INFN BOLOGNA
Istituto Nazionale di Fisica Nucleare
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Theory and Phenomenology
of Fundamental Interactions
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# NNLO QCD corrections to ttW production

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Based on JHEP 07 (2025) 001, 2606.09503 [hep-ph] and WIP
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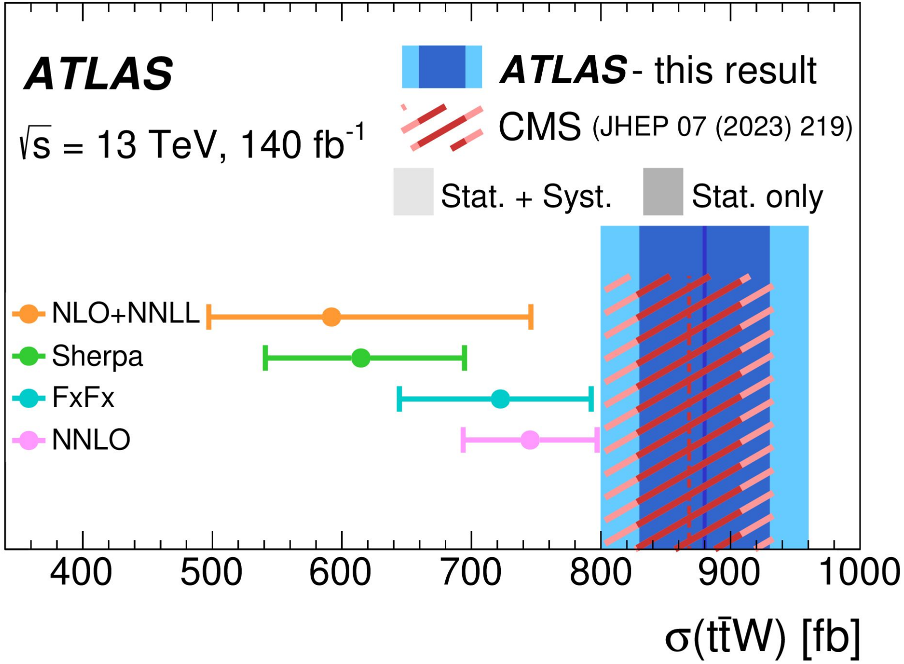
# Why ttW production @ NNLO QCD

$$\bar{u}(p_1) + d(p_2) \rightarrow t(p_3) + \bar{t}(p_4) + W^-(p_5)$$

- **Background to key SM process**: $t\bar{t}H$ and $t\bar{t}t\bar{t}$ production
- Relevant for many **searches** of Physics **beyond the SM**
- **Better description** of experimental **data**



**Theory predictions underestimate measurements**

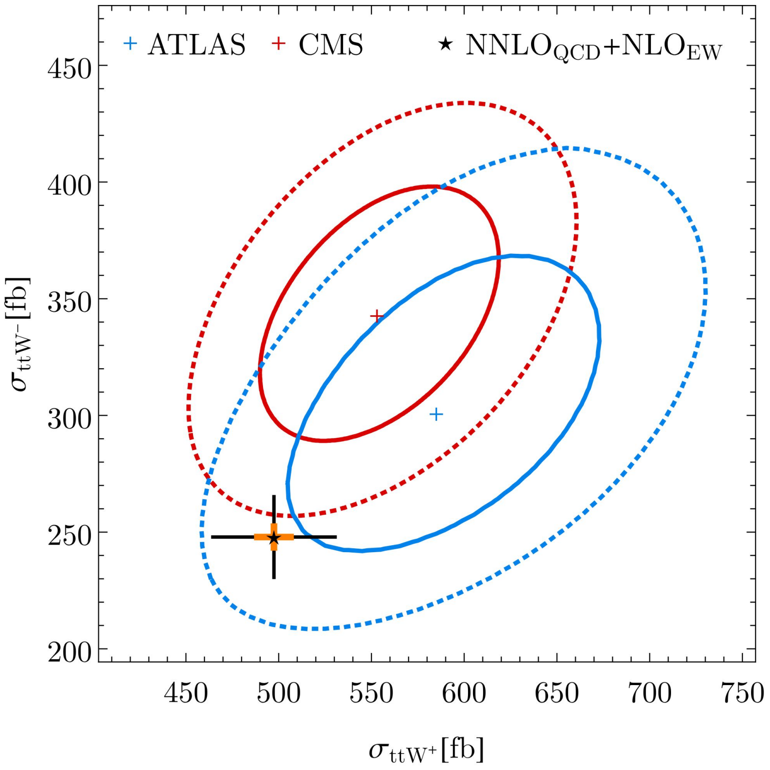
# Approximated NNLO QCD predictions for ttW

