Novel cluster-algebraic letters for 5- and 6-point QCD processes

Georgios Papathanasiou

NATIONAL & KAPODISTRIAN
UNIVERSITY OF ATHENS
Department of Physics
Est. 1837

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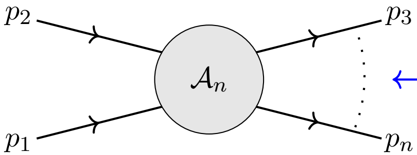
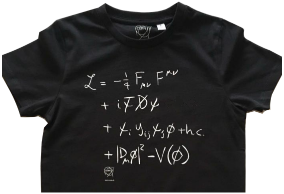
2603.16743 with Rigers Aliaj, Gab Dian, to appear in JHEP
PRL 126 091603 (2021) with Dima Chicherin, Johannes Henn

Motivation: Scattering Amplitudes $\mathcal{A}_n$ in Quantum Field Theory





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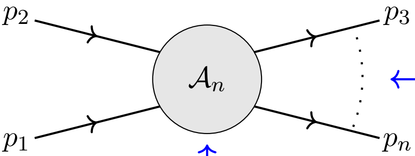
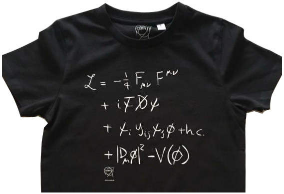
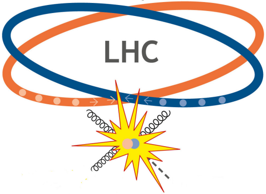




Collider Experiments



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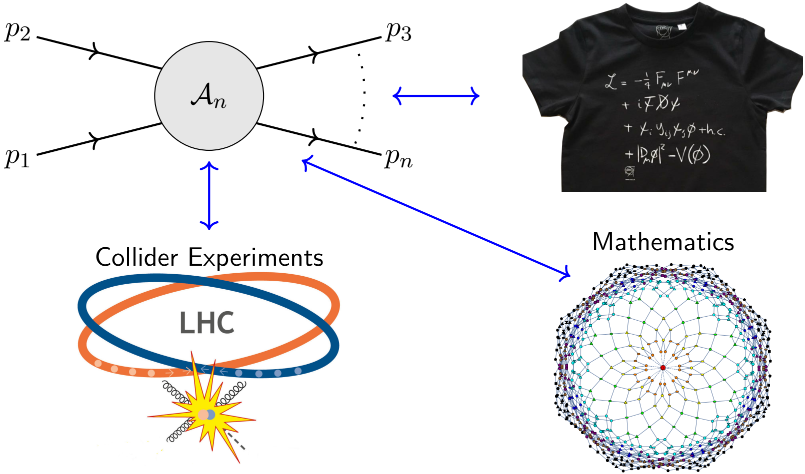

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- Theoretical predictions for outcome of elementary particle collisions, central for experiments such as the LHC & High-Luminosity upgrade
- Exhibit remarkably deep mathematical structures

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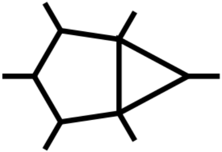
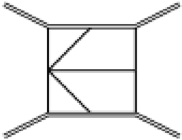
Applied to recent state of the art calculations for collider and gravitational wave physics.

[Abreu,Monni,Page,Usovitsch'24]
[Henn,Matijašić,Miczajka,Peraro,Xu,Zhang'25]



[Bern,Jackman,Mansfield,Ruf'26]



The Role of Cluster Algebras

Tremendously successful in describing singularities of n-particle planar amplitudes in $\mathcal{N} = 4$ SYM.
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$\Rightarrow$ results for $n = 6, 7$ to unprecedented loop order. [Drummond, GP,Spradlin'14]
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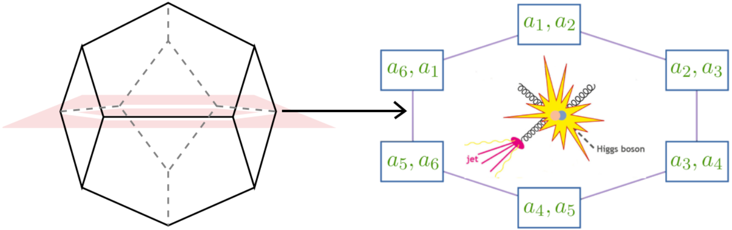

