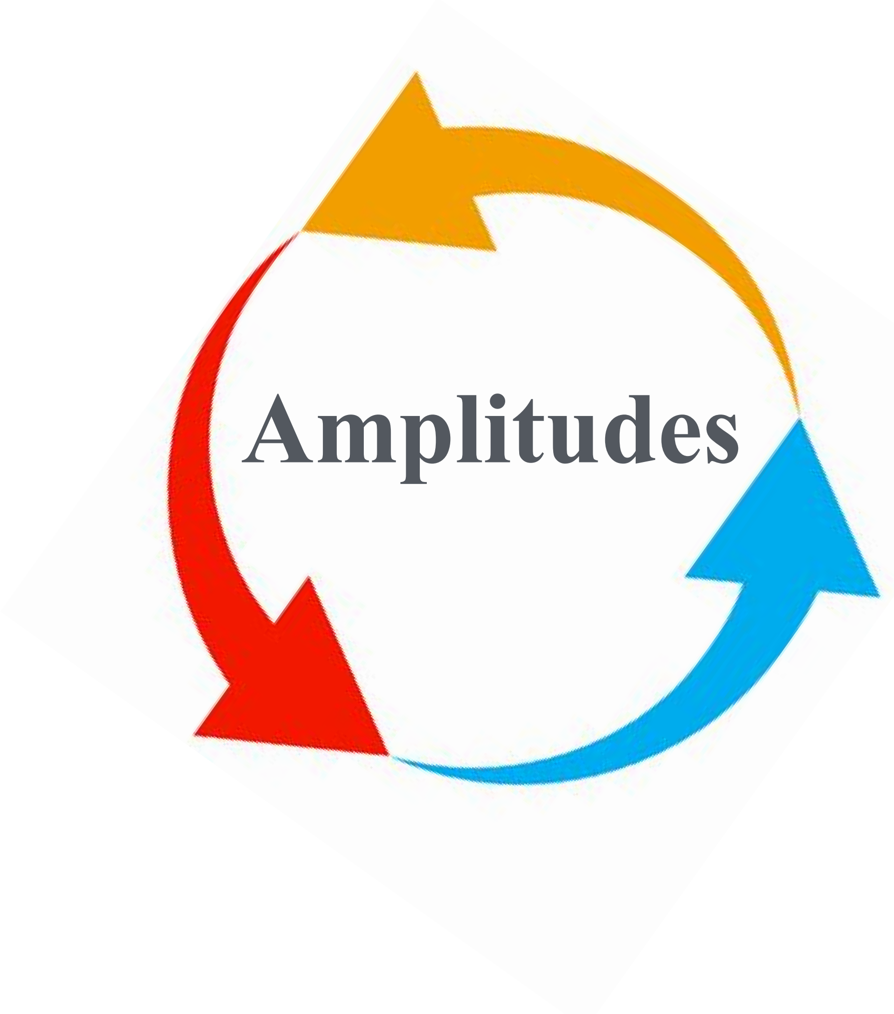

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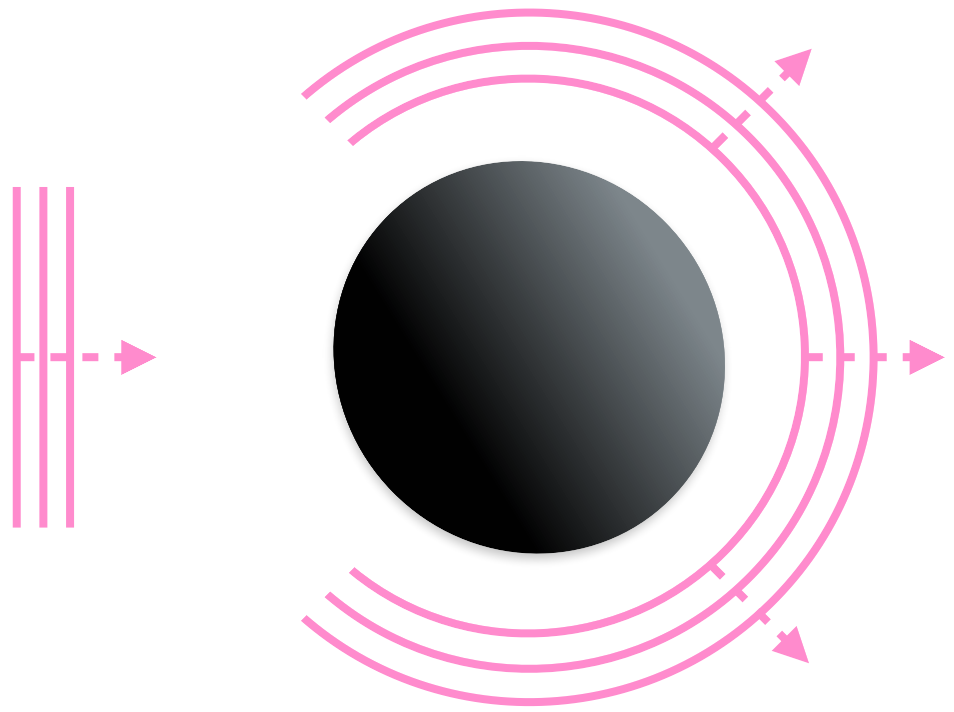