- **NNLO QCD** results obtained using kinematic approximations for the two-loop scattering amplitude [Buonocore et al, ’23]:

  1) **Soft-Boson approximation** $m_W \ll m_t$ [Catani et al, ’23].

  2) **Massification** [Mitov & Moch, ’07] of $b\bar{b}W$ amplitudes [Badger et al, ’21].

**Exact NNLO QCD necessary for validation & improvement of approximations + obtain differential distributions**



# Challenges for exact NNLO QCD

$$d\hat{\sigma}^{\text{NNLO}}_{\text{QCD}} \sim \int d\Phi^{(n)} |\mathcal{M}^{(0)}_{n+2}|^2 + \int d\Phi^{(n-1)} \mathcal{M}^{(1)}_{n+1} \mathcal{M}^{(0)*}_{n+1} + \int d\Phi^{(n-2)} (|\mathcal{M}^{(1)}_n|^2 + \boxed{\mathcal{M}^{(2)}_n} \mathcal{M}^{(0)*}_n)$$

- **Main Bottleneck:** computation of the **two-loop scattering amplitude**!

$$\mathcal{M}^{(2)}_n = \sum \text{Feynman Graphs}$$

- Why? → 7 kinematic variables + massive propagators in the loops

**Substantial Analytic Complexity**
(Elliptic functions + Nested square roots)

**Enormous Algebraic Complexity**
(High-degree polynomials)

This talk: **Computation of two-loop amplitude in the leading color approximation** (LC)

# Leading color approximation

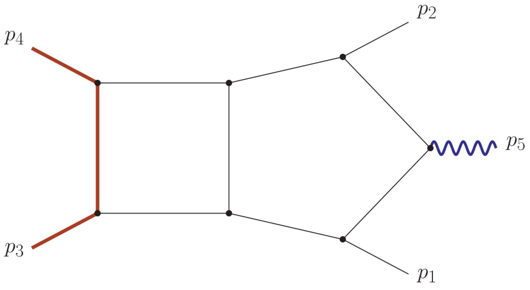
- The **amplitude** is decomposed into **color factors** × **kinematic dependent parts**

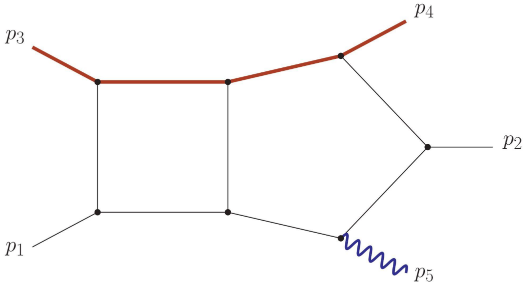
$$\mathcal{M}_n^{(2)} = N_C^2 \mathcal{A}^{(2)}_{\{2,0\}} + n_f^2 \mathcal{A}^{(2)}_{\{0,2\}} + N_C n_f \mathcal{A}^{(2)}_{\{1,1\}} + N_C \mathcal{A}^{(2)}_{\{1,0\}} + n_f \mathcal{A}^{(2)}_{\{0,1\}} + \dots$$

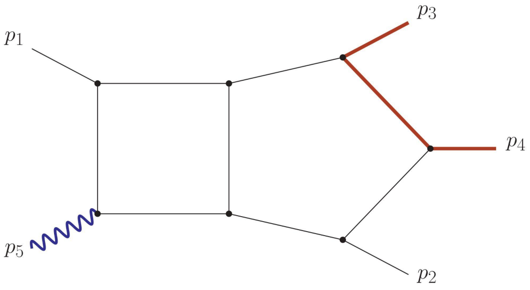
- **LC@2Loop**: Keep only terms proportional to $\{N_c^2, N_c n_f, n_f^2\}$ → great **simplifications**:

1) **Fewer Feynman graphs** (from 722 we get 210)!