Recently observed to underlie analytic structure of Feynman integrals and realistic processes such as Higgs+jet production in Quantum Chromodynamics (QCD)! [Chicherin,Henn, GP;PRL 126 091603 (2021)][Aliaj,GP'24].

This talk

Cluster algebras in QCD: From discovery to prediction

Starting from $\mathcal{N}=4$ SYM theory and breaking certain symmetry, will show how we can make predictions for QCD processes with

1. Six particles where one is massive
   - Beyond current state of the art
   - New qualitative features
2. Six massless particles
   - Essentially contain those of recent 2-loop planar integral calculations! [Abreu,Monni,Page,Usovitsch'24] [Henn,Matijašić,Miczajka,Peraro,Xu,Zhang'25]
   - Also go beyond them
3. Five particles where two are massive
   - Highly nontrivial overlap with recent 2-loop planar integral calculations [Abreu,Chicherin,Sotnikov,Zoia'24]
   - Also go beyond them

Outline

What Are Cluster Algebras? [Fomin,Zelevinsky'02]

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$$a_5 = \frac{1 + a_1}{a_2}, \quad a_6 = a_1$$

Feynman integrals and symbol letters

- Amplitudes$=\sum$ Feynman integrals.
- Generically diverge, regularise by taking dimension $D = 4 - 2\epsilon$.

Evaluation method of choice for basis of integrals $\mathbf{f}$: Differential equations

[Kotikov'91] [Gehrmann,Remiddi'99] [Henn'13] [Talks by Hartanto,Pozzoli,Canko]

$$d\mathbf{f}(\vec{z};\epsilon) = \epsilon\, d\tilde{\mathbf{A}}(\vec{z}) \cdot \mathbf{f}(\vec{z};\epsilon)$$

where $\vec{z}$ their kinematic variables, $\tilde{\mathbf{A}}$ a matrix and $d = \sum_j dz_j \partial_{z_j}$.

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For wide class of polylogarithmic integrals,

$$\tilde{\mathbf{A}}(\vec{z})=\epsilon\sum_i \mathbf{A}_i\log\alpha_i(\vec{z})\,,$$

where $\mathbf{A}_i$ constant matrices, α_i are *(symbol) letters* capturing singularities of $\mathbf{f}$, and $\Phi=\{\alpha_i\}$ is the *alphabet*. [Goncharov,Spradlin,Vergu,Volovich'10]

Cluster Algebras and $\mathcal{N} = 4$ SYM symbol alphabets

Symbol alphabet of SYM amplitudes $\mathcal{A}_n =$ variables $\{a_m\}$ of a *Grassmannian* $Gr(4, n)$ *cluster algebra* for $n = 6, 7$.
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Main idea: Limits of infinite mutation sequences. E.g. for $a_{m+1} = \frac{1+a_m^2}{a_{m-1}}$,

1. Linearise to $$a_{m+1} = a_m \mathcal{P} - a_{m-1}$$ with $\mathcal{P} = \frac{1+a_i^2+a_{i+1}^2}{a_i a_{i+1}}$, $\forall\ i \in \mathbb{Z}$.
2. Take $m \to \infty$, $$\beta = \mathcal{P} - \frac{1}{\beta}, \quad \beta = \lim_{m\to\infty} \frac{a_{m+1}}{a_m},$$
3. Readily solved by $$\beta_\pm = \frac{\mathcal{P} \pm \sqrt{\mathcal{P}^2 - 4}}{2}\ !$$

From n-point SYM to $(n-1)$-point $(k-1)$-mass QCD alphabets

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Dual conformal Invariance (DCI) [Drummond,Henn,Smirnov,Sokatchev'06...]

