Can you make the shape of a drum from its sound?

Ki-Seok Kim

(POSTECH & APCTP)

CAN ONE HEAR THE SHAPE OF A DRUM?

MARK KAC, The Rockefeller University, New York

To George Eugene Uhlenbeck on the occasion of his sixty-fifth birthday

"La Physique ne nous donne pas seulement l'occasion de résoudre des problèmes . . . , elle nous fait presentir la solution." H. POINCARÉ.

Before I explain the title and introduce the theme of the lecture I should like to state that my presentation will be more in the nature of a leisurely excursion than of an organized tour. It will not be my purpose to reach a specified destination at a scheduled time. Rather I should like to allow myself on many occasions the luxury of stopping and looking around. So much effort is being spent on streamlining mathematics and in rendering it more efficient, that a solitary transgression against the trend could perhaps be forgiven.

Microscopic or **first-principles** construction of dual holography from quantum field theory

Nonperturbative generalization of

the Wilsonian RG theoretical framework

→ Path integral reformulation of

the functional RG equation

Dual holography as functional renormalization group

In progress -_-;;

Dual holography as functional renormalization group

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arXiv:2511.05786v5 [hep-th] 26 Apr 2026

Emergent AdS$_{d+1}$ Geometry from Functional Renormalization Group in the Massless Critical Limit

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(Dated: June 22, 2026)

We present a holographic dual description for the O(N) vector model in d-dimensional Euclidean space within the functional renormalization group (FRG) framework. By iterating Wilsonian renormalization group transformations, the extra-dimensional scale coordinate is identified as the radial direction of an emergent $(d+1)$-dimensional bulk spacetime. We construct a bidirectional holographic dictionary that maps non-perturbative fluctuations to the emergent bulk metric warping factors. Under the massless critical configuration, the emergent gravitational vacuum reduces to an Anti-de Sitter (AdS$_{d+1}$) geometry, satisfying the local energy conditions.

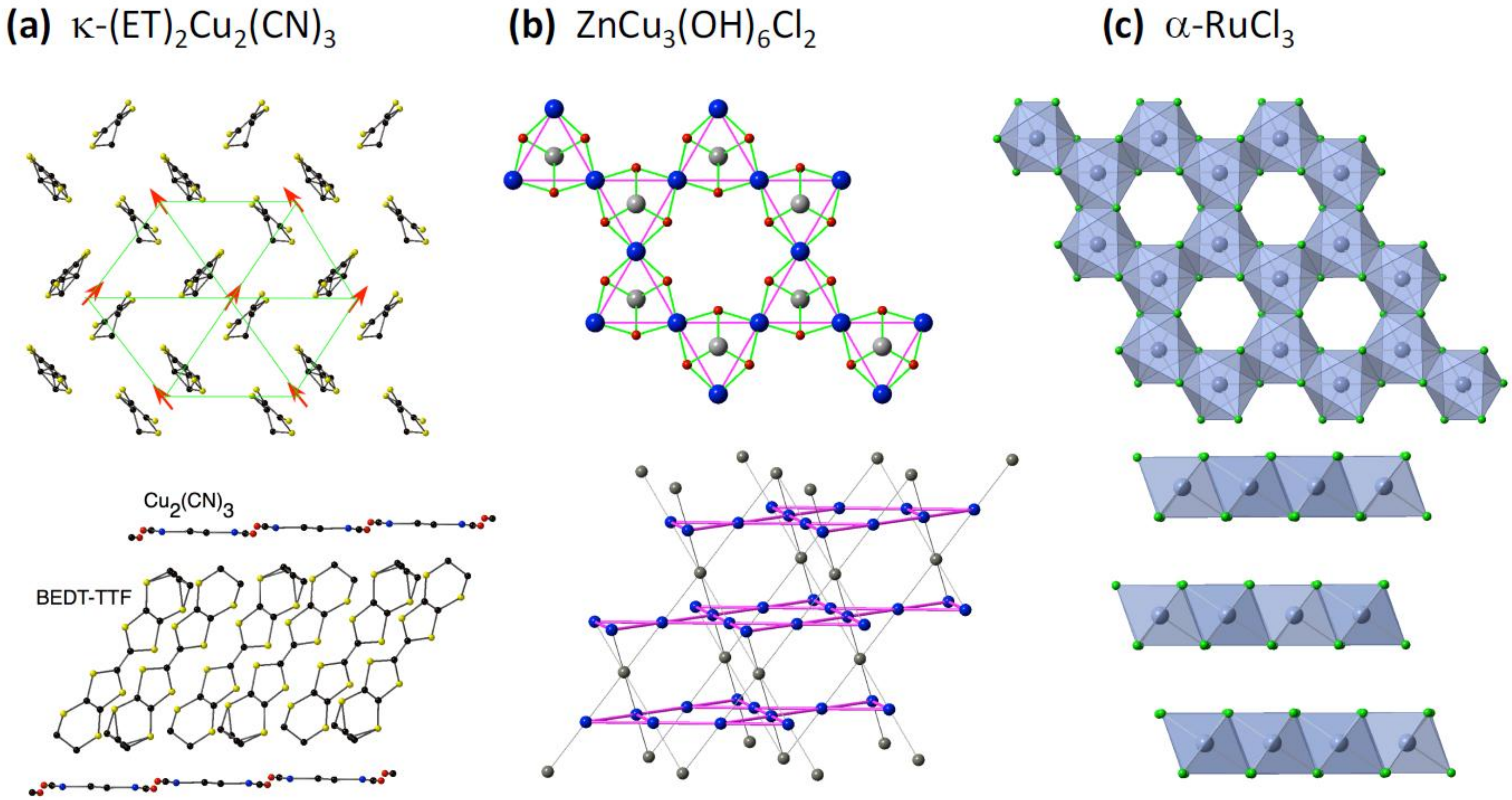
arXiv:2605.17209v3 [hep-th] 10 Jun 2026

Abstract

We investigate the relationship between the functional renormalization group (RG) and the dual holography framework in the path integral formulation, highlighting how each can be understood as a manifestation of the other. Rather than employing the conventional functional RG formalism, we consider a functional RG equation for the probability distribution function, where the RG flow is governed by a Fokker-Planck-type equation. The central idea is to reformulate the solution of Fokker-Planck type functional RG equation in a path integral representation. Within the semiclassical approximation, this leads to a Hamilton-Jacobi equation for an effective renormalized on-shell action. We then examine our framework for an Einstein-Hilbert action coupled to a scalar field. Applying standard techniques, we derive a corresponding functional RG equation for the distribution function, where the dual holographic path integral serves as its formal solution. By synthesizing these two perspectives, we propose a generalized dual holography framework in which the RG flow is explicitly incorporated into the bulk effective action. This generalization naturally introduces RG β-functions and reveals that the RG flow of the distribution function is essentially identical to that of the functional RG equation.

Who am I?

Still, in progress -_-;;



Quantum Spin Liquids

1905.07040

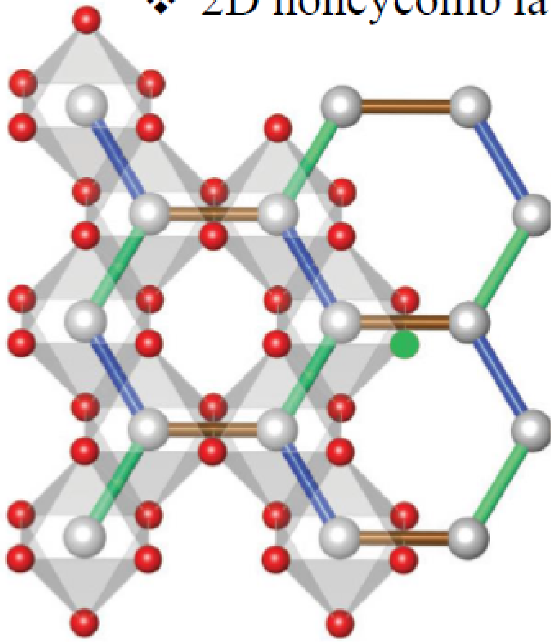
C. Broholm[1], R. J. Cava[2], S. A. Kivelson[3], D. G. Nocera[4], M. R. Norman[5], T. Senthil[6]

• How to host Kitaev

• The prime Kitaev candidate material: α-$RuCl_3$

Seung – Hwan Do's talk in POSTECH

❖ 2D honeycomb la



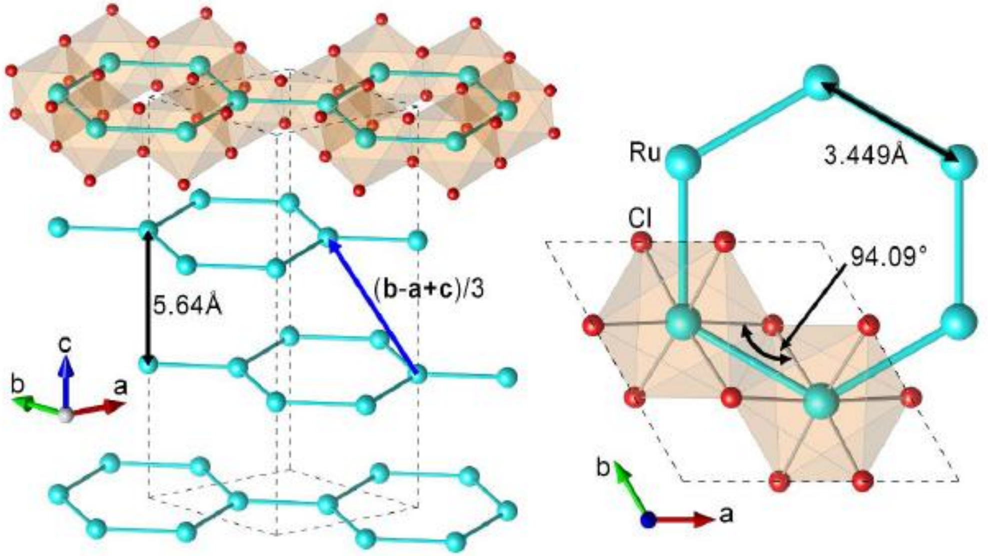
"**Bond dependent**" Ising typ

Majornara fermion excitatior

$$H_{HK\Gamma} = \sum_{<i,j>} J\, \vec{S}_i \cdot \vec{S}_j + \boxed{K S_i^{\gamma_{ij}} S_j^{\gamma_{ij}}}$$

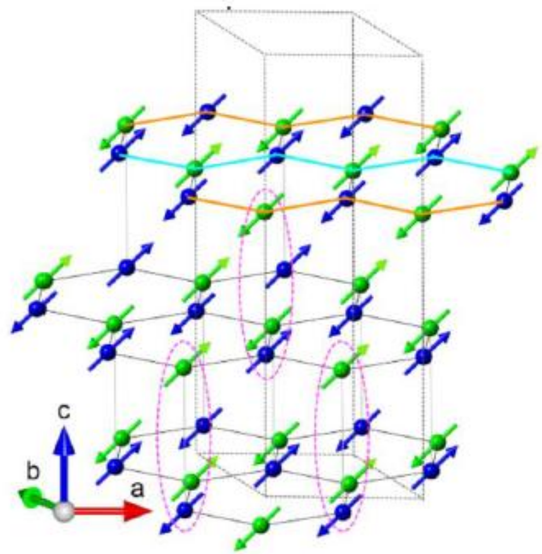
candidates : **α-$RuCl_3$**, Na_2IrO

2D van der Waals honeycomb lattice



Ru^{3+}($4d^5$;J_{eff}=1/2) in "**isotropic**" honeycomb lattice in ***R*-3** (Rhombohedral) at Low *T*.

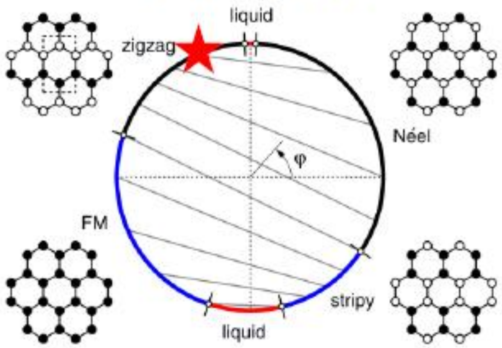
Zig-Zag AFM order at T_N=6.5 K



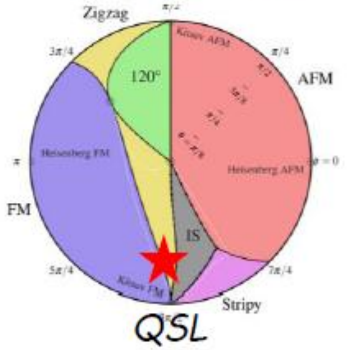
only ~42% of full moment

→ **Large fluctuation** by fractionalized excitation??