2) Only **planar** graphs with at most **2 massive propagators**!







# Projection into scalar form factors

- Decompose $\mathcal{A}^{(2)}_{\{x,y\}}$ in terms of linearly independent **tensor structures** × **form factors** [See Bayu's talk]

$$\mathcal{A}^{(2)}_{\{x,y\}} = \sum_{i=1}^{N} T_i F^{(i)}_{\{x,y\}}$$

- tHV scheme ⟶ **Physical projectors** [Peraro & Tancredi, '19 & '21] (# tensors = # helicity amplitudes): ***N=24***

$$F^{(i)}_{\{x,y\}} = \sum_{pol} \mathcal{P}_i \mathcal{A}^{(2)}_{\{x,y\}} \quad \text{with} \quad \mathcal{P}_i = \sum_{i=1}^{N=24} M^{-1}_{ij} T_i \quad \text{and} \quad M_{ij} = \sum_{pol} T_i^{\dagger} T_j$$

- Form factors are a sum of **8959 Feynman integrals** (FIs) [See Aris & Costas talks for alternative approach]

$$F = \sum_t C_t I_t \quad \text{with} \quad I_t = \int \mathrm{d}^d k_1 \mathrm{d}^d k_2 \prod_{i \in t} \frac{1}{D_i^{a_i}} \quad \text{and} \quad D_i = (k_i + p_i)^2 - m_i^2$$

# Reduction to master integrals

- FIs satisfy **integration-by-part relations** (IBPs) [Chetyrkin & Tkachov, ’81; Laporta ’04]

$$\int \mathrm{d}^d k_1 \mathrm{d}^d k_2 \frac{\partial}{\partial k_l^{\mu}} \prod_{i \in t} \frac{1}{D_i^{a_i}} = 0 \longrightarrow \boxed{\{\mathcal{J}_i\}}$$

Set of independent **Master integrals** (MIs)

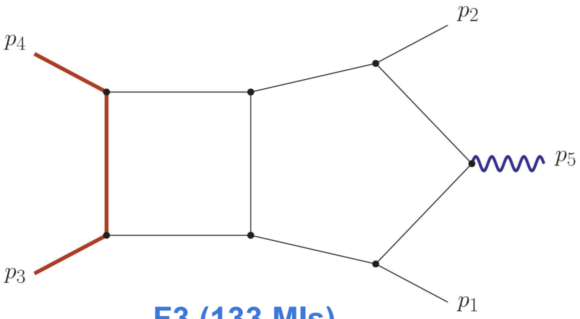
- Use **NeatIBP** [Wu et al, ’21] to generate a compact IBP system.
- Employ **FiniteFlow** [Peraro, ’19] to solve it using **finite-field arithmetic** [von Manteuffel et al, ’14; Peraro, ’16].
- Thus each form factor is expressed as a sum of **330 rational coefficients × MIs**

$$F = \sum_{i=1}^{330} \tilde{C}_i(\vec{x}, \epsilon) \mathcal{J}_i(\vec{x}, \epsilon)$$

where $\vec{x} = \{s_{13}, s_{34}, s_{24}, s_{25}, s_{15}, m_t^2, m_w^2\}$ and $s_{ij} = (p_i + p_j)^2$.

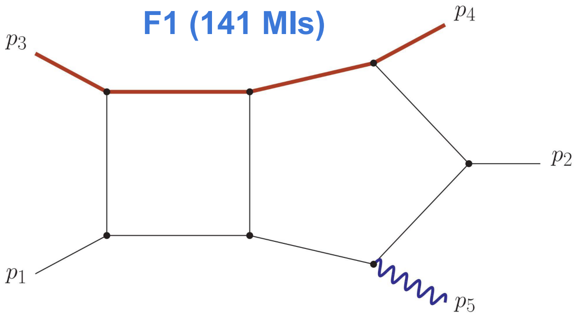
# Integral families

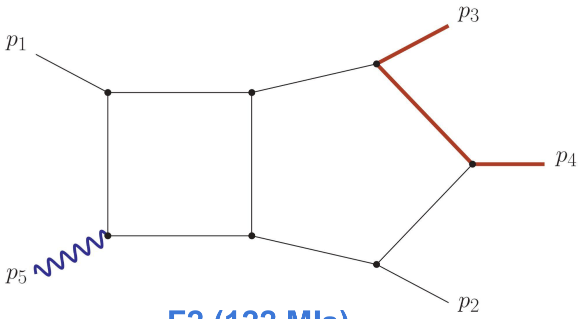
- Group into **integral families** according to loop and kinematic structure