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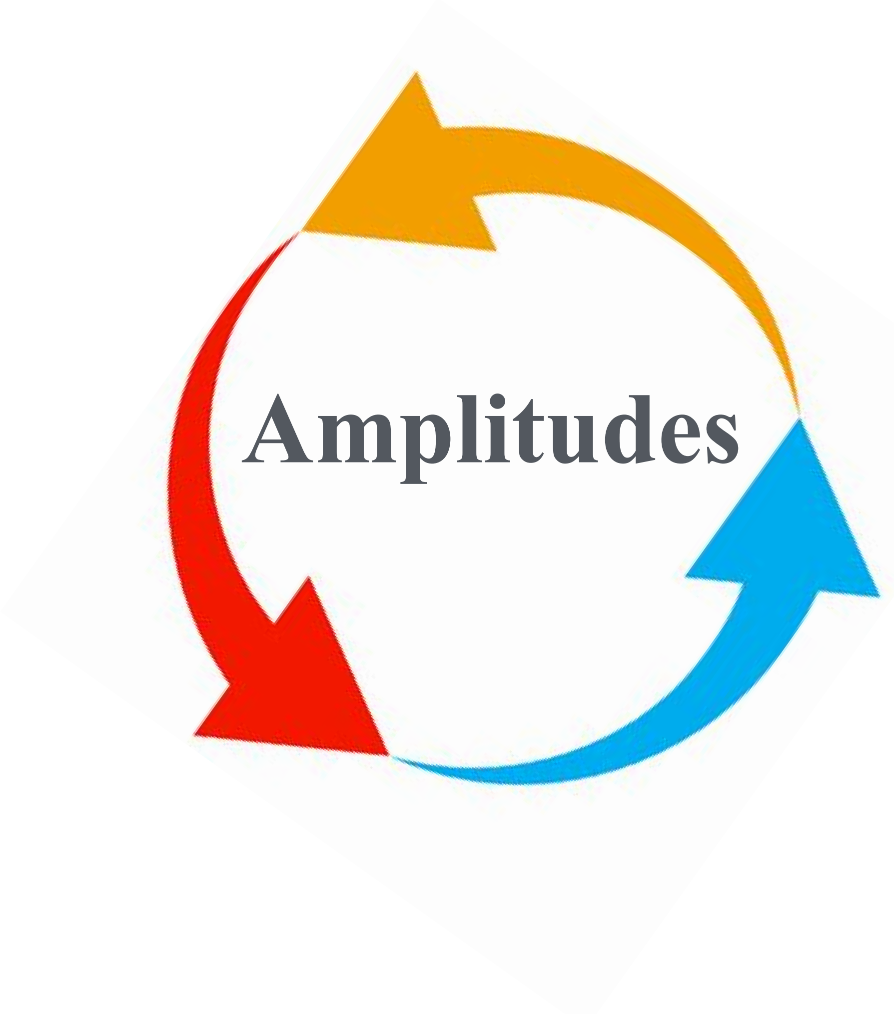