Appropriately normalised MSYM amplitudes $\mathcal{A}_n$ & contributing integrals are invariant under conformal transf$^{\text{ns}}$ of *dual x-coordinates*,

$$p_i =: x_{i+1} - x_i := x_{i+1i}\,, \quad \text{with} \quad x_{i+n} \equiv x_i$$

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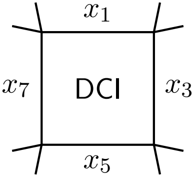
Example: 1-loop 4-mass box*



$= \Phi(u,v)$

Depends on 2 *cross-ratios*

$$u = \frac{x_{13}^2 x_{57}^2}{x_{15}^2 x_{37}^2}\,, v = \frac{x_{35}^2 x_{17}^2}{x_{15}^2 x_{37}^2}$$

instead of 6 *Lorentz-invariant* (LI) x_{ij}^2

*Form massive external legs $k_i = p_{2i-1} + p_{2i}$ as sums of massless ones, $p_i^2 = x_{ii+1}^2 = 0$, to make contact with massless field content of $\mathcal{N}=4$ SYM.

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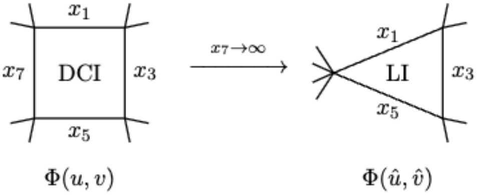
$$\Phi(u,v) = 2\mathrm{Li}_2(-\rho u) + 2\mathrm{Li}_2(-\rho v) + \log\frac{v}{u}\log\frac{1+\rho v}{1+\rho u} + \log(\rho u)\log(\rho v) + \tfrac{1}{3}\pi^2$$

$$\rho = \frac{2}{1-u-v+\Delta}\,, \quad \Delta = \sqrt{(1-u-v)^2 - 4uv}\,,$$

Main idea: Breaking dual conformal symmetry

In previous example, may choose a reference frame where e.g. $x_7 \to \infty$,



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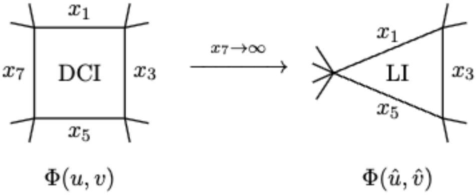

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4-mass DCI box = 3-mass LI triangle!

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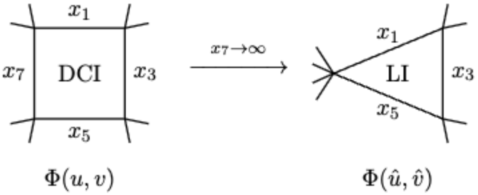

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n-point* DCI $=$ $(n-1)$-point LI integral $\forall$ loop order!

*With at least two adjacent massive legs

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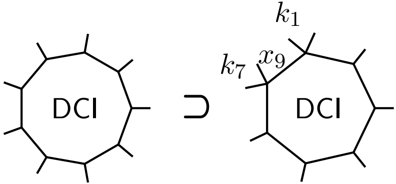

7-pt 2-mass 'hard' DCI subalphabet
(From now on diagrams represent kinematics irrespective of loop order)

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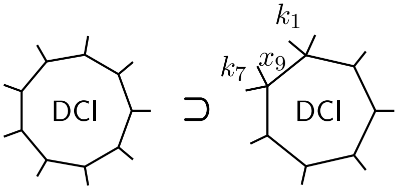

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Can identify as subspace of letters annihilated by certain operators

$\Downarrow$

Nullspace computation

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Obtain reduction to all $(9-k)$-point k-mass DCI subalphabets.