**F3 (133 MIs)**

**F1 (141 MIs)**





**F2 (122 MIs)**

- Each family consists of **11 propagators**, including **3** associated with irreducible scalar products (**ISPs**).
- **All MIs in the amplitude** = MIs of these families + permutations + 1L$^2$ integrals.

# Differential equations for MIs

- MIs fulfill **differential equations** (DEs) [Barucchi & Ponzano, ’73; Kotikov, ’91; Bern et al, ’94; Gehrmann et al, ’00]

$$\mathrm{d}\vec{\mathcal{J}}_F(\vec{x},\epsilon) = \mathrm{d}A^{(F)}(\vec{x},\epsilon)\cdot\vec{\mathcal{J}}_F(\vec{x},\epsilon)$$

- Basis not unique: Good choice makes DEs easier → **canonical DEs** (CDE) [Henn, ’14]

$$\boxed{\mathrm{d}\vec{\mathcal{J}}_F = \epsilon\,\mathrm{d}\tilde{A}(\vec{x})\cdot\vec{\mathcal{J}}_F}$$

- MIs are Laurent-expanded around $\varepsilon$ and computed up to a desired order

$$\vec{\mathcal{J}}_F(\vec{x},\epsilon) = \sum_{i=0}^{4}\epsilon^{i-4}\vec{\mathcal{J}}_F^{(i)}(\vec{x})$$

# Poly-logarithmic case

- Poly-logarithmic functional space → **canonical dlog-form**

$$\mathrm{d}\tilde{A}(\vec{s}) = \sum a_i \, \mathrm{d}\log w_i(\vec{s})$$

constant matrices

letters (alphabet)

- How to construct canonical bases? → No general algorithm exists → Conjecture [Arkani-Hamed et al, '12]:

Integrands with at most simple poles and constant **leading singularities (LS)**

$$\mathcal{J} = \frac{1}{\mathrm{LS}(\vec{x})} \int d\log(\ldots) \wedge \ldots \wedge d\log(\ldots) \longrightarrow$$

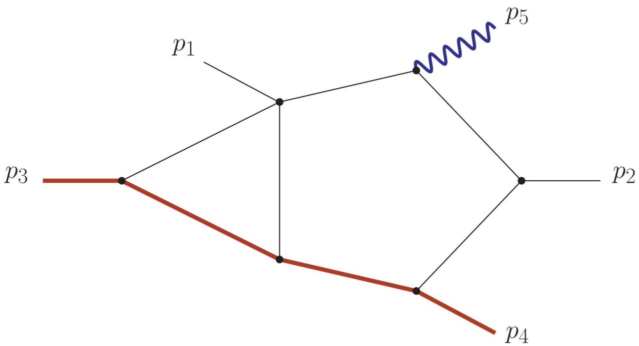
Good Choice: $\mathrm{LS}(\vec{x}) \times \mathcal{J}$

- Find such MIs: 1) **Integrand analysis** (Baikov representation) 2) **Magnus Exponential** [See Mattia's talk]

**LS are not necessarily rational, they may involve square roots and more complicated structures!**

# Nested square roots

- **LS** can contain also **nested square roots**: this is the case for 2 out of the 3 MIs of the following sector



$$\mathrm{NS}_\pm = \pm \frac{\sqrt{b + 6am_t^2 \pm c\, r_1}\sqrt{b - 2am_t^2 \pm c\, r_1}}{r_1}$$

$$r_1 = \sqrt{\mathrm{G}(p_1, p_2, p_3, p_4)}$$

- Observed also in ttH [Febres Cordero et al], ttj [Badger et al, '24] & 5- 6-point QCD processes [Aliaj et al, '26].

  [See Georgios' talk]

- Nested square roots are not supported from public packages that solve DEs!