J-K model



J-K-Γ model



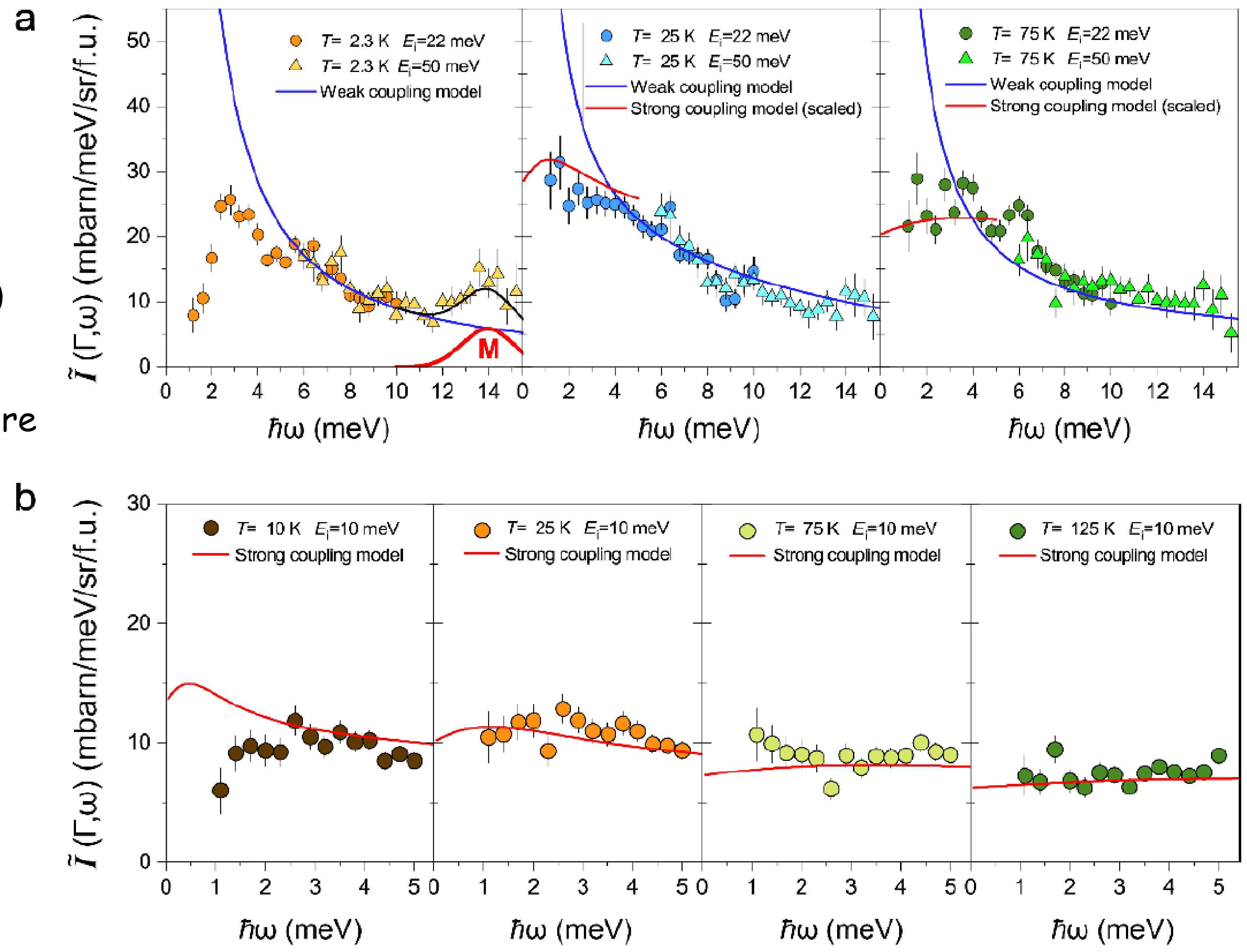
Zig-Zag AFM proximate to QSL state!!

S.-Y. Park, S.-H. Do et al, arXiv:1609.05690
J. Chaloupka et al, PRL.110.097204(2013)
J. M. Winter et al, PRB.93.214431(2016)
J.G.Rau et al PRL.112.077204(2014)

Inelastic neutron scattering (INS) measurements as a function of transferred energy and temperature at zero transferred momentum:

High energy INS spectrum vs.

Low energy INS spectrum



$$
\begin{aligned}
S &= S_{\mathrm{K}} + S_{\mathrm{AF}} + S_{\mathrm{K-AF}}, \\
S_{\mathrm{K}} &= \int \frac{d^3k}{(2\pi)^3} \left(\overline{\Psi}(k) \mathrm{i}\gamma \cdot k \Psi(k) + \overline{X}(k)(\mathrm{i}\gamma_0 k_0 + \mathrm{i}\boldsymbol{\gamma} \cdot \mathbf{M}) X(k)\right), \\
S_{\mathrm{K-AF}} &= -\mathrm{i}g \int \frac{d^3k d^3q}{(2\pi)^6} \phi(q) \left(\overline{X}(k+q)\Psi(k) - \overline{\Psi}(k) X(k-q)\right), \\
S_{\mathrm{AF}} &= \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \phi(-k)(k^2 + r)\phi(k) + \frac{\lambda}{4!} \int \frac{d^3k d^3p d^3q}{(2\pi)^9} \phi(-k)\phi(-p-q)\phi(p)\phi(k+q).
\end{aligned}
\tag{10}
$$

$$\Sigma_X(\boldsymbol{k}) = i v_X \boldsymbol{\gamma} \cdot \boldsymbol{k}$$

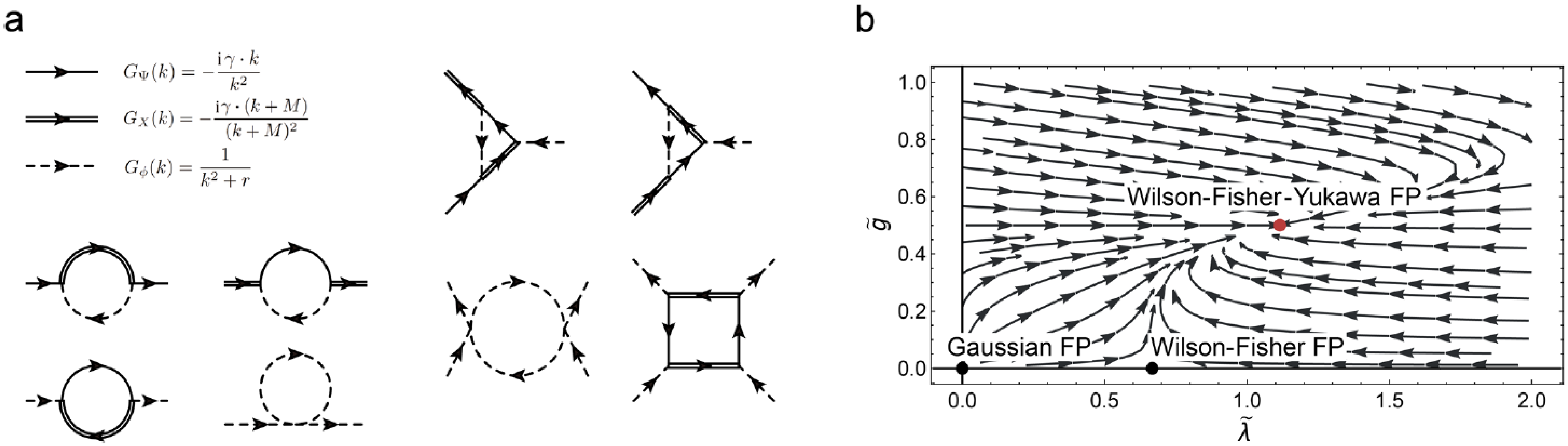
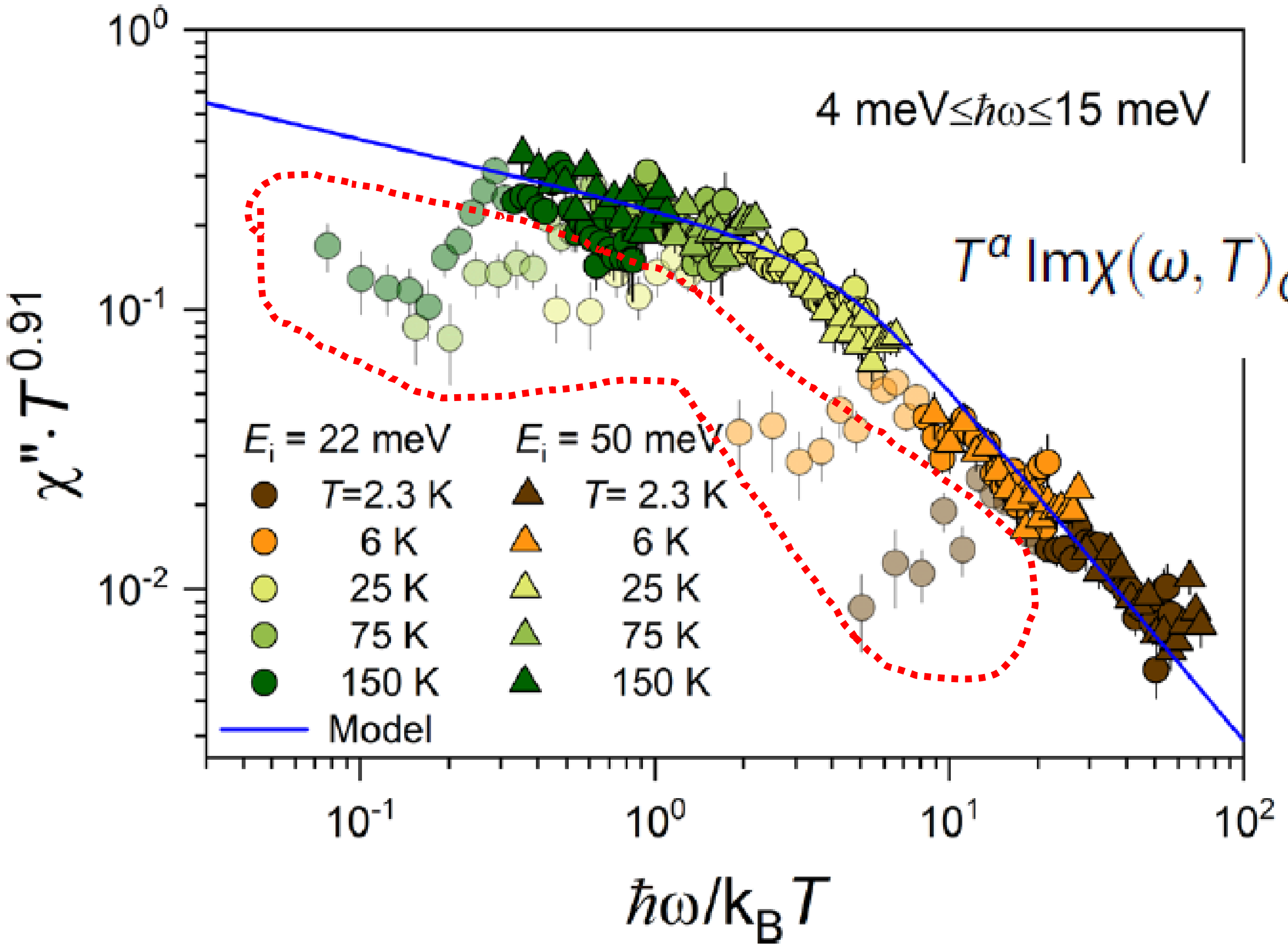


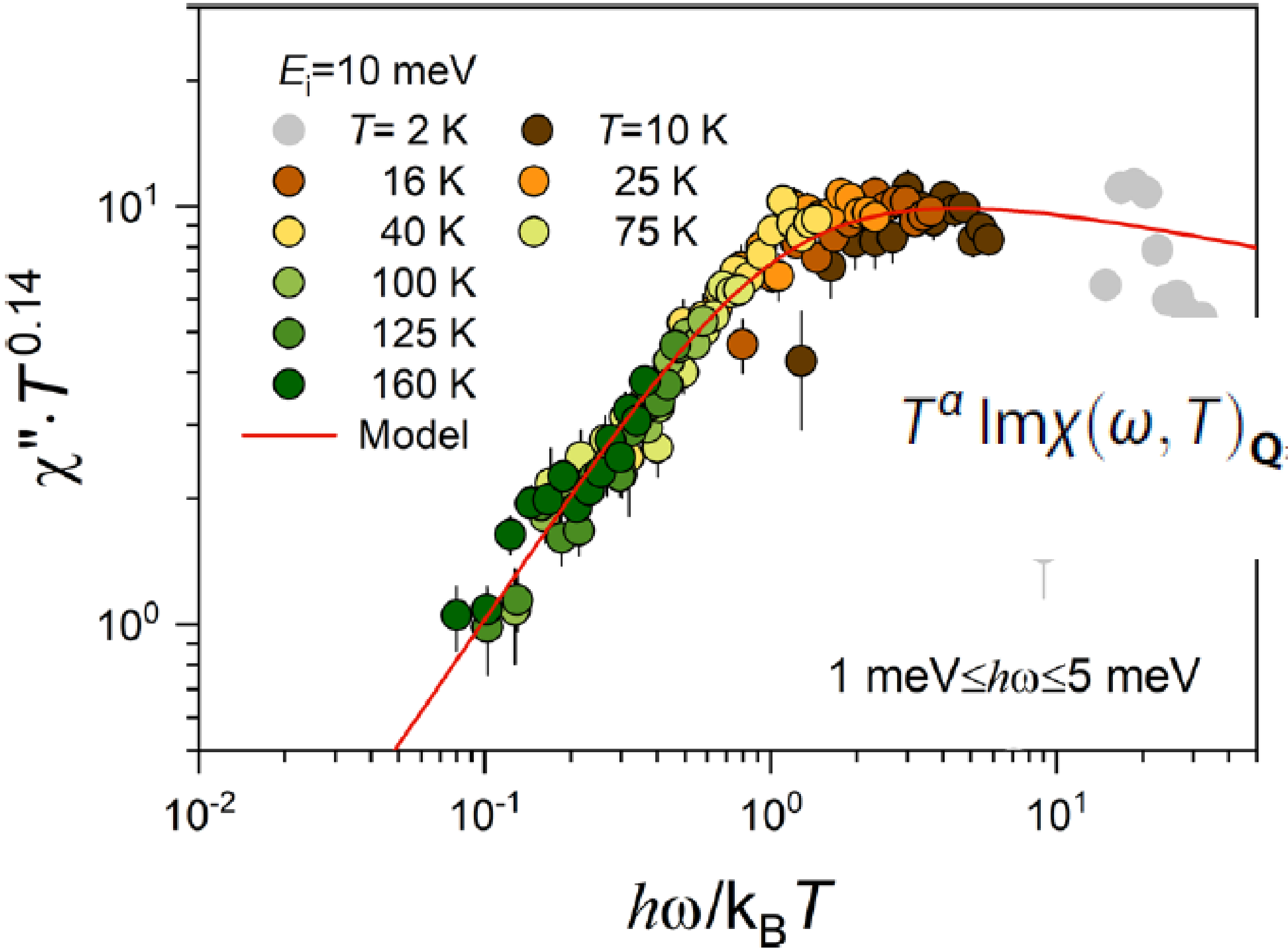
Fig. 4 Feynmann diagrams and the renormalization group flow diagram of coupling constants. a Green functions and one-loop Feynman diagrams for self-energies and vertex functions. **b** RG flows in a $\tilde{g}$-$\tilde{\lambda}$ coupling space with $\tilde{g}^2 = g^2/8\pi^2$ and $\tilde{\lambda} = \lambda/8\pi^2$ by solving the coupled β-functions. Various fixed points (FPs) are indicated, two unstable FPs (black dots) and one stable interacting FP (red dot).