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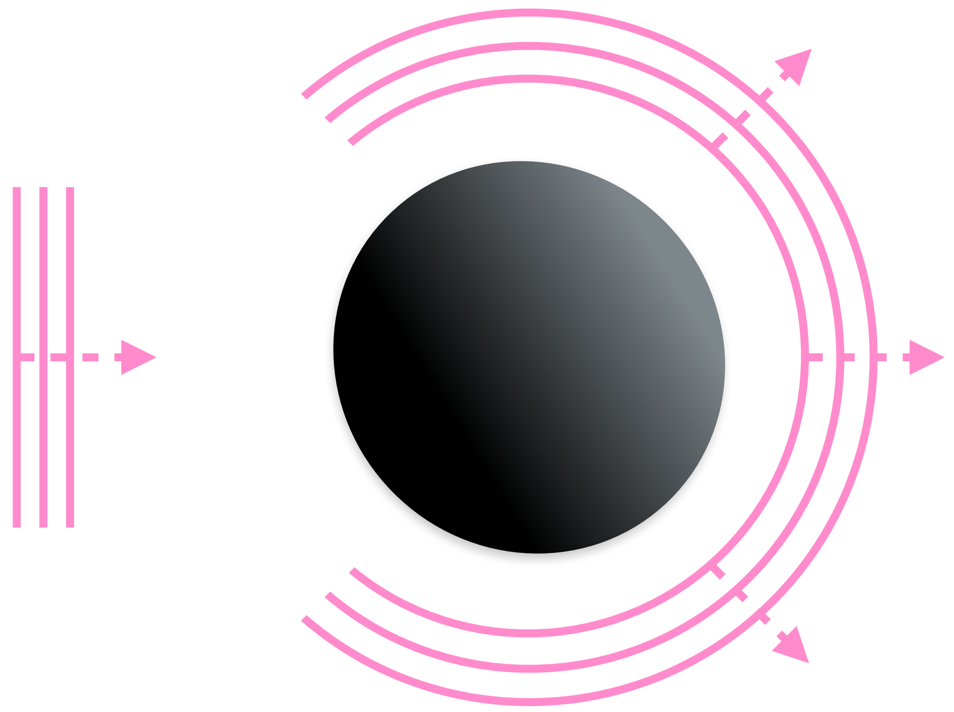
New avenue for analytical calculations of high-order-in-G two-body observables

Current state-of-the-art for the two-body observables: G^5

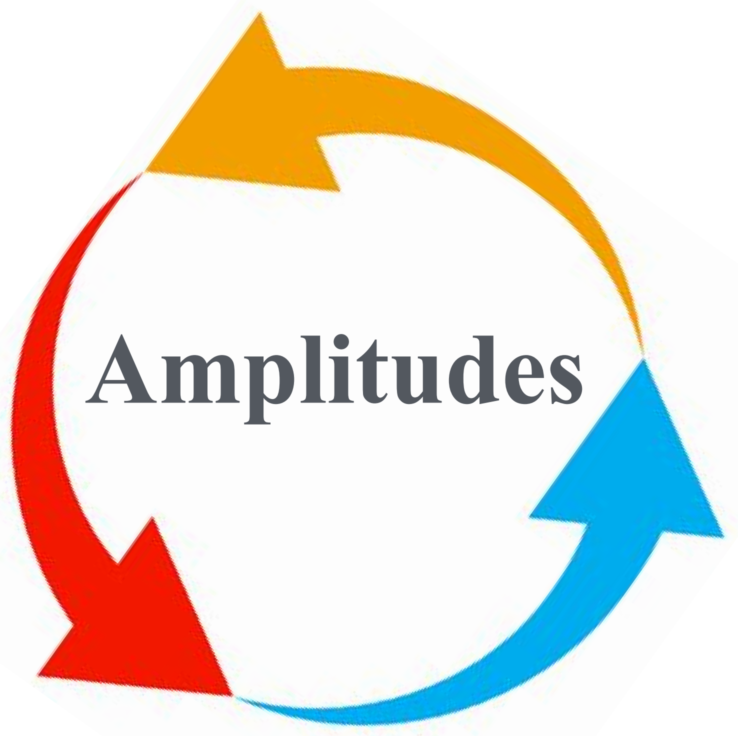
[Driesse, Jakobsen, Mogull, Plefka, Sauer, Usovitsch (2024), +Nega (2026)]
[Bern, Herrmann, Roiban, Ruf, Smirnov, Smith, Zeng (2025)]
[Dlapa, Kälin, Liu, Porto (2025)]