- We keep DEs linear in $\varepsilon$ without introducing $\mathrm{NS}_\pm$. Anyway we could not obtain canonical DEs …

# Elliptic integrals

- In **complicated cases** the functional space is extended beyond poly-logs: **elliptic integrals**! [Sabry, 1962]
- In such cases, elliptic curves appear during the integrand analysis of the MIs

$$\mathcal{J} = \int \frac{dz}{\sqrt{\mathcal{P}_4(z)}} \wedge d\log(\ldots) \wedge \ldots \wedge d\log(\ldots)$$

where $\mathcal{P}_4(z) = (z - a_1)(z - a_2)(z - a_3)(z - a_4)$.

**CDEs can be constructed** in the elliptic case: see e.g. [Görges et al, 2023; ε collaboration, 2025].

↓

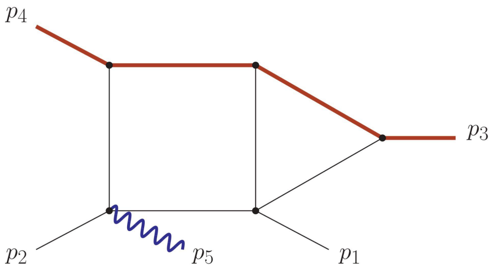
Introduction of **transcendental functions in the DEs** (elliptic periods, Logs).

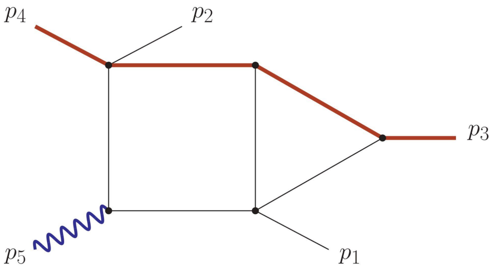
↓

Not available public tools for their numerical evaluation, efficiency also remains an open challenge!

# Elliptic MIs in ttW

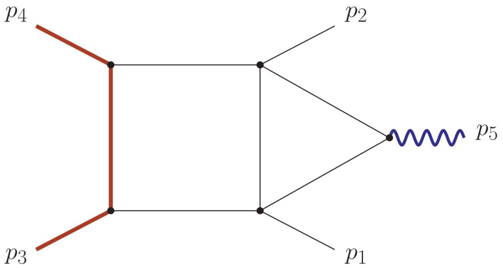
- 4-Point elliptic sectors with 3 MIs:





$$\mathcal{P}_{\text{4-pt.}}(z) = (z+m_t^2)(z-3m_t^2)\mathcal{P}_2(z)$$

- 5-Point elliptic sector with 7 MIs:



$$\left(\frac{\mathrm{d}z\wedge\mathrm{d}\log[\alpha'(z,z_9)]}{\sqrt{\mathcal{P}_{\text{5-pt.}}(z)}} - \frac{\mathrm{d}z\wedge\mathrm{d}\log[\alpha'(z,z_9)^\dagger]}{\sqrt{\mathcal{P}_{\text{5-pt.}}(z)^\dagger}}\right)$$

$$\mathcal{P}^{\text{W.}}_{\text{5-pt.}}(z) = 4\,z^3 + g_1\,z + g_0$$

DEs size dominated by this sector!

First study of such complicated elliptic sector!

Degree-14 polynomial in the DE denominators and discriminant: predicted by Landau singularities [Correia et al., ’26].

# Strategy for the construction of MI-bases

- **Goal**: compact and simplified DEs without introducing transcendental functions [Badger et al, '24]

For the polylogarithmic MIs we obtain CDEs using tools described in Mattia's talk.

For the nested square root MIs we construct DEs that are linear in $\varepsilon$.

For the elliptic MIs we build DEs that are quadratic in $\varepsilon$.

differential one-forms

$$\mathrm{d}A^{(F)}(\vec{x},\epsilon) = \sum_{k=0}^{2} \epsilon^{k} \left[\sum_{\alpha} c_{k\alpha}^{(F)}\, \mathrm{d}\log\big(W_\alpha(\vec{x})\big) + \sum_{\beta} d_{k\beta}^{(F)}\, \boxed{\omega_\beta}(\vec{x})\right]$$

- **Elliptic/Nested** structures appear only in the **finite part** → choose **corresponding MIs** to be **finite**

| | MIs | elliptic | roots | nested | entries | letters | one-forms | one-forms size |
|---|---|---|---|---|---|---|---|---|
| $F_1$ | 141 | 2 | 8 | 1 | 2339 | 101 | 119 | 6.7 MB |
| $F_2$ | 122 | 1 | 11 | 0 | 2027 | 122 | 84 | 311 MB |
| $F_3$ | 131 | 1 | 12 | 0 | 2333 | 137 | 96 | 317 MB |

ttj DEs < 1MB

# Construction of special function basis

- $\vec{\mathcal{J}}_F^{(i)}$ are not **algebraically independent** → construct **special function basis** (SFs): **298**.