$$T^{\alpha}\,\mathrm{Im}\chi(\omega, T)_{Q=\Gamma} = \chi_0 M^2 \left(\frac{T}{\omega}\right)^{1.25} \tanh\frac{\omega}{4T}$$

Low energy region data

Fig. 5 Scaling plot in $\hbar\omega/k_B T$ for the dynamic spin susceptibility at high energies, $\hbar\omega \geq 4$ meV. Blurred circles present the data in the low-energy range $\hbar\omega = [1, 4]$ meV, which are out of the universal scaling behavior. Error bars represent 1 standard deviation.



$$T^{\alpha}\,\mathrm{Im}\chi(\omega,T)_{\mathbf{Q}=\Gamma} = \frac{A}{\left(a^2T^2+\omega^2\right)^{\alpha/2}}\sin\left\{\alpha\tan^{-1}\left(\frac{\omega}{aT}\right)\right\}$$

Fig. 6 Scaling plot for the dynamic spin susceptibility obtained at low energies, $\hbar\omega \leq 5$ meV. The quantity $\chi'' T^{\alpha}$ with $\alpha = 0.14$ is plotted against $\hbar\omega/k_BT$. The red line is a fit to the local quantum criticality model as described in the text. The gray circles presenting the 2 K data deviate from the universal scaling due to the dominant magnon excitation at a low-energy AFM state below $T_N \approx 6.5$ K. Error bars represent 1 standard deviation.

$$\chi(\omega,T,H) = \frac{A}{(aT-\mathrm{i}\omega)^{\alpha}+a^{\alpha}T^{*\alpha}}$$

Condensed matter physics phenomenology (CMP-PH): To understand the spectrum of quantum material based on effective field theory

$Nonperturbative\ RG\ flow$

$from\ a\ weak-coupling\ fixed\ point$

$to\ a\ strong\ coupling\ one$

Towards a nonperturbative theoretical framework:

Functional Renormalization Group (FRG)

$$
\begin{aligned}
&\frac{d}{d\ln\Lambda}P_\Lambda[\phi] \\
&= \int_M d^dx \int_M d^dy \left\{ C_\Lambda^{Pol.}(x,y)\frac{\delta^2 P_\Lambda[\phi]}{\delta\phi(x)\delta\phi(y)} + \frac{\delta}{\delta\phi(x)}\left(P_\Lambda[\phi] C_\Lambda^{Pol.}(x,y)\frac{\delta V_\Lambda^{Pol.}[\phi]}{\delta\phi(y)}\right)\right\} \\
&\equiv \Delta P_\Lambda[\phi] + \mathrm{div}\left(P_\Lambda[\phi]\mathrm{grad}_{C_\Lambda^{Pol.}} V_\Lambda^{Pol.}[\phi]\right).
\end{aligned}
$$

$$
\delta_{\ln\Lambda} O = \ln\Lambda\left[\int_M d^dx \int_M d^dy \left(C_\Lambda^{Pol.}(x,y)\frac{\delta^2}{\delta\phi(x)\delta\phi(y)} + \frac{\delta}{\delta\phi(x)}\left\{ C_\Lambda^{Pol.}(x,y)\frac{\delta V_\Lambda^{Pol.}[\phi]}{\delta\phi(y)}\right\}\right), O\right]
$$

$$
\begin{aligned}
C_\Lambda^{Pol.}(p^2) &= (2\pi)^d G(p^2)\frac{\partial K_\Lambda(p^2)}{\partial\ln\Lambda}, \\
V_\Lambda^{Pol.} &= \int \frac{d^dp}{(2\pi)^d}\phi(p)G^{-1}(p^2)K_\Lambda^{-1}(p^2)\phi(-p) \ .
\end{aligned}
$$

2.4 Remarks on the path integral reformulation of the functional RG equation

Recalling the path integral construction for the formal solution of the Schrodinger equation, we can write down the probability density at IR as follows:

$$P_\Lambda[\phi(x,r_{IR});r_{IR}] = \int D\phi(x,r_{UV}) \exp\Big\{ -\int_{r_{UV}}^{r_{IR}} dr \hat{\mathcal{H}}\Big(\frac{\delta}{\delta\phi(x,r)}, \phi(x,r)\Big)\Big\} P_\Lambda[\phi(x,r_{UV});r_{UV}]. \tag{2.35}$$

Here, one may easily read out the Hamiltonian kernel for the Wilsonian RG transformation from the functional RG equation (2.5) as

$$\begin{aligned} \hat{\mathcal{H}}\Big(\frac{\delta}{\delta\phi(x,r)}, \phi(x,r)\Big) = \int d^d x \, \Bigg(& - \int d^d y \frac{\delta}{\delta\phi(x,r)} C_\Lambda^{Pol.}(x,y,r) \frac{\delta V_\Lambda^{Pol.}[\phi]}{\delta\phi(y,r)} \\ & + \frac{1}{2}\int d^d y \frac{\delta}{\delta\phi(x,r)} C_\Lambda^{Pol.}(x,y,r) \frac{\delta}{\delta\phi(y,r)} \Bigg) . \end{aligned} \tag{2.36}$$

$P_\Lambda[\phi(x, r_{UV}); r_{UV}] = Z_\Lambda^{-1} \exp\Big\{ - S_{UV}[\phi(x, r_{UV})]\Big\}$ is the probability density at UV, given the field configuration $\phi(x, r_{UV})$, where the UV action is

$$S_{UV}[\phi(p, r_{UV})] = \frac{1}{2}\int \frac{d^dp}{(2\pi)^d}\phi(p, r_{UV})G^{-1}(p^2)K_\Lambda^{-1}(p^2, r_{UV})\phi(-p, r_{UV}) + S_{UV}^{int}[\phi(p, r_{UV})] \tag{2.37}$$

with the UV partition function,

$$Z_\Lambda[\phi(x, r_{UV}); r_{UV}] = \int D\phi(x, r_{UV}) \exp\Big\{ - S_{UV}[\phi(x, r_{UV})]\Big\}. \tag{2.38}$$

Dual holography as functional renormalization group: Review

Published for SISSA by Springer

Received: April 24, 2015
Accepted: July 19, 2015
Published: August 21, 2015

Ab initio holography

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Explicit derivation based on Wilsonian RG

Abstract: We apply the quantum renormalization group to construct a holographic dual for the U(N) vector model for complex bosons defined on a lattice. The bulk geometry becomes dynamical as the hopping amplitudes which determine connectivity of space are promoted to quantum variables. In the large N limit, the full bulk equations of motion for the dynamical hopping fields are numerically solved for finite systems. From finite size scaling, we show that different phases exhibit distinct geometric features in the bulk. In the insulating phase, the space gets fragmented into isolated islands deep inside the bulk, exhibiting *ultra-locality*. In the superfluid phase, the bulk exhibits a horizon beyond which the geometry becomes *non-local*. Right at the horizon, the hopping fields decay with a universal power-law in coordinate distance between sites, while they decay in slower power-laws with continuously varying exponents inside the horizon. At the critical point, the bulk exhibits a *local* geometry whose characteristic length scale diverges asymptotically in the IR limit.

Keywords: AdS-CFT Correspondence, Holography and condensed matter physics (AdS/CMT), Spontaneous Symmetry Breaking, Renormalization Group

ArXiv ePrint: 1503.06474

Published for SISSA by Springer

Received: April 8, 2016
Revised: August 22, 2016
Accepted: August 27, 2016
Published: September 8, 2016

Horizon as critical phenomenon

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Application of functional RG kernel

Abstract: We show that renormalization group flow can be viewed as a gradual wave function collapse, where a quantum state associated with the action of field theory evolves toward a final state that describes an IR fixed point. The process of collapse is described by the radial evolution in the dual holographic theory. If the theory is in the same phase as the assumed IR fixed point, the initial state is smoothly projected to the final state. If in a different phase, the initial state undergoes a phase transition which in turn gives rise to a horizon in the bulk geometry. We demonstrate the connection between critical behavior and horizon in an example, by deriving the bulk metrics that emerge in various phases of the U(N) vector model in the large N limit based on the holographic dual constructed from quantum renormalization group. The gapped phase exhibits a geometry that smoothly ends at a finite proper distance in the radial direction. The geometric distance in the radial direction measures a complexity: the depth of renormalization group transformation that is needed to project the generally entangled UV state to a direct product state in the IR. For gapless states, entanglement persistently spreads out to larger length scales, and the initial state can not be projected to the direct product state. The obstruction to smooth projection at charge neutral point manifests itself as the long throat in the anti-de Sitter space. The Poincare horizon at infinity marks the critical point which exhibits a divergent length scale in the spread of entanglement. For the gapless states with non-zero chemical potential, the bulk space becomes the Lifshitz geometry with the dynamical critical exponent two. The identification of horizon as critical point may provide an explanation for the universality of horizon. We also discuss the structure of the bulk tensor network that emerges from the quantum renormalization group.

Keywords: AdS-CFT Correspondence, Holography and condensed matter physics (AdS/CMT), Renormalization Group

ArXiv ePrint: 1603.08509

5 Examples: vector model

As a concrete application, we consider the D-dimensional vector model,

$$\mathcal{S} = m^2 \sum_i (\phi_i^* \cdot \phi_i) - \sum_{ij} t_{ij}^{(0)} (\phi_i^* \cdot \phi_j) + \frac{\lambda}{N} \sum_i (\phi_i^* \cdot \phi_i)^2 \,. \tag{5.1}$$

Here ϕ is complex boson field with N flavours, m is the mass of the bosons, and λ is the quartic coupling. $t^{(0)}$ is the hopping parameter, which gives the kinetic term. i, j refer to site indices in a D-dimensional lattice. They are promoted to continuous coordinates in continuum space. For other holographic approaches to the vector models and related conjectures see refs. [32–46].