Summary



New method (Shell EFT) that employs
the **all-orders-in-G solutions** to Black Hole Perturbation Theory

New results for dynamical Love numbers
- New results to $\mathcal{O}\left(G^9\right)$
- No integration needed



New structure for black-hole Love numbers
- Conjecture to all orders in G
- Resummation of the perturbative series

New avenue for analytical calculations of high-order-in-G two-body observables

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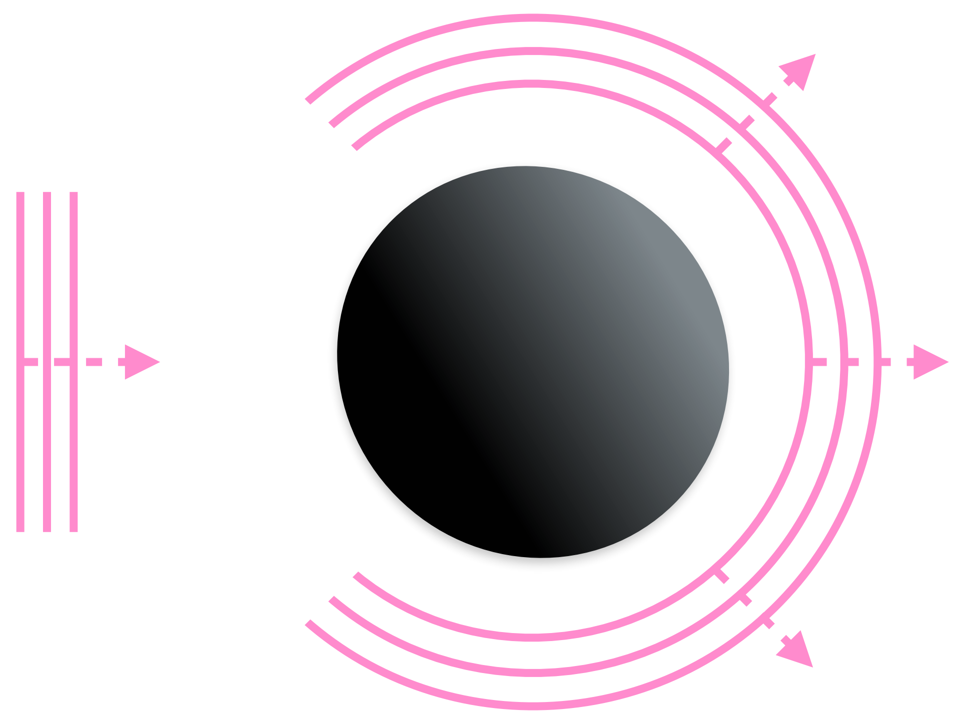
Target for next-gen. observatories: G^7

[Buonanno, Khalil, O'Connell, Roiban, Solon, Zeng (2022)]

Thank you

Backup Slides

In-in formalism for dissipative dynamics



[Schwinger (1961), Keldysh (1964)]

Doubling of the fields: $\phi \rightarrow \phi$ & φ

Classical field — Fluctuation

In-in effective action (allows for dissipation): $S_{\text{in-in}} = \int d^4x\sqrt{-g}\,\varphi\,\text{EoM}[\phi]$