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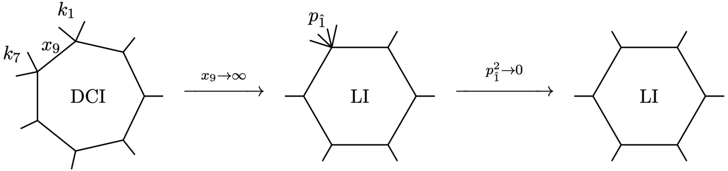
Obtain reduction to all $(9-k)$-point k-mass DCI subalphabets. Focus on

1. 7-pt 2-mass DCI → 6-pt 1-mass and massless LI alphabet



Application: DCI to LI alphabets II

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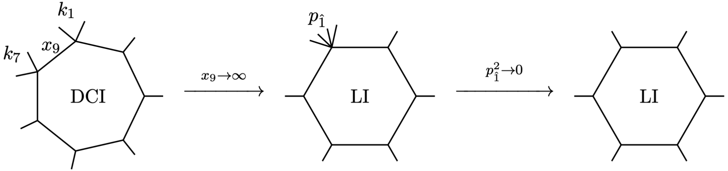
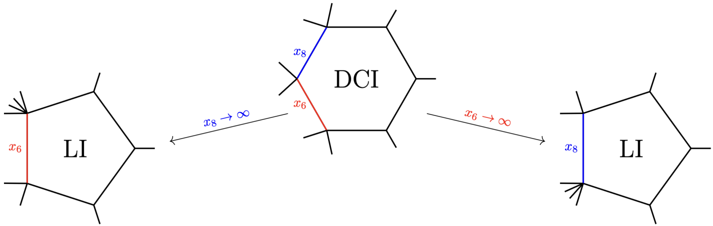


2. 6-pt 3-mass DCI → 5-pt 2-mass LI alphabet



similarly to previously performed 8-point reductions
[Chicherin,Henn, GP;PRL 126 091603 (2021)]

The 9-particle $\mathcal{N} = 4$ SYM alphabet

- Obtained by taming the infinity $Gr(4, 9)$ cluster algebra
  [Henke, GP'21] [Ren,Spradlin,Volovich'21]

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- These rationalise certain square roots of momentum variables, e.g.

$$\epsilon_{1234} = 4\sqrt{\det(p_i \cdot p_j)}, \quad i, j = 1, \dots, 4\,,$$

  with obvious generalisation to ϵ_{ijkl}. Denote these as *rationalisable*.

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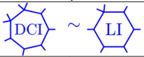
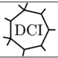
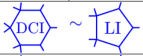
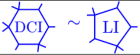
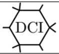
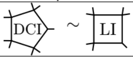
- From (1), can identify and add by hand any overall square root letters that typically also appear $\Rightarrow$ denote these as *trivially recoverable*

Results: Subalphabets for $(9-k)$-point k-mass DCI kinematics

Kinematics	Dimensionless variables	Rational letters	Rationalisable letters	Square-root letters
	12	1737	1341	2349
BCFW shift	11	979	271	692
	10	421	271	395
	8	119	59	68
	8	113	53	46
	8	128	64	81
	6	39	12	23
	6	40	12	19
	6	30	6	0
	4	12	0	8

6-point 1-mass LI letters: New Results

Find 178 rational and 59 square-root letters, out of which 17 and 12 genuinely 6-pt (not part of 5-pt 2-mass etc. subalphabets).

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$$l_\pm^i = \frac{A_i \pm B_i \epsilon_{2356} - C_i\sqrt{\Delta_\pm}}{A_i \pm B_i \epsilon_{2356} + C_i\sqrt{\Delta_\pm}}, \qquad \Delta_\pm = F \pm \epsilon_{2356} G\,, \qquad i = 1, \ldots, 6,$$

where A_i, B_i, C_i, F, G rational expressions of LI scalars (Mandelstam invariants) and ϵ_{2356} another (pseudoscalar) square root.

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Our results strongly suggest that *nested square roots are ubiquitous even in QFTs with massless propagators, and for quantities purely expressible in terms of polylogarithms*.