We write the partition function as

$$Z = \left\langle S_0^* \middle| t^{(0)} \right\rangle, \tag{5.2}$$

where

$$\begin{aligned} \left|S_0\right\rangle &= \int D\boldsymbol{\phi}\; e^{-m^2 \sum_i \boldsymbol{\phi}_i^* \cdot \boldsymbol{\phi}_i} \left|\boldsymbol{\phi}\right\rangle, \\ \left|t^{(0)}\right\rangle &= \int D\boldsymbol{\phi}\; e^{\;\sum_{ij} t_{ij}^{(0)} \boldsymbol{\phi}_i^* \cdot \boldsymbol{\phi}_j - \frac{\lambda}{N} \sum_i \left(\boldsymbol{\phi}_i^* \cdot \boldsymbol{\phi}_i\right)^2} \left|\boldsymbol{\phi}\right\rangle. \end{aligned} \tag{5.3}$$

Because S_0 is real, $\left|S_0^*\right\rangle = \left|S_0\right\rangle$. In the vector model, the single-trace operators are the bi-local operators, $(\boldsymbol{\phi}_i^* \cdot \boldsymbol{\phi}_j)$ [33, 34, 39]. In $\left|t^{(0)}\right\rangle$, one might try to remove the quartic

For the generator of coarse graining transformation, we choose a Hamiltonian,

$$\delta_{\ln\Lambda} O = \ln\Lambda \left[\int_M d^d x \int_M d^d y \left(C_\Lambda^{Pol.}(x,y) \frac{\delta^2}{\delta\phi(x)\delta\phi(y)} + \frac{\delta}{\delta\phi(x)} \left\{ C_\Lambda^{Pol.}(x,y) \frac{\delta V_\Lambda^{Pol.}[\phi]}{\delta\phi(y)} \right\} \right), O \right] \tag{5.5}$$

where $\hat{\boldsymbol{\pi}}_i, \hat{\boldsymbol{\pi}}_i^\dagger$ are conjugate operators of $\hat{\boldsymbol{\phi}}_i, \hat{\boldsymbol{\phi}}_i^\dagger$ with the commutation relation $[\hat{\phi}_{ia}, \hat{\pi}_{jb}] = [\hat{\phi}^\dagger_{ia}, \hat{\pi}^\dagger_{jb}] = i\delta_{ij}\delta_{ab}$ with all other commutators being zero. Here $a, b = 1, 2, \ldots, N$ are flavor indices. $\left|S_0\right\rangle$ is the ground state of $\hat{H}^\dagger$ with energy 0. This guarantees that the partition function is independent of z in $Z = \left\langle S_0 \middle| e^{-z\hat{H}} \middle| t^{(0)} \right\rangle$. Multi-trace operators generated in each infinitesimal time evolution are removed such that

$$e^{-dz\hat{H}}\left|t^{(0)}\right\rangle = \int Dt^{(1)} \;\; e^{-N\sum_{ij} t_{ij}^{*(1)}(t_{ij}^{(1)} - t_{ij}^{(0)}) - dzN\mathcal{H}[t^{*(1)}, t^{(0)}]}\left|t^{(1)}\right\rangle. \tag{5.6}$$

$$\left|t^{(0)}\right\rangle = \int D\boldsymbol{\phi} Dt \; e^{\;\sum_{ij} t_{ij}\boldsymbol{\phi}_i^* \cdot \boldsymbol{\phi}_j - Nt_{ij}^*(t_{ij} - t_{ij}^{(0)}) - N\lambda\sum_i (t_{ii}^*)^2}\left|\boldsymbol{\phi}\right\rangle.$$

Baker–Campbell–Hausdorff formula

Contents hide

From Wikipedia, the free encyclopedia

In mathematics, the **Baker–Campbell–Hausdorff formula** gives the value of Z that solves the equation

$$e^X e^Y = e^Z$$

for possibly noncommutative X and Y in the Lie algebra of a Lie group. There are various ways of writing the formula, but all ultimately yield an expression for Z in Lie algebraic terms, that is, as a formal series (not necessarily convergent) in X and Y and iterated commutators thereof. The first few terms of this series are:

$$Z = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] + \frac{1}{12}[Y,[Y,X]]$$

where "$\cdots$" indicates terms involving higher commutators of X anc
algebra $\mathfrak{g}$ of a Lie group G, the series is convergent. Meanwhile, eve
expressed as $g = e^X$ for a small X in $\mathfrak{g}$. Thus, we can say that *near*
$e^X e^Y = e^Z$—can be expressed in purely Lie algebraic terms. The B
comparatively simple proofs of deep results in the Lie group–Lie alg

If X and Y are sufficiently small $n \times n$ matrices, then Z can be cor
and the logarithm can be computed as power series. The point of th
nonobvious claim that $Z := \log\left(e^X e^Y\right)$ can be expressed as a seri

Modern expositions of the formula can be found in, among other pl

History [edit]

The formula is named after Henry Frederick Baker, John Edward Ca
i.e. that only commutators and commutators of commutators, ad in

$$\begin{aligned} Z(X,Y) &= \log(\exp X \exp Y) \\ &= \boxed{X} + Y \boxed{+ \frac{1}{2}[X,Y]} + \frac{1}{12}\left([X,[X,Y]] \boxed{+ [Y,[Y,X]]}\right) \\ &\quad - \frac{1}{24}[Y,[X,[X,Y]]] \\ &\quad \boxed{- \frac{1}{720}\left([Y,[Y,[Y,[Y,X]]]]\right.} + \left.[X,[X,[X,[X,Y]]]]\right) \\ &\quad + \frac{1}{360}\left([X,[Y,[Y,[Y,X]]]] + [Y,[X,[X,[X,Y]]]]\right) \\ &\quad + \frac{1}{120}\left([Y,[X,[Y,[X,Y]]]] + [X,[Y,[X,[Y,X]]]]\right) \\ &\quad + \frac{1}{240}\left([X,[Y,[X,[Y,[X,Y]]]]]\right) \\ &\quad + \frac{1}{720}\left([X,[Y,[X,[X,[X,Y]]]]] - [X,[X,[Y,[Y,[X,Y]]]]]\right) \\ &\quad + \frac{1}{1440}\left([X,[Y,[Y,[Y,[X,Y]]]]] - [X,[X,[Y,[X,[X,Y]]]]]\right) + \cdots \end{aligned}$$

Appearance hide

Birthday mode (Baby Globe)

- Disabled
- Enabled

Learn more about Birthday mode

Text

Here $\mathcal{H}[t^*, t]$ is the coherent state representation of the quantum Hamiltonian,

$$\hat{\mathcal{H}} = \sum_i \left[-\frac{2}{m^2} t_{ii} + \frac{4\lambda\left(1+\frac{1}{N}\right)}{m^2} t^\dagger_{ii} - 4\lambda\left(t^\dagger_{ii}\right)^2 - \frac{8\lambda^2}{m^2}\left(t^\dagger_{ii}\right)^3 \right]$$
$$+ \sum_{ij}\left[2 + \frac{4\lambda}{m^2}(t^\dagger_{ii} + t^\dagger_{jj})\right] t^\dagger_{ij}\ t_{ij} - \frac{2}{m^2}\sum_{ijk}\left[t^\dagger_{kj} t_{ki} t_{ij}\right], \tag{5.7}$$

where $t^\dagger_{ij}$ and t_{ij} satisfy the commutation relation, $[t_{ij}, t^\dagger_{kl}] = \frac{1}{N}\delta_{ik}\delta_{jl}$. t_{ij} ($t^\dagger_{ij}$) is an operator that annihilates (creates) a connection between sites i and j. t_{ij}'s describe dynamical geometry of the D-dimensional space as the distance between two points is determined by the connectivity formed by the hopping fields. This will become clear through an explicit derivation of metric from the hopping field in the following subsections. For finite N, the strength of the connection is quantized. Therefore, $\hat{\mathcal{H}}$ describes the evolution of the quantum geometry under the RG flow. The partition function is given by

$$Z = \lim_{z\to\infty} \left\langle S_0 \middle| e^{-z\hat{H}} \middle| t^{(0)} \right\rangle = \int Dt(z)\ \ \left\langle S_0 \middle| t(\infty) \right\rangle\ e^{-N\int_0^\infty dz\left(t^*_{ij}\partial_z t_{ij} + \mathcal{H}[t^*,t]\right)}\Big|_{t_{ij}(0)=t^{(0)}_{ij}}, \tag{5.8}$$

where $\int Dt(z)$ sums over all RG paths defined in the space of the single-trace operators.

t^*_{ij} as $\bar{t}_{ij}$, $\bar{p}_{ij}$. The equations of motion for $\bar{t}_{ij}$ and $\bar{p}_{ij}$ are

$$\begin{aligned}
\partial_z \bar{t}_{ij} \;&=\; -2\bigg\{ \frac{2\lambda\,\delta_{ij}}{m^2} - \delta_{ij}\left[4\lambda + \frac{12\lambda^2}{m^2}\bar{p}_{ii}\right]\bar{p}_{ii} + \frac{2\lambda\,\delta_{ij}}{m^2}\sum_k\left(\bar{t}_{ik}\bar{p}_{ik} + \bar{t}_{ki}\bar{p}_{ki}\right) \\
&\qquad + \left[1 + \frac{2\lambda}{m^2}\left(\bar{p}_{ii} + \bar{p}_{jj}\right)\right]\bar{t}_{ij} - \frac{1}{m^2}\sum_k \bar{t}_{ik}\bar{t}_{kj}\bigg\}, \\
\partial_z \bar{p}_{ij} \;&=\; 2\left\{-\frac{\delta_{ij}}{m^2} + \left[1 + \frac{2\lambda}{m^2}\left(\bar{p}_{ii} + \bar{p}_{jj}\right)\right]\bar{p}_{ij} - \frac{1}{m^2}\sum_k\left(\bar{p}_{ik}\bar{t}_{jk} + \bar{t}_{ki}\bar{p}_{kj}\right)\right\} \qquad (5.9)
\end{aligned}$$

with the two boundary conditions

$$\bar{t}_{ij}(0) \;=\; t^{(0)}_{ij}, \qquad (5.10)$$

$$\bar{p}_{ij}(\infty) \;=\; \frac{1}{N}\,\frac{\partial \ln\left\langle S_0 \middle| \bar{t}\right\rangle}{\partial \bar{t}_{ij}}\bigg|_{z=\infty}. \qquad (5.11)$$

5.1 Zero density

$$\bar{T}_q(z) = \frac{2\lambda}{m^2} + m^2 + \frac{2\lambda}{m^2} e^{-2z}(m^2 \bar{p}_0(0) - 1) - m^2 \frac{\delta^2 + q^2}{(1 - e^{-2z})(q^2 + \delta^2) + m^2 e^{-2z}},$$
$$\bar{P}_q(z) = \frac{e^{-2z}}{q^2 + \delta^2} + \frac{1 - e^{-2z}}{m^2}. \tag{5.21}$$

$$\delta^2 \equiv -\bar{T}_0(0) + m^2 + 2\lambda \bar{p}_0(0)$$

From the saddle point solution in eq. (5.21), one can compute the metric in the bulk. In order to extract the metric, it is useful to consider fluctuations of the hopping fields around the saddle point,

$$\tilde{t}_{ij} = t_{ij} - \bar{t}_{ij}, \quad \tilde{p}_{ij} = t^*_{ij} - \bar{p}_{ij}. \tag{5.22}$$

The quadratic action for the fluctuations is given by

$$\begin{aligned} S_2 = \int dz \Bigg\{ & -4\lambda \sum_i \left(1 + \frac{6\lambda}{m^2}\bar{p}_0\right) \tilde{p}_{ii}^2 \\ & + \sum_{ij} \left[\tilde{p}_{ij} \left(\partial_z + 2 + \frac{8\lambda}{m^2}\bar{p}_0\right) \tilde{t}_{ij} + \frac{4\lambda}{m^2}(\tilde{p}_{ii} + \tilde{p}_{jj}) \left(\bar{p}_{ij}\tilde{t}_{ij} + \bar{t}_{ij}\tilde{p}_{ij}\right)\right] \\ & - \frac{2}{m^2} \sum_{ijk} \left(\bar{p}_{ij}\tilde{t}_{ik}\tilde{t}_{kj} + \bar{t}_{ik}\tilde{p}_{ij}\tilde{t}_{kj} + \bar{t}_{kj}\tilde{p}_{ij}\tilde{t}_{ik}\right) \Bigg\}. \end{aligned} \tag{5.23}$$

Here $\tilde{p}_{ij}$ is the conjugate momentum of $\tilde{t}_{ij}$. The action includes terms that are quadratic

The equations of motion for $\tilde{t}_{ij}, \tilde{p}_{ij}$ are

$$\left(\partial_z + 2 + \frac{8\lambda}{m^2}\bar{p}_0(z)\right)\tilde{t}_{ij} - \frac{2}{m^2}\left(\bar{t}_{ik}\tilde{t}_{kj} + \bar{t}_{kj}\tilde{t}_{ik}\right) + \frac{4\lambda}{m^2}\bar{t}_{ij}(\tilde{p}_{ii} + \tilde{p}_{jj}) \tag{5.24}$$

$$+\delta_{ij}\left[-8\lambda\left(1 + \frac{6\lambda}{m^2}\bar{p}_0(z)\right)\tilde{p}_{ii} + \frac{4\lambda}{m^2}\left(\bar{p}_{ik}(\tilde{t}_{ik} + \tilde{t}_{ki}) + \bar{t}_{ik}(\tilde{p}_{ik} + \tilde{p}_{ki})\right)\right] = 0,$$

$$\left(-\partial_z + 2 + \frac{8\lambda}{m^2}\bar{p}_0(z)\right)\tilde{p}_{ij} + \frac{4\lambda}{m^2}\bar{p}_{ij}(\tilde{p}_{ii} + \tilde{p}_{jj}) - \frac{2}{m^2}\left(\bar{p}_{ik}\tilde{t}_{jk} + \bar{p}_{kj}\tilde{t}_{ki} + \bar{t}_{ki}\tilde{p}_{kj} + \bar{t}_{jk}\tilde{p}_{ik}\right) = 0. \tag{5.25}$$

Once the equation of motion for $\tilde{p}_{ij}$ is solved, the equations for $\tilde{t}_{ij}$ in general include second order derivatives in z due to the quadratic kinetic term in eq. (5.23). However, we don't

5.1.1 Gapped state

$$\left(m\sqrt{g^{zz}}\partial_z + 4\delta^2 - \frac{2m^2}{1-\frac{g^{\mu\nu}\partial_\mu\partial_\nu}{m^2}} - \frac{2m^2}{1-\frac{g^{\mu\nu}\partial'_\mu\partial'_\nu}{m^2}}\right)\tilde{t}^A(x,x',z) = 0, \tag{5.33}$$

where $g^{zz} = m^2 A(z)^{-2}$, $g^{00} = g^{11} = \ldots = g^{D-1,D-1} = B(z)$. eq. (5.33) describes the bi-local field that propagates diffusively with the diffusion coefficient $D_0 \sim 1/m$ in a curved space [47] with the metric,

$$ds^2 = \left(\frac{1}{1+\left(\frac{\delta}{m}e^z\right)^2}\right)^2 \frac{dz^2}{m^2} + \left(\left(\frac{\delta}{m}\right)^2 + e^{-2z}\right)\sum_{\mu=0}^{D-1} dx^\mu dx^\mu. \tag{5.34}$$