**Non-elliptic (polylogarithmic) MIs**

Explore iterated structure of CDEs to find relations between components

[Gehrmann et al, ’18; Chicherin et al, ’20 & ’21]

**Elliptic + Nested-Root MIs**

No iterated structure, construct a potentially over-complete basis for i=4

[Badger et al, ’25]

$$\mathcal{J}_j^{(i)}(\vec{x}) = \sum_j c_j \, f_j^{(i)}(\vec{x})$$

- This basis allows for **analytic subtraction of poles**: direct evaluation of finite remainder

$$\boxed{\mathcal{R} = \sum_j d_j(\vec{x}) f_j(\vec{x})}$$

# Differential equations for SFs

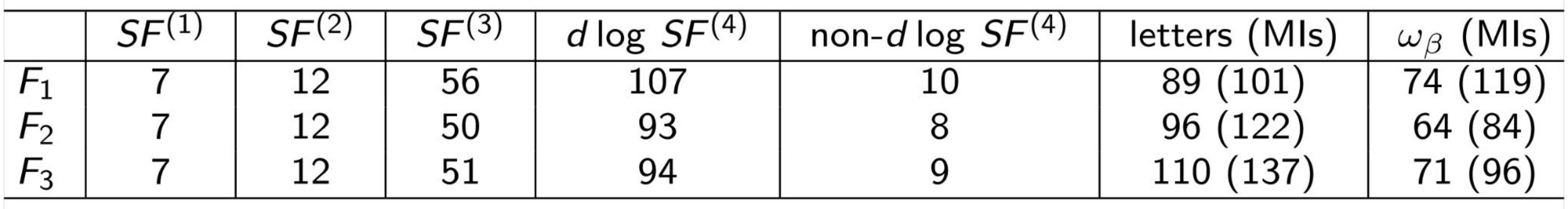
- **SFs satisfy linear DEs** which we derived from the MIs ones

$$\mathrm{d}\vec{f}(\vec{x}) = \mathcal{B}(\vec{x}) \cdot \vec{f}(\vec{x})$$

- **SFs are not a basis of the DEs**: we add extra functions (combinations of SFs) to close the system

$$\begin{aligned}\mathrm{d}f_9^{(4*)} = &\,\omega_1^{(r6)} f_8^{(4*)} + \omega_2^{(r6)} f_{10}^{(4*)} + \frac{1}{4}\left(\frac{1}{2}\mathrm{d}\log W_{32}^{(r6)} + \frac{7}{3}\omega_3^{(r6)}\right) f_{49}^{(3)} \\ &+ \mathrm{d}\log W_{28}^{(r6)}\left[4 f_2^{(1)} f_{12}^{(2)} - 4\left(f_2^{(1)}\right)^3 - \frac{3}{4}\zeta_2 f_5^{(1)}\right] + \ldots\end{aligned}$$

- **SFs DEs are simpler:** 1) $\varepsilon$-independent, 2) sparser, 3) many letters and one-forms drop out.



| | $SF^{(1)}$ | $SF^{(2)}$ | $SF^{(3)}$ | $d\log SF^{(4)}$ | non-$d\log SF^{(4)}$ | letters (MIs) | $\omega_\beta$ (MIs) |
|---|---|---|---|---|---|---|---|
| $F_1$ | 7 | 12 | 56 | 107 | 10 | 89 (101) | 74 (119) |
| $F_2$ | 7 | 12 | 50 | 93 | 8 | 96 (122) | 64 (84) |
| $F_3$ | 7 | 12 | 51 | 94 | 9 | 110 (137) | 71 (96) |