Conservative example: $S_{\text{Bulk}} = \frac{1}{2}\int d^4x\sqrt{-g}\,g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \rightarrow -\int d^4x\sqrt{-g}\,\varphi\,\Box\,\phi$

Dissipative example: $\int d\tau\,\varphi\,\partial_\tau\phi$ or $\int d^3y\sqrt{\gamma}\,\varphi\,(u\cdot\nabla)\phi$

Intuition about the shell

(Shell) $$\int_{\text{Shell}} \varphi\, D_\ell \phi \quad \sim \quad \int_{WL} \nabla_{\langle \ell \rangle} \varphi \, \nabla_{\langle \ell \rangle} \phi$$ (Worldline)

Intuition from flat space: $\varphi(t,0)\phi(t,0) = \lim_{R\to 0} \int \frac{d\hat{n}}{4\pi} \varphi\left(t, R\hat{n}\right) \phi\left(t, R\hat{n}\right)$

$$\lim_{R\to 0} \int \frac{d\hat{n}}{4\pi} \varphi\left(t, R\hat{n}\right) D_1 \phi\left(t, R\hat{n}\right) = -\lim_{R\to 0} \frac{3}{2R^2} \int \frac{d\hat{n}}{4\pi} \varphi\left(t, R\hat{n}\right) \nabla^2_\Omega \phi\left(t, R\hat{n}\right)$$

$$= -\lim_{R\to 0} \frac{3}{2R^2} \int \frac{d\hat{n}}{4\pi} \left(\varphi|_0 + R\hat{n}\cdot\nabla\varphi|_0 + \mathcal{O}\left(R^2\right)\right) \nabla^2_\Omega \left(\phi|_0 + R\hat{n}\cdot\nabla\phi|_0 + \mathcal{O}\left(R^2\right)\right)$$

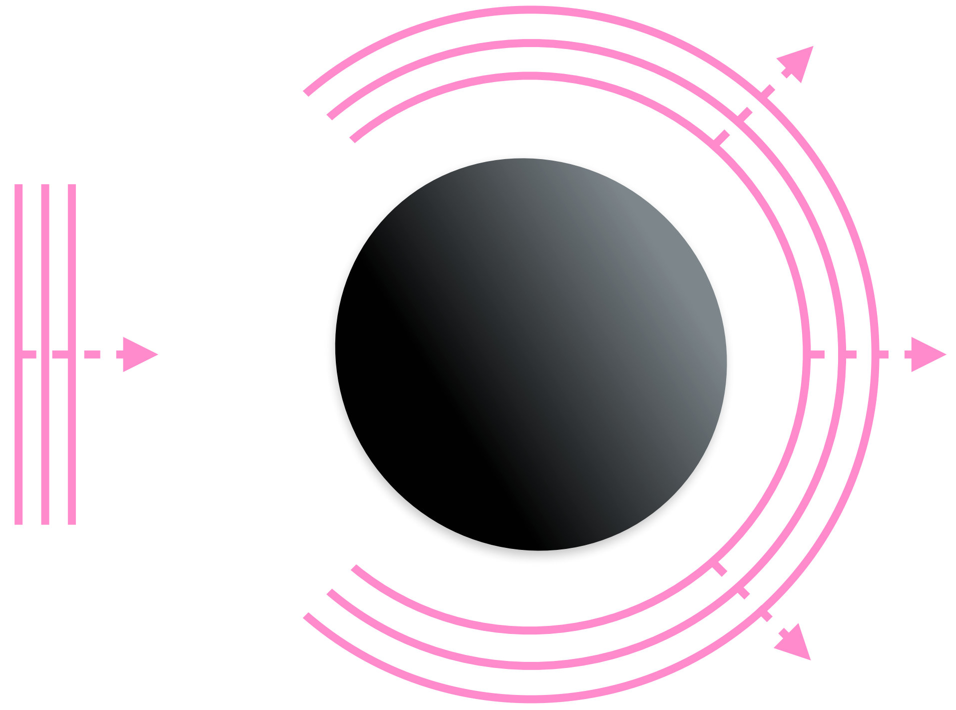
$$= \lim_{R\to 0} \frac{3}{R} \int \frac{d\hat{n}}{4\pi} \left(\varphi|_0 + R\hat{n}\cdot\nabla\varphi|_0\right) \hat{n}\cdot\nabla\phi|_0 \qquad = \left(\nabla\varphi\cdot\nabla\phi\right)|_0$$