5.1.2 Gapless states

infinitely sharp, and $\bar{T}_q(z)$ becomes non-analytic at $q = 0$. In other words, $z \to \infty$ and $q \to 0$ limits do not commute. If one takes the small q limit first at a large but finite z, one obtains

$$\bar{T}_q(z) = \bar{T}_0(z) - e^{2z}q^2 + \frac{(e^{2z}q^2)^2}{m^2} + \dots \tag{5.38}$$

The coefficients of the q dependent terms grow indefinitely with increasing z. This implies that $\bar{t}_{ij}$ becomes non-local in the large z limit as can be seen by setting $\delta = 0$ in eq. (5.30).

the hopping field satisfy the same equation in eq. (5.33). With $\delta = 0$, eq. (5.34) is reduced to the AdS_{D+1} metric,

$$ds^2 = \frac{dz^2}{m^2} + e^{-2z} \sum_{\mu=0}^{D-1} dx^\mu dx^\mu. \tag{5.39}$$

The curvature length scale of the bulk space is $1/m$. Higher derivative terms in eq. (5.33)

5.2 Finite density

Now we consider the case with a non-zero chemical potential. In the D-dimensional Euclidean lattice, we pick one of the direction to be an imaginary time. A chemical potential enters in the hopping field as an imaginary gauge potential along the Euclidean time direction. With a chemical potential μ, the hopping field is modified as

$$\bar{T}_q(0) = \bar{T}_0(0) - (q_0 - i\mu)^2 - |\vec{q}|^2 \tag{5.40}$$

at the UV boundary, where $\vec{q} = (q_1, q_2, \ldots, q_{D-1})$ represents the $(D-1)$-dimensional momentum. The hopping field in the bulk becomes

$$\bar{T}_q(z) = \frac{2\lambda}{m^2} + m^2 + \frac{2\lambda}{m^2} e^{-2z}(m^2 p_0(0) - 1) - m^2 \frac{\delta^2 + (q_0 - i\mu)^2 + |\vec{q}|^2}{(1 - e^{-2z})\left((q_0 - i\mu)^2 + |\vec{q}|^2 + \delta^2\right) + m^2 e^{-2z}}. \tag{5.41}$$

If $\delta > \mu$, the gap remains non-zero, and the system is in the insulating state. The behaviour of $\bar{t}_{ij}$ in the bulk is essentially the same as the one in the insulating state at zero

To extract the metric in the bulk, we again consider fluctuations around the saddle point, $\tilde{t}_{ij} = t_{ij} - \bar{t}_{ij}$. With a non-zero chemical potential, $\bar{t}_{ij}$ is no longer identical to $\bar{t}_{ji}$. Since the anti-symmetric component is not decoupled from the symmetric component, eq. (5.26) does not hold. However, eq. (5.25) is still simplified in the large $|i-j|$ limit, where only the first two terms are important. The continuum field defined by $\tilde{t}(x, x', z) = e^{2z}\tilde{t}_{ij}(z)$ satisfies

$$\left(m^2\partial_z - \frac{2m^2}{1-\frac{1}{m^2}\left(2\mu\sqrt{g^{00}}\partial_0 + g^{\alpha\beta}\partial_\alpha\partial_\beta\right)} - \frac{2m^2}{1-\frac{1}{m^2}\left(2\mu\sqrt{g^{00}}\partial_0' + g^{\alpha\beta}\partial_\alpha'\partial_\beta'\right)}\right)\tilde{t}(x, x', z) = 0 \tag{5.43}$$

in the large $|x - x'|$ limit, where $g^{00} = e^{4z}$, $g^{11} = g^{22} = \ldots = g^{D-1,D-1} = e^{2z}$, and α, β runs from 1 to $D-1$. Therefore the fluctuation fields propagate in the Lifshitz background [54–56] whose metric is given by

$$ds^2 = \frac{dz^2}{m^2} + e^{-4z}(dx^0)^2 + e^{-2z}d\vec{x}^2. \tag{5.44}$$

The geometry is invariant under the anisotropic scaling with the dynamical critical exponent 2,

$$z \to z + s, \quad x^0 \to s^2 x^0, \quad \vec{x} \to s\vec{x}. \tag{5.45}$$

The non-locality of the hopping amplitude in the large z limit gives rise to the horizon at $z = \infty$.

Some remarks

- Black hole geometry has not been derived yet. Cardy formula is not checked out explicitly in this case.
- Unfortunately, the resulting physics (phase diagram, correlation functions, critical exponents, and etc.) are given by mean-field types (i.e., $N \to \infty$) essentially, and thus, no gains practically speaking.
- We need to develop a nonperturbative theoretical framework, of course.

Beyond the mean-field theory

2.5 Nonperturbative generalization

As seen in the previous section, it is disappointing to have just a mean-field theory result in this dual holographic construction. Certainly, we need to improve the RG scheme in the microscopic derivation. In this section, we propose a self-consistent RG scheme, introducing an auxiliary field into the holographic dual effective field theory and performing the RG transformation in a self-consistent way. An idea is to replace the Hamiltonian kernel in

$$P_\Lambda[\phi(p,r_{IR}),\Sigma(p,r_{IR});r_{IR}] = \int D\phi(p,r_{UV})D\Sigma(p,r_{UV})\ \mathcal{J}[\Sigma(p,r)]$$
$$\exp\Big\{-\int_{r_{UV}}^{r_{IR}} dr\hat{\mathcal{H}}\Big(\frac{\delta}{\delta\phi(p,r)},\phi(p,r),\frac{\delta}{\delta\Sigma(p,r)},\Sigma(p,r)\Big)\Big\}P_\Lambda[\phi(p,r_{UV}),\Sigma(p,r_{UV});r_{UV}] \tag{2.49}$$

Explicit FRG derivation

II. DERIVATION OF THE PATH INTEGRAL FORMULATION FOR THE FUNCTIONAL RG ANALYSIS

In this section, we present explicit derivation of the path integral formulation for the functional RG analysis. We start from an O(N) vector model,

$$Z = \int D\phi_\alpha(x) \exp\Big[- \int d^d x \Big\{ \sum_{\alpha=1}^{N} \Big(\partial_\mu \phi_\alpha(x)\Big)^2 + m^2 \sum_{\alpha=1}^{N} \phi_\alpha^2(x) + \frac{u}{2N} \sum_{\alpha=1}^{N} \phi_\alpha^2(x) \sum_{\beta=1}^{N} \phi_\beta^2(x) \Big\}\Big]. \tag{1}$$

Here, α and β are flavor indexes for the O(N) symmetry, which run from 1 to N.

$$Z = \int D\phi_\alpha(x) D\varphi(x) \exp\Big[- \int d^d x \Big\{ \phi_\alpha(x)\Big(- \partial_\mu^2 + m^2 - i\varphi(x)\Big)\phi_\alpha(x) + \frac{N}{2u}\varphi^2(x) \Big\}\Big].$$

This is our starting point for the RG transformation.

To investigate the scale-dependent behavior of the theory, we implement the Wilsonian RG transformation. The matter field $\phi_\alpha(x)$ is decomposed into low-energy and high-energy modes. By performing a Gaussian integration over the high-energy fluctuations within an energy window scaled by the RG interval dz, we obtain the following effective action for the low-energy modes:

$$
Z = \int D\phi_\alpha(x) D\varphi(x) \exp\Big[- \int d^d x \Big\{ \phi_\alpha(x) \Big(-\partial_\mu^2 + m^2 - i\varphi(x) \Big) \phi_\alpha(x) + \frac{N}{2u}\varphi^2(x)
$$

$$
+ 2\alpha dz \frac{N}{4} \ln\Big(-\partial_\mu^2 + m^2 - i\varphi(x) \Big) \Big\} \Big]. \tag{2.3}
$$

Next, we address the scale-dependent evolution of the collective sector. The field $\varphi(x)$ is replaced with an initial configuration $\varphi^{(0)}(x)$, and its corresponding fluctuations are introduced. Integrating out the high-energy collective modes and shifting the field via $\varphi^{(1)}(x) \to \varphi^{(1)}(x) - \varphi^{(0)}(x)$ yield:

$$Z = \int D\phi_\alpha(x) D\varphi^{(1)}(x) D\varphi^{(0)}(x) \exp\Big[- \int d^d x \Big\{ \phi_\alpha(x) \Big(-\partial_\mu^2 + m^2 - i\varphi^{(1)}(x) \Big) \phi_\alpha(x) + \frac{N}{2u} \varphi^{(0)2}(x)$$
$$+ \frac{1}{2\beta dz} \frac{N}{2} \Big(\varphi^{(1)}(x) - \varphi^{(0)}(x) \Big) \Big\{ \frac{1}{u} + \alpha\beta(dz)^2 \Big(-\partial_\mu^2 + m^2 - i\varphi^{(0)}(x) \Big)^{-2} \Big\} \Big(\varphi^{(1)}(x) - \varphi^{(0)}(x) \Big)$$
$$-2\alpha dz \frac{N}{4} \Big(-\partial_\mu^2 + m^2 - i\varphi^{(0)}(x) \Big)^{-1} i \Big(\varphi^{(1)}(x) - \varphi^{(0)}(x) \Big) + 2\alpha dz \frac{N}{4} \ln\Big(-\partial_\mu^2 + m^2 - i\varphi^{(0)}(x) \Big) \Big\} \Big], \quad (2.4)$$

where β serves as the control parameter for the RG flow of the collective field. The quadratic term in $\big(\varphi^{(1)} - \varphi^{(0)}\big)$ contains the inverse of the Random Phase Approximation (RPA) propagator, which includes the one-loop level RG corrections [30, 31, 32]. This recursive structure defines the step-by-step evolution for the continuous bulk dynamics.

$$
\begin{aligned}
Z = \int D\phi_\alpha(x) D\varphi(x,z) \exp\Bigg[-\int d^d x \Bigg\{ \phi_\alpha(x)\Big(-\partial_\mu^2 + m^2 - i\varphi(x,z_f)\Big)\phi_\alpha(x) + \frac{N}{2u}\varphi^2(x,0) \Bigg\} \\
-N \int_0^{z_f} dz \int d^d x \Bigg\{ \frac{1}{4\beta}\Big(\partial_z\varphi(x,z)\Big)\left(\frac{1}{u} + \alpha\beta\varepsilon^2\Big(-\partial_\mu^2 + m^2 - i\varphi(x,z)\Big)^{-2}\right)\Big(\partial_z\varphi(x,z)\Big) \\
-i\alpha\frac{\varepsilon}{2}\Big(-\partial_\mu^2 + m^2 - i\varphi(x,z)\Big)^{-1}\Big(\partial_z\varphi(x,z)\Big) + \frac{\alpha}{2}\ln\Big(-\partial_\mu^2 + m^2 - i\varphi(x,z)\Big) \Bigg\}\Bigg],
\end{aligned}
$$

Based on this derivation, the holographic dual effective field theory is defined as:

$$
\begin{aligned}
Z = \int D\phi_\alpha(x) D\varphi(x,z) \exp \Big[- \int d^d x \Big\{ \phi_\alpha(x) G^{-1}(x, z_f) \phi_\alpha(x) + \frac{N}{2u} \varphi^2(x,0) \Big\} \\
- N \int_0^{z_f} dz \int d^d x \ \mathcal{L}_{bulk} \Big],
\end{aligned} \tag{2.6}
$$

where the bulk Lagrangian density $\mathcal{L}_{\text{bulk}}$ is given by:

$$
\mathcal{L}_{bulk} = \frac{1}{4\beta} \left(\partial_z \varphi(x,z) \right) D^{-1}(x,z) \left(\partial_z \varphi(x,z) \right) - i\alpha \frac{\epsilon}{2} G(x,z) \left(\partial_z \varphi(x,z) \right) + \frac{\alpha}{2} \ln G^{-1}(x,z). \tag{2.7}
$$

$$
\begin{aligned}
\Big(- \partial_\mu^2 + m^2 - i\varphi(x,z) \Big) G(x,z) = \delta^{(d)}(x), \\
\Big(\frac{1}{u} + \alpha\beta\varepsilon^2 G^2(x,z) \Big) D(x,z) = \delta^{(d)}(x).
\end{aligned}
$$

3. Summary of equation of motion

$$-\partial_z\left\{D^{-1}(x,z)\Big(\partial_z\varphi(x,z)\Big)\right\} = i\alpha\beta G(x,z).$$

$$\partial_{z_f}\varphi(x,z_f) = i\beta(1+\varepsilon)D(x,z_f)G(x,z_f),$$
$$[\partial_z\varphi(x,z)]_{z=0} = \frac{2\beta}{u}D(x,0)\varphi(x,0) + i\varepsilon\beta D(x,0)G(x,0).$$