# Evaluation of SFs and rational coefficients

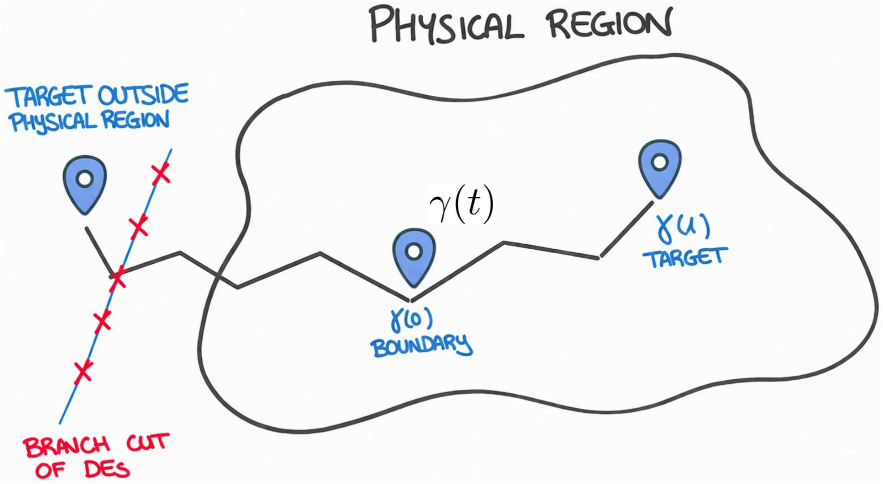
- Numerical evaluation of DEs: **generalised series expansions** [Pozzorini & Remiddi, 2006; Moriello, 2020]

Divide path into segments according to singularity structure of DEs

↓

Expand and solve the DEs in each segment

$$\vec{I}(t) \sim \sum c_w (t-\tau)^{w_1} \log(t-\tau)^{w_2}$$



Boundary values obtained with AMFlow [Liu & Ma, '22].

Employ AMFlow solver

**SFs evaluation ~ 20-120 mins**

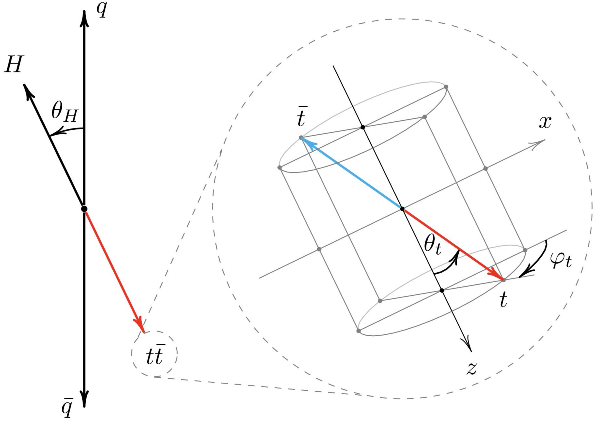
- Analytic computation of $d_j(\vec{x})$ very expensive ➔ numerical evaluations by rational reconstructions

**Finite Remainder coefficients evaluated in ~ 10 mins**

# Towards the Cross Section: interpolation grid

Construct of **interpolation grid of 224640 rationalised points** with fixed masses and 5 variables:

$$m_t = 173.2\,\text{GeV} \qquad \theta_t \in [0,\pi] \qquad \theta_W \in [0,\pi] \qquad \text{frac}_{s_{t\bar{t}}} \in [0,1]$$
$$m_W = 80.385\,\text{GeV} \qquad \phi_t \in [0,2\pi] \qquad \beta \in [0,1]$$



[Agrawal et al, 2024]

Validation @ One-Loop against:
1) OpenLoops [Buccioni et al, ’19]
2) Recola [Actis et al, ’17]
3) Analytic results [Becchetti et al, ’25]