$$D_\ell = \frac{(-1)^\ell (2\ell+1)}{2^\ell f(R)^\ell R^{2\ell}} \prod_{j=0}^{\ell-1} \left(\nabla^2_\Omega + j(j+1)\right)$$

$$\nabla^2_\Omega \hat{n} = -2\,\hat{n}$$

From the shell to the worldline

The source of the background metric



$$S_{\text{Shell}} = \int \frac{d^3y\sqrt{\gamma}}{4\pi R^2}\left(-\hat{M} + \sum_{\ell,n} \frac{\hat{c}_{\ell,n}}{\ell\,!}\varphi\,(u\cdot\nabla)^n D_\ell\,\phi\right) \qquad \hat{M} = \alpha\,R^{(3)} + \beta$$

Ricci scalar of induced metric γ : $R^{(3)} = -\dfrac{2}{R^2}$

$$\alpha = \frac{R^3\left(R_S\left(2-3\sqrt{f(R)}\right)+2R\left(-1+\sqrt{f(R)}\right)\right)}{8G(R-R_S)} = -\frac{MR^2}{2}\left(1+\mathcal{O}\left(R_S/R\right)\right)$$

$$\beta = \frac{R_S+2R\left(-1+\sqrt{f(R)}\right)}{4G\sqrt{f(R)}} = -\frac{M}{8}\frac{R_S}{R}\left(1+\mathcal{O}\left(R_S/R\right)\right)$$