FRG formulation

$$\mathcal{H}_{RG} = N\int d^dx\left\{-\beta\frac{\delta}{\delta\varphi(x,z)}D(x,z)\frac{\delta}{\delta\varphi(x,z)} + \frac{\delta}{\delta\varphi(x,z)}\varphi(x,z) - \alpha\frac{\delta}{\delta\phi(x,z)}G(x,z)\frac{\delta}{\delta\phi(x,z)} + \frac{\delta}{\delta\phi(x,z)}\phi(x,z)\right\}$$

$$\delta_{\ln\Lambda}O = \ln\Lambda\left[\int_M d^dx\int_M d^dy\left(C_\Lambda^{Pol.}(x,y)\frac{\delta^2}{\delta\phi(x)\delta\phi(y)} + \frac{\delta}{\delta\phi(x)}\left\{C_\Lambda^{Pol.}(x,y)\frac{\delta V_\Lambda^{Pol.}[\phi]}{\delta\phi(y)}\right\}\right), O\right]$$

$$\frac{\partial}{\partial z}P[\varphi(x,z),\phi(x,z)] = N\int d^dx\left\{-\beta\frac{\delta}{\delta\varphi(x,z)}D(x,z)\frac{\delta}{\delta\varphi(x,z)} + \frac{\delta}{\delta\varphi(x,z)}\varphi(x,z)\right.$$
$$\left.-\alpha\frac{\delta}{\delta\phi(x,z)}G(x,z)\frac{\delta}{\delta\phi(x,z)} + \frac{\delta}{\delta\phi(x,z)}\phi(x,z)\right\}P[\varphi(x,z),\phi(x,z)].$$

$$P[\varphi(x,z),\phi(x,z)] = \int D\varphi(x,0)D\phi(x,0)\exp\left\{-\int_0^z dw\mathcal{H}_{RG}\right\}P[\varphi(x,0),\phi(x,0)].$$

$$\mathcal{H}_{RG} = N\int d^dx\Big\{-\beta\Pi_\varphi(x,w)D(x,w)\Pi_\varphi(x,w) + \Pi_\varphi(x,w)\varphi(x,w) - \alpha\Pi_\phi(x,w)G(x,w)\Pi_\phi(x,w) + \Pi_\phi(x,w)\phi(x,w)\Big\}$$

$$P[\varphi(x,0),\phi(x,0)] = \exp\Big[-N\int d^dx\Big\{\phi(x,0)G^{-1}(x,0)\phi(x,0) + \frac{1}{2u}\varphi^2(x,0)\Big\}\Big]$$

$$Z = \int D\Pi_\varphi(x,z)D\varphi(x,z)\exp\Bigg[-N\int d^dx\Big\{\frac{1+\varepsilon\alpha}{2}\ln G^{-1}(x,z_f) - \frac{\varepsilon\alpha}{2}\ln G^{-1}(x,0) + \frac{1}{2u}\varphi^2(x,0)\Big\}$$
$$-N\int_0^{z_f}dz\int d^dx\Big\{\beta\Pi_\varphi(x,z)D(x,z)\Pi_\varphi(x,z) - i\Pi_\varphi(x,z)\Big(\partial_z\varphi(x,z)\Big) + \frac{\alpha}{2}\ln G^{-1}(x,z)\Big\}\Bigg].$$

Normal modes

$$\varphi(x,z) = \varphi_c(x,z) + \delta\varphi(x,z)$$

$$-\partial_z\Big\{D^{-1}(x,z)\Big(\partial_z\delta\varphi(x,z)\Big)\Big\} = -\alpha\beta G^2(x,z)\delta\varphi(x,z)$$

$$\Big(-\partial_\mu^2 + m^2 - i\varphi_c(x,z)\Big)G(x,z) = \delta^{(d)}(x),$$
$$\Big(\frac{1}{u} + \alpha\beta\varepsilon^2 G^2(x,z)\Big)D(x,z) = \delta^{(d)}(x).$$

How can we reconstruct the corresponding geometry from both single-particle and two-particle spectra?

JHEP

Published for SISSA by Springer

Received: March 21, 2020

Revised: October 18, 2020

Accepted: May 11, 2021

Published: May 28, 2021

Entanglement transfer from quantum matter to classical geometry in an emergent holographic dual description of a scalar field theory

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Abstract: Applying recursive renormalization group transformations to a scalar field theory, we obtain an effective quantum gravity theory with an emergent extra dimension, described by a dual holographic Einstein-Klein-Gordon type action. Here, the dynamics of both the dual order-parameter field and the metric tensor field originate from density-density and energy-momentum tensor-tensor effective interactions, respectively, in the recursive renormalization group transformation, performed approximately in the Gaussian level. This linear approximation in the recursive renormalization group transformation for the gravity sector gives rise to a linearized quantum Einstein-scalar theory along the z-directional emergent space. In the large N limit, where N is the flavor number of the original scalar fields, quantum fluctuations of both dynamical metric and dual scalar fields are suppressed, leading to a classical field theory of the Einstein-scalar type in $(D+1)$-spacetime dimensions. We show that this emergent background gravity describes the renormalization group flows of coupling functions in the UV quantum field theory through the extra dimension. More precisely, the IR boundary conditions of the gravity equations correspond to the renormalization group β-functions of the quantum field theory, where the infinitesimal distance in the extra-dimensional space is identified with an energy scale for the renormalization group transformation. Finally, we also show that this dual holographic formulation describes quantum entanglement in a geometrical way, encoding the transfer of quantum entanglement from quantum matter to classical gravity in the large N limit. We claim that this entanglement transfer serves as a microscopic foundation for the emergent holographic duality description.

Keywords: AdS-CFT Correspondence, Renormalization Group, Holography and condensed matter physics (AdS/CMT), Resummation

ArXiv ePrint: 2003.00165

1. Introduction of RG flows

2. UV & IR boundary conditions

https://doi.org/10.1007/JHEP05(2021)260

$$Z = \int D\phi_\alpha(x) \exp\Big\{ - S_{\mathrm{UV}}[\phi_\alpha(x); g^B_{\mu\nu}(x)]\Big\} \tag{2.1}$$

and the effective action is

$$S_{\mathrm{UV}}[\phi_\alpha(x); g^B_{\mu\nu}(x)] = \int d^D x \sqrt{g_B}\Big\{ g_B^{\mu\nu}(\partial_\mu\phi_\alpha)(\partial_\nu\phi_\alpha) + m^2\phi_\alpha^2 + \xi R_B \phi_\alpha^2 + \boxed{\frac{u}{2N}\phi_\alpha^2\phi_\beta^2} + \boxed{\frac{\lambda}{2N}T_{\mu\nu}T^{\mu\nu}}\Big\}. \tag{2.2}$$

$$Z = Z_\Lambda \int D\varphi(x,z) Dg_{\mu\nu}(x,z) \exp\Big\{ - S_{\mathrm{UV}}[g_{\mu\nu}(x,0), \varphi(x,0)] - S_{\mathrm{IR}}[g_{\mu\nu}(x,z_f), \varphi(x,z_f)] - S_{\mathrm{Bulk}}[g_{\mu\nu}(x,z), \varphi(x,z)]\Big\}. \tag{2.4}$$

$$\varphi \leftrightarrow \phi_\alpha^2 \quad \& \quad g_{\mu\nu} \leftrightarrow T^{\mu\nu}$$

$$e^{-\int d^D x \sqrt{g_B}\, \frac{u}{2N}\phi_\alpha^2\phi_\beta^2} = \int D\varphi\, e^{-\int d^D x \sqrt{g_B}\, \left(\frac{N}{2u}\varphi^2 - i\varphi\phi_\alpha^2\right)}$$

$$\varphi \rightarrow \boxed{\varphi^{(0)}} + \delta\varphi, \qquad g_{\mu\nu} \rightarrow \boxed{g_{\mu\nu}^{(0)}} + \delta g_{\mu\nu}, \qquad \phi_\alpha \rightarrow \phi_\alpha + \delta\phi_\alpha$$

$$\int D\delta\varphi \rightarrow \phi_\alpha^2\phi_\beta^2 \rightarrow \varphi^{(1)}, \qquad \int D\delta g_{\mu\nu} \rightarrow T_{\mu\nu}^2 \rightarrow g_{\mu\nu}^{(1)}, \qquad \int D\delta\phi_\alpha \rightarrow V_{eff}[\varphi^{(0)}, g_{\mu\nu}^{(0)}]$$

Heat kernel calculation

$$\varphi^{(1)} \rightarrow \boxed{\varphi^{(1)}} + \delta\varphi, \qquad g_{\mu\nu}^{(1)} \rightarrow \boxed{g_{\mu\nu}^{(1)}} + \delta g_{\mu\nu}, \qquad \phi_\alpha \rightarrow \phi_\alpha + \delta\phi_\alpha$$

$$\int D\delta\varphi \rightarrow \phi_\alpha^2\phi_\beta^2 \rightarrow \varphi^{(2)}, \qquad \int D\delta g_{\mu\nu} \rightarrow T_{\mu\nu}^2 \rightarrow g_{\mu\nu}^{(2)}, \qquad \int D\delta\phi_\alpha \rightarrow V_{eff}[\varphi^{(1)}, g_{\mu\nu}^{(1)}]$$

Heat kernel calculation

Just doing Gaussian integral
in the presence of general background fields
$\varphi^{(k-1)}$ & $g_{\mu\nu}^{(k-1)}$ *for every RG step* (k)

PUBLISHED FOR SISSA BY SPRINGER
RECEIVED: *March 21, 2020*
REVISED: *October 18, 2020*
ACCEPTED: *May 11, 2021*
PUBLISHED: *May 28, 2021*

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ABSTRACT: Applying recursive renormalization group transformations to a scalar field theory, we obtain an effective quantum gravity theory with an emergent extra dimension, described by a dual holographic Einstein-Klein-Gordon type action. Here, the dynamics of both the dual order-parameter field and the metric tensor field originate from density-density and energy-momentum tensor-tensor effective interactions, respectively, in the recursive renormalization group transformation, performed approximately in the Gaussian level. This linear approximation in the recursive renormalization group transformation

$$dz \sum_{k=1}^{f} \frac{g_{\mu\nu}^{(k)}(x) - g_{\mu\nu}^{(k-1)}(x)}{dz[k-(k-1)]} \to \int_0^{z_f} dz\, \partial_z g_{\mu\nu}(x,z)$$

group flows of coupling functions in the UV quantum field theory through the extra dimension. More precisely, the IR boundary conditions of the gravity equations correspond to the renormalization group β-functions of the quantum field theory, where the infinitesimal distance in the extra-dimensional space is identified with an energy scale for the renormalization group transformation. Finally, we also show that this dual holographic formulation describes quantum entanglement in a geometrical way, encoding the transfer of quantum entanglement from quantum matter to classical gravity in the large N limit. We claim that this entanglement transfer serves as a microscopic foundation for the emergent holographic duality description.

KEYWORDS: AdS-CFT Correspondence, Renormalization Group, Holography and condensed matter physics (AdS/CMT), Resummation

ARXIV EPRINT: 2003.00165

1. Introduction of RG flows
2. UV & IR boundary conditions

https://doi.org/10.1007/JHEP05(2021)260

$$Z = \int D\phi_\alpha(x) \exp\Big\{ - S_{\rm UV}[\phi_\alpha(x); g_{\mu\nu}^B(x)] \Big\} \tag{2.1}$$

and the effective action is

$$S_{\rm UV}[\phi_\alpha(x); g_{\mu\nu}^B(x)] = \int d^D x \sqrt{g_B} \Big\{ g_B^{\mu\nu} (\partial_\mu \phi_\alpha)(\partial_\nu \phi_\alpha) + m^2 \phi_\alpha^2 + \xi R_B \phi_\alpha^2 + \boxed{\frac{u}{2N} \phi_\alpha^2 \phi_\beta^2} + \boxed{\frac{\lambda}{2N} T_{\mu\nu} T^{\mu\nu}} \Big\}. \tag{2.2}$$

$$Z = Z_\Lambda \int D\varphi(x,z) D g_{\mu\nu}(x,z) \exp\Big\{ - S_{\rm UV}[g_{\mu\nu}(x,0), \varphi(x,0)] - S_{\rm IR}[g_{\mu\nu}(x,z_f), \varphi(x,z_f)] - S_{\rm Bulk}[g_{\mu\nu}(x,z), \varphi(x,z)] \Big\}. \tag{2.4}$$

$\varphi \leftrightarrow \phi_\alpha^2$ & $g_{\mu\nu} \leftrightarrow T^{\mu\nu}$

$$S_{\rm Bulk}[g_{\mu\nu}(x,z), \varphi(x,z)] = N \int_0^{z_f} dz \int d^D x \sqrt{g(x,z)} \Big\{ + \boxed{\frac{1}{2u} [\partial_z \varphi(x,z)]^2} + \boxed{\frac{\mathcal{C}_\varphi}{2} g^{\mu\nu}(x,z) [\partial_\mu \varphi(x,z)][\partial_\nu \varphi(x,z)] + \mathcal{C}_\xi R(x,z) [\varphi(x,z)]^2} - \boxed{\frac{1}{2\lambda} \Big(\partial_z g^{\mu\nu}(x,z) - g^{\mu\nu'}(x,z) (\partial_{\nu'} \partial_{\mu'} G_{xx'}[g_{\mu\nu}(x,z), \varphi(x,z)])_{x' \to x} g^{\mu'\nu}(x,z) \Big)^2} + \boxed{\frac{1}{2\kappa} \Big(R(x,z) - 2\Lambda \Big)} \Big\}. \tag{2.5}$$