6-point 1-mass LI letters: Broader implications

At the moment nested square roots pose a challenge for existing partial algorithms for determining alphabets directly from the integrals, such as

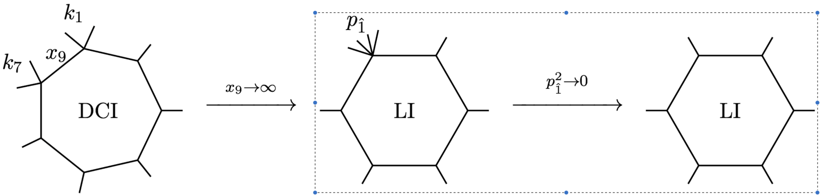
- `baikovletters` [Jiang,Liu,Xu,Yang'24]
- `Effortless` [Matijašić,Miczajka'25]
- `SOFIA` [Correia,Giroux,Mizera'25]

Warrants their improvement, possibly by incorporating momentum twistor variables where nesting disappears.

Comparison of our prediction with Landau singularities underway [Demaku,GP;WIP]

The massless limit

Taking $p_{\hat{1}}^2 \to 0$ produces 140 rational + 20 square-root letters.



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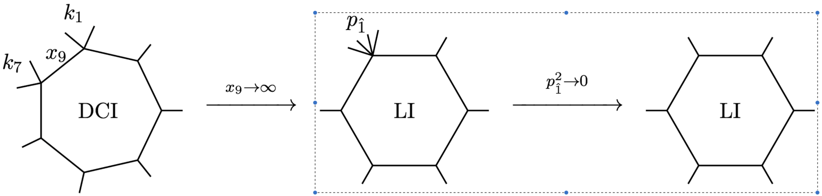

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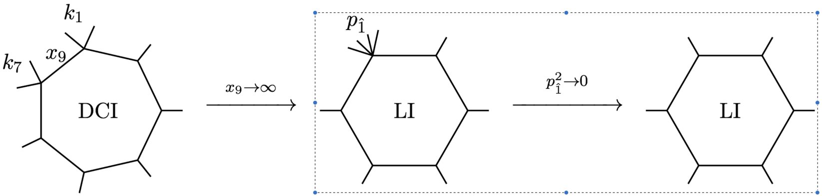


Comparison with literature: Alphabet of 2-loop 6-point massless planar

1. integrals in 4 dimensions: 245 'CDE' letters
   [Abreu,Monni,Page,Usovitsch'24] [Henn,Matijašić,Miczajka,Peraro,Xu,Zhang'25]

$\cup$

2. finite amplitudes: 167 'Amplitude' letters

$\cup$

3. $\mathcal{N} = 4$ SYM and QCD (MHV) quantities: 136 'MHV' letters

Results: 6-point massless alphabet

Comparison with literature

# 6-pt massless letters in:	Amplitude	CDE	This work	New letters
Rational	99 ✓	117 ✓	225	108
Rationalisable (RB)	36 ✓	72	114	60
RB overall root $\{\epsilon_{ijkl}, \Delta_6\}$	7 (✓)	16 (✓)	-	-
Non-rationalisable (NR)	23 ✓	35 ✓	35	-
NR overall root $\{r_i\}$	2 (✓)	5 (✓)	-	-
Total	167	245	374	168

Matching & trivially recoverable letters are denoted by ✓ & (✓), respectively. Last column contains our new predictions.

- Amplitude letters essentially contained in our results
- Their trivially recoverable letters absent from MHV letters

Strong consistency check!

Comparison with other cluster algebraic approaches

Candidate letters for n-particle massless LI scattering also obtained by identifying a so-called partial flag variety inside the $Gr(n-2, 2n-4)$ cluster algebra. [Bossinger,Drummond,Glew'22] [Glew'23] [Bossinger:2024apm]

Recently applied to $n = 6$: [Pokraka,Spradlin,Volovich,Weng'25] [Bossinger,Drummond,Glew,Gürdoğan,Wright'25]

- From simpler $Gr(4,8)$ rather $Gr(4,9)$ case.
- Lacks a stopping criterion ⇒ can only make postdictions.