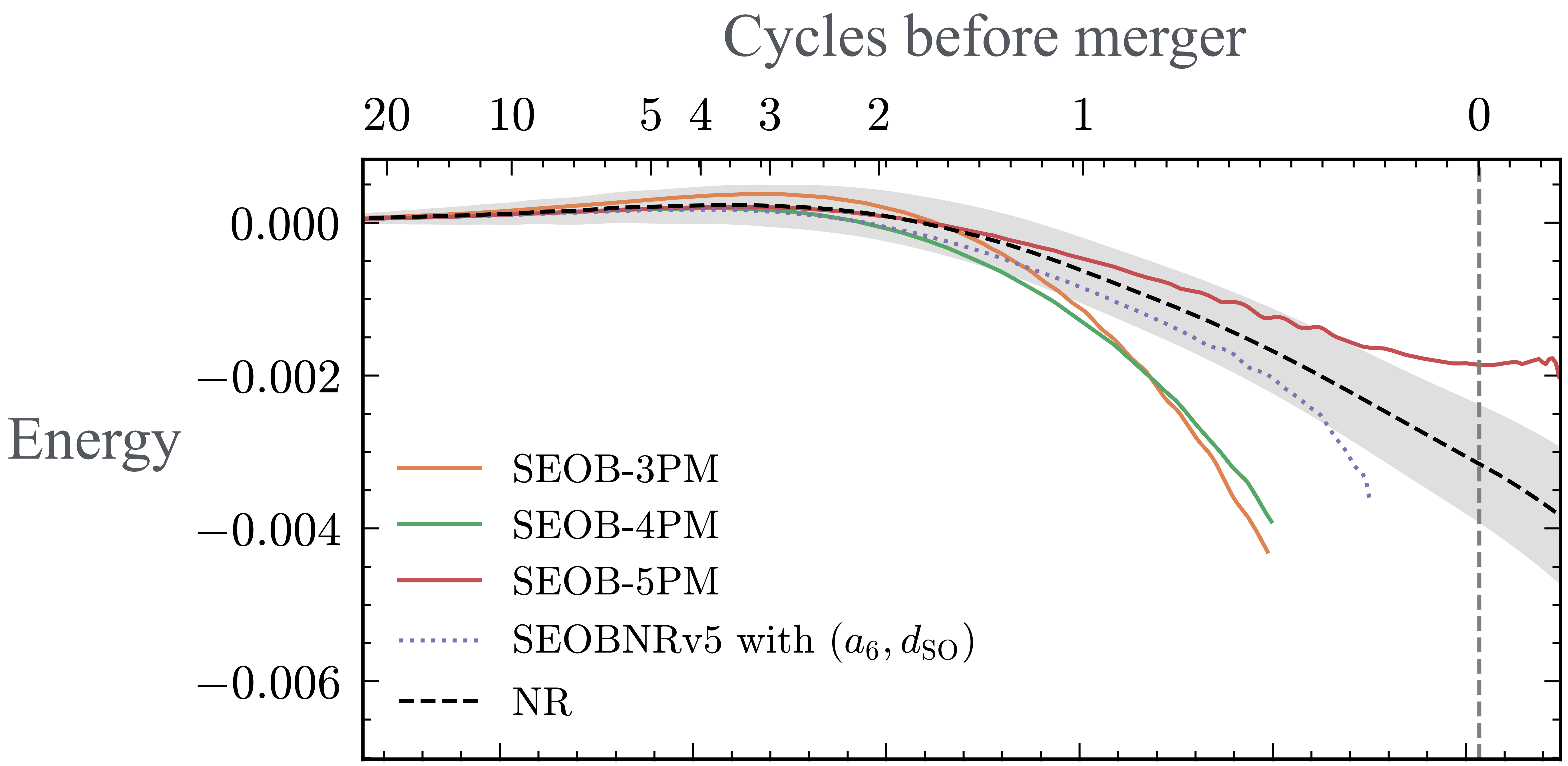
$$f(R) = 1-\frac{R_S}{R}$$

$$\hat{M} = M + \mathcal{O}\left(R_S/R\right)$$

The junction conditions yield the source of the background metric

Theoretical gravitational-wave analysis

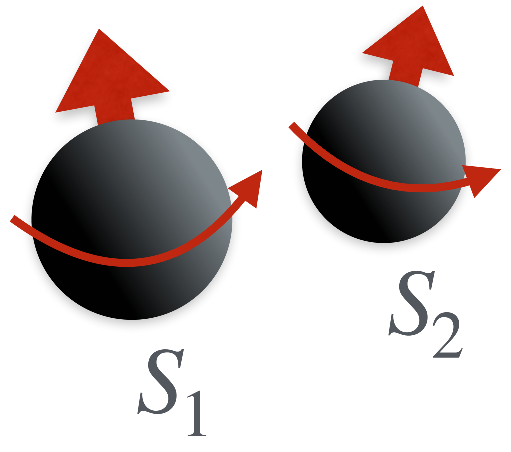
Comparison: EOB & NR



[Buonanno, Mogull, Patil, Pompili (2024)]