Heat kernel calculation

RG − improved
Ginzburg − Landau theory
on emergent curved spacetime
with an extradimension

$$S_{\rm IR}[g_{\mu\nu}(x,z_f), \varphi(x,z_f)] = N \int d^D x \sqrt{g(x,z_f)} \Big\{ \frac{\mathcal{C}_\varphi^f}{2} g^{\mu\nu}(x,z_f) [\partial_\mu \varphi(x,z_f)][\partial_\nu \varphi(x,z_f)] + \mathcal{C}_\xi^f R(x,z_f) [\varphi(x,z_f)]^2 + \frac{1}{2\kappa_f} \Big(R(x,z_f) - 2\Lambda_f \Big) \Big\}, \tag{2.8}$$

Eq. of the scalar (density) field

$$ds^2 = g_{zz}dz^2 + g_{\mu\nu}dx^\mu dx^\nu .$$

$$\frac{1}{\sqrt{g}}\partial_z\Big(\sqrt{g}g^{zz}\partial_z\delta\varphi(x,z)\Big) + \frac{1}{\sqrt{g}}\partial_\mu\Big(\sqrt{g}g^{\mu\nu}\partial_\nu\delta\varphi(x,z)\Big) = m^2_{eff}(x,z)\delta\varphi(x,z)$$

$$\partial_z\Big\{D^{-1}(x,z)\Big(\partial_z\delta\varphi(x,z)\Big)\Big\} + \partial^2_\mu\delta\varphi(x,z) = \alpha\beta G^2(x,z)\delta\varphi(x,z).$$

$$g_{zz}(x,z) = [D(x,z)]^{\frac{d-2}{d-1}},$$

$$g_{xx}(x,z) = [D(x,z)]^{-\frac{1}{d-1}}, \qquad g_{\mu\nu}(x,z) = g_{xx}(x,z)\delta_{\mu\nu}.$$

$$m^2_{eff}(x,z) = \frac{\alpha\beta}{\sqrt{g}} G^2(x,z)$$

$$
\begin{aligned}
R_{zz}(x,z) =& \frac{d}{2(d-1)}\frac{\partial_z^2 D(x,z)}{D(x,z)} - \frac{3d}{4(d-1)}\frac{\left[\partial_z D(x,z)\right]^2}{D(x,z)^2} - \frac{d-2}{2(d-1)}\delta^{\mu\nu}\partial_\mu\partial_\nu D(x,z) \\
&+ \frac{d-2}{2(d-1)}\frac{\delta^{\mu\nu}\left[\partial_\mu D(x,z)\right]\left[\partial_\nu D(x,z)\right]}{D(x,z)},
\end{aligned} \tag{3.29}
$$

$$
R_{z\mu}(x,z) = \frac{\partial_z\partial_\mu D(x,z)}{2D(x,z)} - \frac{3d-4}{4(d-1)}\frac{\left[\partial_z D(x,z)\right]\left[\partial_\mu D(x,z)\right]}{D(x,z)^2}, \tag{3.30}
$$

$$
\begin{aligned}
R_{\mu\nu}(x,z) =& -\frac{d-2}{4(d-1)}\frac{\left[\partial_\mu D(x,z)\right]\left[\partial_\nu D(x,z)\right]}{D(x,z)^2} + \frac{\delta_{\mu\nu}}{2(d-1)}\left[\frac{\partial_z^2 D(x,z)}{D(x,z)^2} - 2\frac{\left[\partial_z D(x,z)\right]^2}{D(x,z)^3}\right. \\
&\left. + \frac{\delta^{\rho\sigma}\partial_\rho\partial_\sigma D(x,z)}{D(x,z)} - \frac{\delta^{\rho\sigma}\left[\partial_\rho D(x,z)\right]\left[\partial_\sigma D(x,z)\right]}{D(x,z)^2}\right].
\end{aligned} \tag{3.31}
$$

$$
\begin{aligned}
R(x,z) =& \frac{d}{d-1} D(x,z)^{-\frac{2d-3}{d-1}} \partial_z^2 D(x,z) - \frac{7d}{4(d-1)} D(x,z)^{-\frac{3d-4}{d-1}} \left[\partial_z D(x,z)\right]^2 \\
&+ \frac{D(x,z)^{-\frac{d-2}{d-1}} \delta^{\mu\nu} \partial_\mu \partial_\nu D(x,z)}{d-1} - \frac{d+2}{4(d-1)} D(x,z)^{-\frac{2d-3}{d-1}} \delta^{\mu\nu} \left[\partial_\mu D(x,z)\right] \left[\partial_\nu D(x,z)\right] .
\end{aligned}
\tag{3.32}
$$

Einstein Eq. $$R_{ab}(x,z) - \frac{1}{2} g_{ab}(x,z) R(x,z) = T_{ab}(x,z)$$

$$T_{zz}(x,z) = -\frac{1}{2}\partial_\mu \partial^\mu D(x,z) + \frac{5d-6}{8(d-1)} \frac{\partial_\mu D(x,z)\,\partial^\mu D(x,z)}{D(x,z)} + \frac{d}{8(d-1)} \frac{\left[\partial_z D(x,z)\right]^2}{D^2(x,z)}. \tag{3.34}$$

$$T_{z\mu} = \frac{\partial_z \partial_\mu D(x,z)}{2D(x,z)} - \frac{3d-4}{4(d-1)} \frac{\partial_z D(x,z)\,\partial_\mu D(x,z)}{D^2(x,z)}, \tag{3.35}$$

$$\begin{aligned} T_{\mu\nu}(x,z) = &- \frac{d-2}{4(d-1)} \frac{\partial_\mu D(x,z)\,\partial_\nu D(x,z)}{D^2(x,z)} + \delta_{\mu\nu} \Bigg[\frac{d-2}{8(d-1)} \frac{\partial_\lambda D(x,z)\,\partial^\lambda D(x,z)}{D^2(x,z)} \\ &- \frac{1}{2} \frac{\partial_z^2 D(x,z)}{D^2(x,z)} + \frac{7d-8}{8(d-1)} \frac{\left[\partial_z D(x,z)\right]^2}{D^3(x,z)} \Bigg]. \end{aligned} \tag{3.36}$$

Approximate solution

$$-\partial_z^2 \varphi(x,z) = i\alpha\beta u G(x,z)$$

$$\partial_{z_f} \varphi(x,z_f) = i\beta u G(x,z_f),$$
$$[\partial_z \varphi(x,z)]_{z=0} = 2\beta\varphi(x,0).$$

$$\varphi(p,z) = i\left(e^{-\frac{\mathcal{C}_1(p)}{2\alpha\beta u}} e^{-\left\{ InverseErf\left[\sqrt{\frac{2\alpha\beta u}{\pi}} e^{\frac{\mathcal{C}_1(p)}{2\alpha\beta u}} \left(z + \mathcal{C}_2(p) \right) \right] \right\}^2} - \left(p^2 + m^2 \right) \right).$$

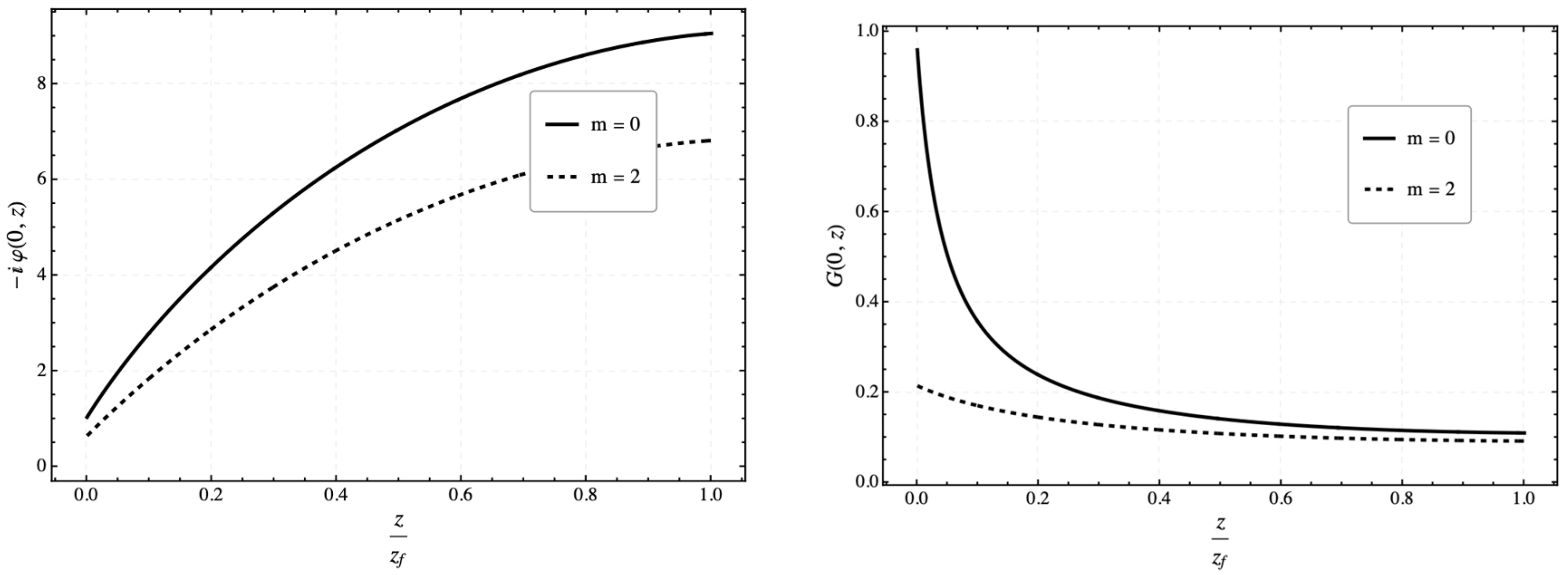
$$G(p,z) = e^{\frac{\mathcal{C}_1(p)}{2\alpha\beta u}} e^{\left\{ InverseErf\left[\sqrt{\frac{2\alpha\beta u}{\pi}} e^{\frac{\mathcal{C}_1(p)}{2\alpha\beta u}} \left(z + \mathcal{C}_2(p) \right) \right] \right\}^2}.$$

$$\mathcal{C}_1(p) = -2\alpha\beta u \ln\left[- \frac{\beta u}{\sqrt{2\alpha\beta u} R_f(p)} e^{R_f^2(p)} \right],$$

$$\mathcal{C}_2(p) = -\frac{\sqrt{\pi}}{2\alpha} \frac{e^{R_f^2(p)}}{R_f(p)} Erf[R_0(p)],$$

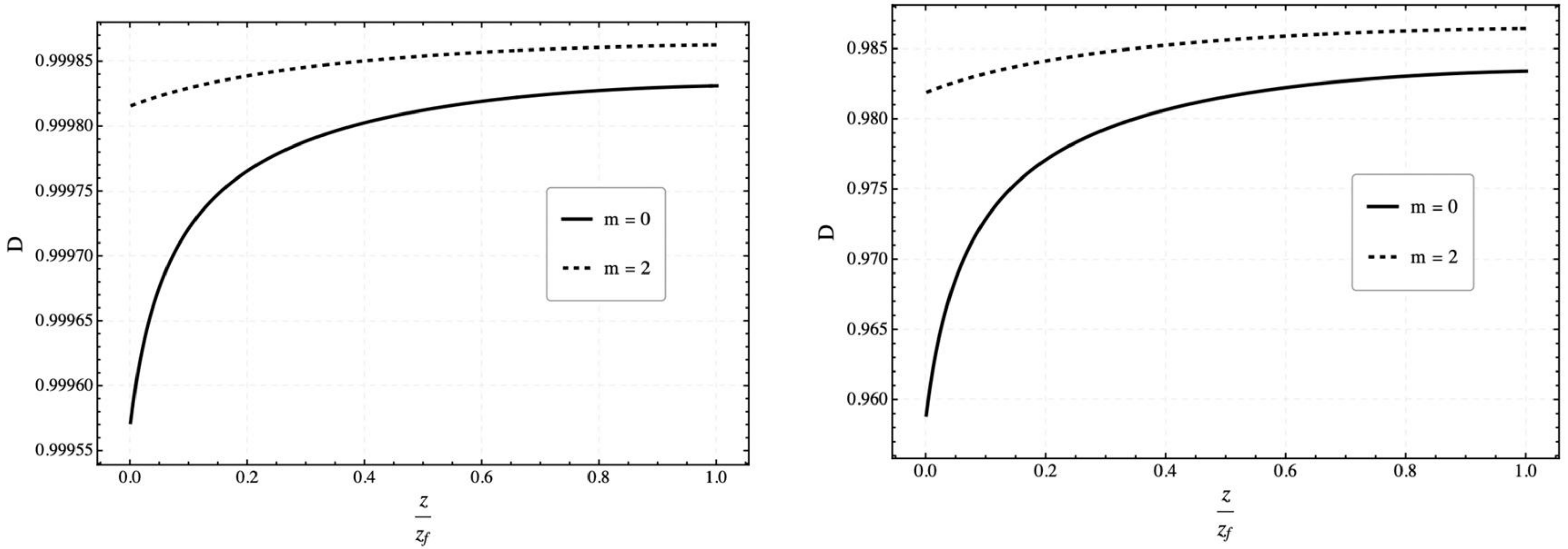
$$\mathrm{Erf}[R_f(p)] - \mathrm{Erf}[R_0(p)] = -\frac{2\alpha z_f}{\sqrt{\pi}} R_f(p) e^{-R_f(p)^2},$$

$$-\sqrt{2\alpha\beta u}\, R_0(p) = 2\beta \left[-\frac{\beta u}{\sqrt{2\alpha\beta u}\, R_f(p)} e^{R_f(p)^2 - R_0(p)^2} - (p^2 + m^2) \right].$$