Both methods the 16 rationalisable overall roots and 18 CDE letters

$$\sum_{I,J} s_I \epsilon_J \in W_{[139,156]},$$

We reproduce an additional 36 amplitude letters, and this approach the 6 non-rationalisable (trivially recoverable) overall roots.

Results: New 6-point massless letters

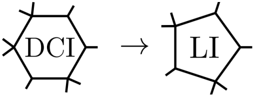
Last but not least, our approach yields another 84 letters that might appear at higher loop or dimensional regularisation parameter orders!

Arranged in 18 rational and 10 rationalisable orbits of size 6. One rational orbit appeared erroneously in the past. [Henn,Peraro,Xu,Zhang'21]
Rest are genuinely new, e.g. rational orbits generated by

$$
\begin{aligned}
\beta_1 &= s_{12}s_{45} - (s_{45} + s_{56})\, s_{345},\\
\beta_2 &= s_{12}s_{45} - (s_{34} + s_{45})\, s_{123},\\
\beta_3 &= s_{12}(s_{34} + s_{45}) - (s_{34} + s_{123})\, s_{345},\\
\beta_4 &= s_{12}(s_{45} + s_{56}) - s_{123}(s_{56} + s_{345}),\\
\beta_5 &= s_{12}s_{45} - s_{345}(-s_{45} + s_{123} + s_{345}),\\
&\ \vdots\\
\beta_{17} &= s_{12}\big(s_{45}s_{61}^2 + (s_{34}(s_{45} + s_{56} - s_{123}) + s_{45}(s_{56} - s_{234} - s_{345}))\, s_{61}\\
&\quad + s_{56}(s_{23} - s_{45} - s_{234})\, s_{345}\big) +\\
&\quad + (s_{23}s_{56} - (s_{61} + s_{56})\, s_{123})(s_{61}(s_{34} + s_{345}) - s_{345}(s_{234} + s_{345}))\,.
\end{aligned}
$$

5-point 2-mass procesees: Different cases and symmetrisation

2-mass 'easy' case done as before: 52 rat('ble) + 19 square-root letters



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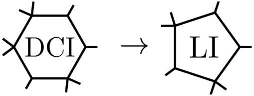
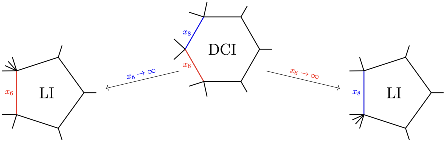


2-mass ‘hard’ case has new feature: Two inequivalent frames



51+23 DCI letters → 72+31 LI letters

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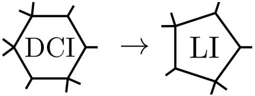

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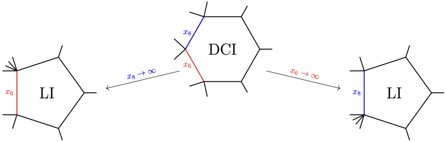

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To compare with literature, close this under certain permutations of particle labels (analogue of cyclic symmetrisation for massless)
[Abreu,Chicherin,Sotnikov,Zoia'24]

⇒ 495 letters arranged in 50 such orbits.

5-point 2-mass alphabet: Comparison with literature

Letter type:	Rational	Δ_5	Δ_3	r_1	r_2	r_3	Double-root	Overall root
Literature	18	9	5	5	4	4	12	5
Matched	17	7	5	4	0	4	5	0
New	5	6	0	0	0	0	0	0

Counts of permutation orbits of 5-point 2-mass letters, classified by the root appearing, if any. First line refers to existing results, second line to their subset matched by our calculation, third line to the new letters of the latter.