nPM = $\mathcal{O}\left(G^a S^b\right)$, with $a + b = n$

Physical Post-Minkowiskian counting



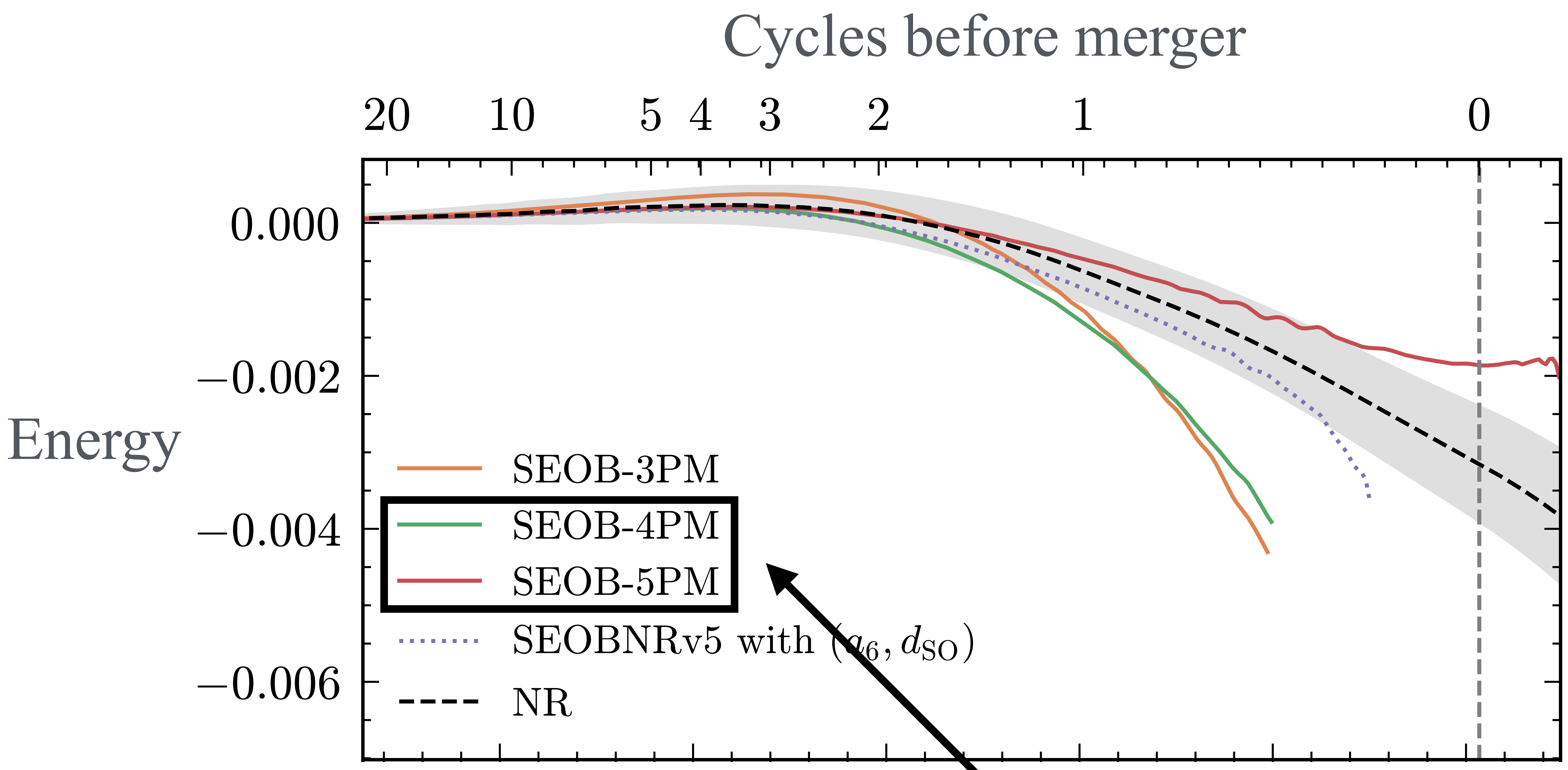
EOB: Effective One Body

NR: Numerical Relativity

Theorists from LIGO-Virgo-KAGRA are interested in the results from Scattering Amplitudes

Theoretical gravitational-wave analysis

Comparison: EOB & NR



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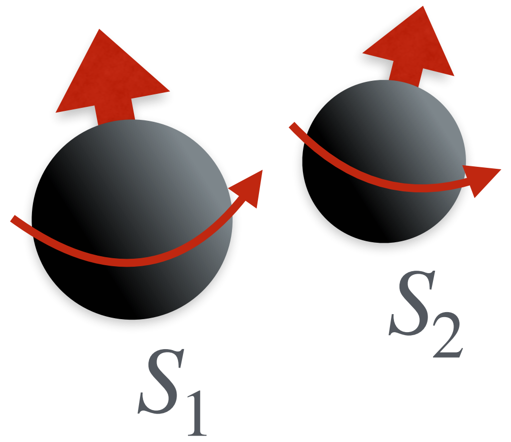

EOB: Effective One Body

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Imports [**DK**, Luna (2021)]

Theorists from LIGO-Virgo-KAGRA are interested in the results from Scattering Amplitudes