**Interpolation uncertainty ~1%**

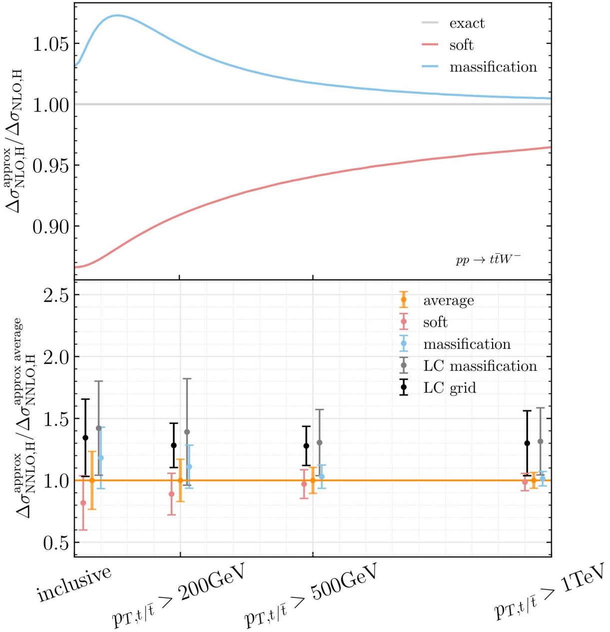
**LC amplitude ~ 22% larger** than the FC one

**Since LC/FC & Two-Loop/One-Loop are flat we adopt 22% as the uncertainty of LC @ two-loop LC**

# Preliminary Results: Cross Section

Cross section obtained using **MATRIX** [Grazzini et al, ’18] (implementing qT slicing [Catani & Grazzini ’07])

| | $\sigma_{t\bar{t}W^+}$ [fb] | $\sigma_{t\bar{t}W^-}$ [fb] |
|---|---|---|
| $\mathrm{NNLO}_{\mathrm{QCD}}^{\mathrm{LC}} + \mathrm{NLO}_{\mathrm{EW}}$ | $512.17^{+6.7\%}_{-7.38\%} \pm 2.51\%$ | $254.49^{+6.94\%}_{-7.64\%} \pm 2.23\%$ |
| $\mathrm{NNLO}_{\mathrm{QCD}}^{*} + \mathrm{NLO}_{\mathrm{EW}}$ | $497.5^{+6.6\%}_{-6.6\%} \pm 1.8\%$ | $247.9^{+7.0\%}_{-7.0\%} \pm 1.8\%$ |
| ATLAS [2401.05299] | $585^{+6.0\%+8.0\%}_{-5.8\%-7.5\%}$ | $301^{+9.3\%+11.6\%}_{-9.0\%-10.3\%}$ |
| CMS [2011.04652] | $533^{+5.4\%+5.4\%}_{-5.4\%-5.4\%}$ | $343^{+7.6\%+7.3\%}_{-7.6\%-7.3\%}$ |



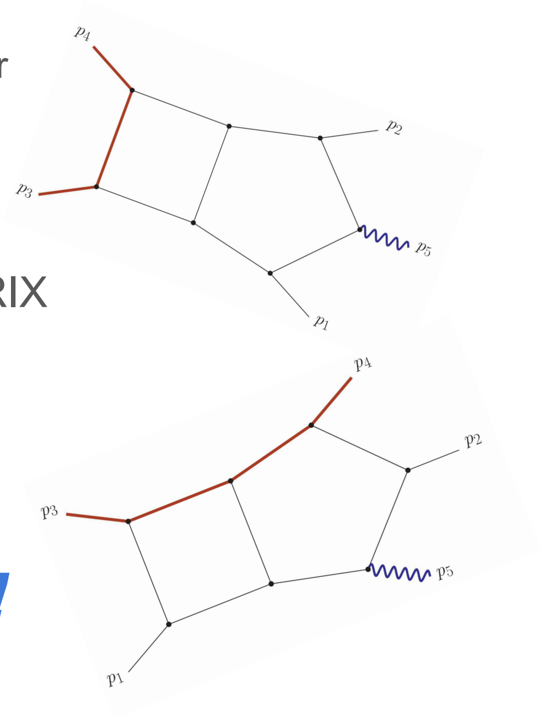
**Full Agreement with previous approximations!**

[Buonocore et al, ’23]

# Conclusions

Computation of two-loop QCD amplitude for ttW production in LC approximation

- Constructed SF-basis that allows for direct access to the finite remainder
- Evaluation of SFs through DEs and generalised series expansions
- Evaluation of amplitude coefficients using finite-field techniques



NNLO cross section computed using an interpolation grid and MATRIX

- Agreement with existing results obtained with kinematic approximations
- Tension between theoretical predictions and experimental data remains

***Thank you very much!***