$$\Delta_5 = (\epsilon_{1234})^2,$$

$$\Delta_3^{(1)} = \lambda(p_4^2, p_5^2, s_{45}), \quad \lambda(a,b,c) = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc$$

$$r_1^{(1)} = p_4^4 s_{23}^2 - 2p_4^2 s_{23}(2p_5^2 - s_{15} + s_{23}) s_{45} + (s_{15} - s_{23})^2 s_{45}^2,$$

$$r_2^{(1)} = p_4^4 s_{12}^2 + 2p_4^2 s_{12}\big(p_5^2 s_{23} + (s_{15} - s_{34}) s_{45}\big) + \big(p_5^2 s_{23} + (s_{34} - s_{15}) s_{45}\big)^2,$$

$$r_3^{(1)} = 4p_4^4 s_{12}(p_5^2 - s_{15}) s_{15} + \big(p_5^2(s_{23} + s_{34}) - s_{15}(s_{34} + s_{45})\big)^2,$$

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Obtain most letters, exceptions include hard box-triangle root r_2 and double-root letters of form [Abreu,Chicherin,Sotnikov,Zoia'24]

$$\frac{a + b\sqrt{\Delta_1 \Delta_2}}{a - b\sqrt{\Delta_1 \Delta_2}},$$

absent by construction in our approach if both Δ_i are non-rationalisable.

- However no analysis of simplifications for finite 2-loop amplitudes made. Previously, cluster algebras elucidated such for 5-point processes with fewer massive legs [Chicherin,Henn, GP;PRL 126 091603 (2021)] [Talk by Hartanto]

Results: New 5-point 2-mass letters

Findings beyond literature: 4 orbits of 54 rational letters generated by

$$\gamma_1 = p_4^2 p_5^2 (s_{23} - s_{45}) - (p_5^2 - s_{45})(p_5^2 s_{23} - s_{45} s_{51})$$

$$\gamma_2 = p_5^2 s_{23} s_{34} + s_{51}\Big(s_{34}(s_{23} - s_{45} - s_{51}) + p_4^2 (s_{12} + s_{51})\Big)$$

$$\begin{aligned}\gamma_3 = s_{34}\Big(&p_5^2 s_{12} s_{23} + p_5^2 (p_5^2 - s_{34})(p_5^2 - s_{34} - s_{45}) \\ &- (p_5^2 - s_{34})(p_5^2 + s_{12} - s_{34} - s_{45}) s_{51}\Big) \\ &- p_4^2 (p_5^2 - s_{34})\Big(p_5^4 + s_{34} s_{51} - p_5^2 (s_{12} + s_{34} + s_{51})\Big)\end{aligned}$$

$$\begin{aligned}\gamma_4 = p_4^2 (p_5^2 - s_{45})\Big(&p_5^2 s_{23} (p_5^2 - s_{12} + s_{34}) - p_5^2 s_{34} s_{45} + (s_{12} - p_5^2) s_{45} s_{51}\Big) \\ &- p_4^4 p_5^2 \Big(p_5^2 (s_{23} - s_{45}) + s_{12} s_{45}\Big) - s_{34}(p_5^2 - s_{45})^2 (p_5^2 s_{23} - s_{45} s_{51}),\end{aligned}$$

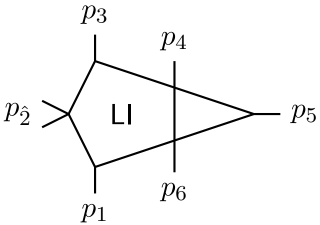
$+$ 4 orbits of 33 rationalisable letters. All results available at

`github.com/GabrieleDian/9-particle-Alphabet-reduction`

Conclusions

Cluster algebras in QCD: From discovery to prediction

New letters for 5- and 6-point state-of-the-art QCD processes



Next Stage

- Bootstrap of QCD amplitudes?
- 10-particle $\mathcal{N} = 4$ SYM amplitude → $(10 - i)$-point i-mass for $i = 1, 2, 3$?
- Letter cancelations in finite amplitudes due to their positivity properties