(a) $\varphi(p=0,z)$ (b) $G(p=0,z)$

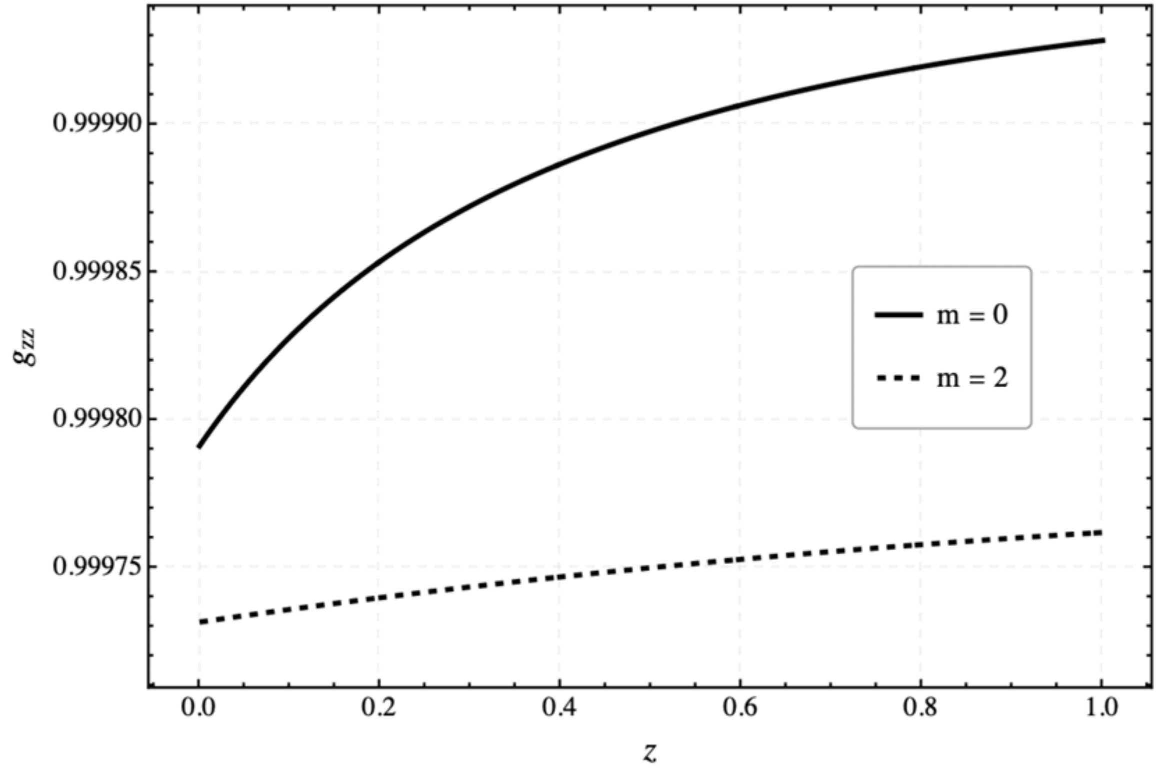
Figure 1. Numeric calculation of the collective mode φ and Green function G for massless and $m=2$ under the parameters $\alpha=\beta=u=1$, and $z_f=10$.



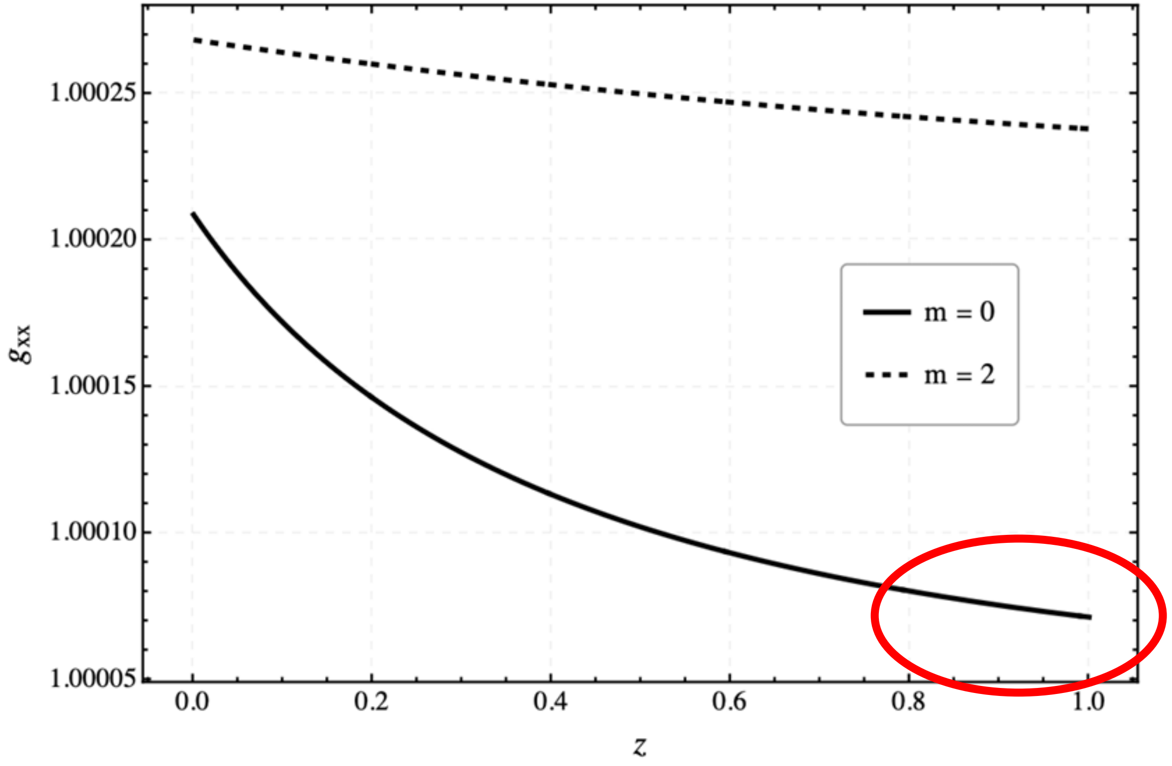
(a) $\epsilon = 0.1$

(b) $\epsilon = 1.0$

Figure 2. Numeric calculation of the collective mode propagator for massless and $m = 2$ under the parameters $\alpha = \beta = u = 1$ $d = 3$, and $z_f = 10$.



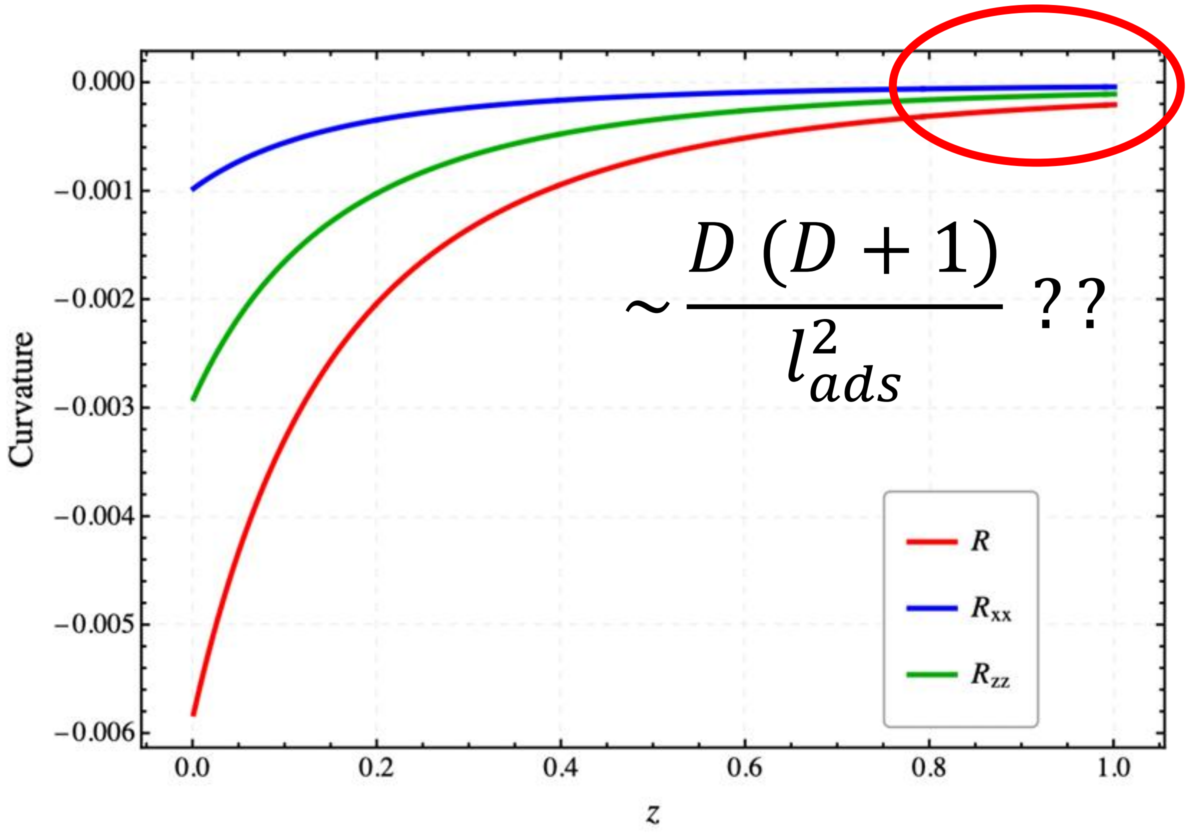
(a) g_{zz}



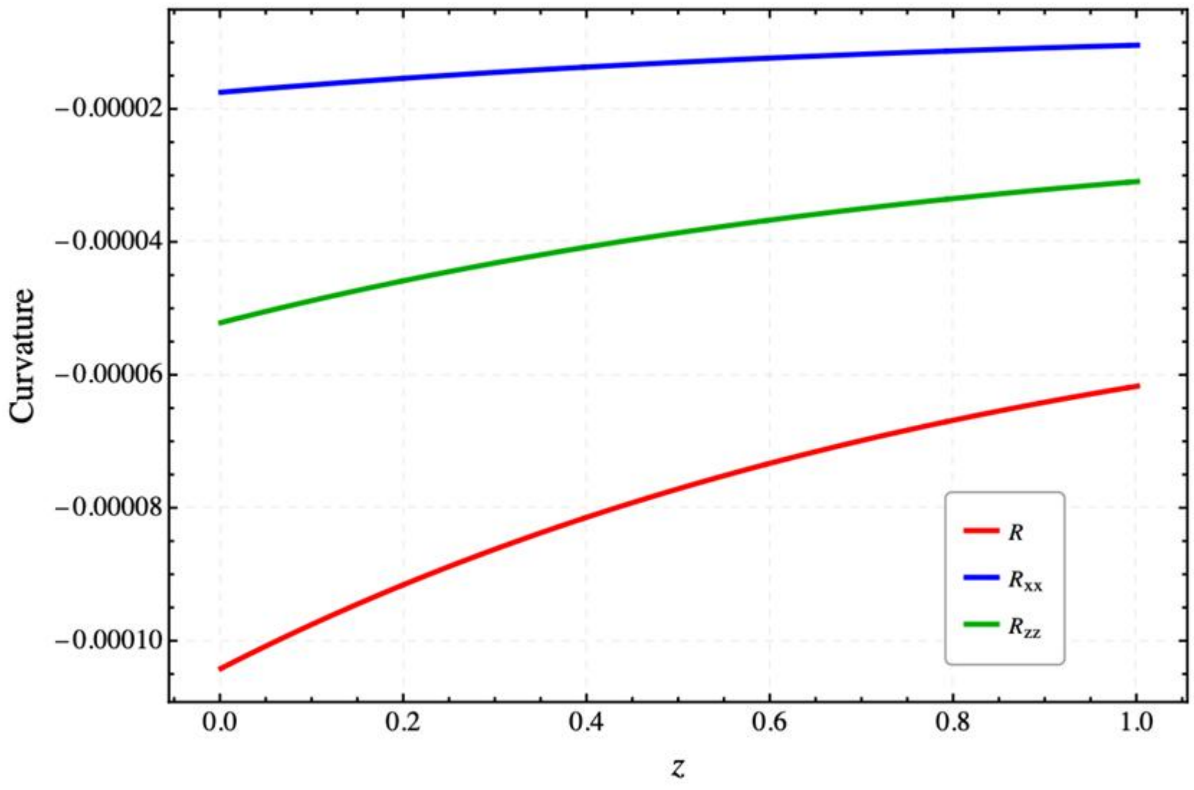
(b) g_{xx}

$\sim e^{-\frac{2z}{l_{ads}}}$??

$\alpha = \beta = u = 1.0$ & $\varepsilon = 0.1$

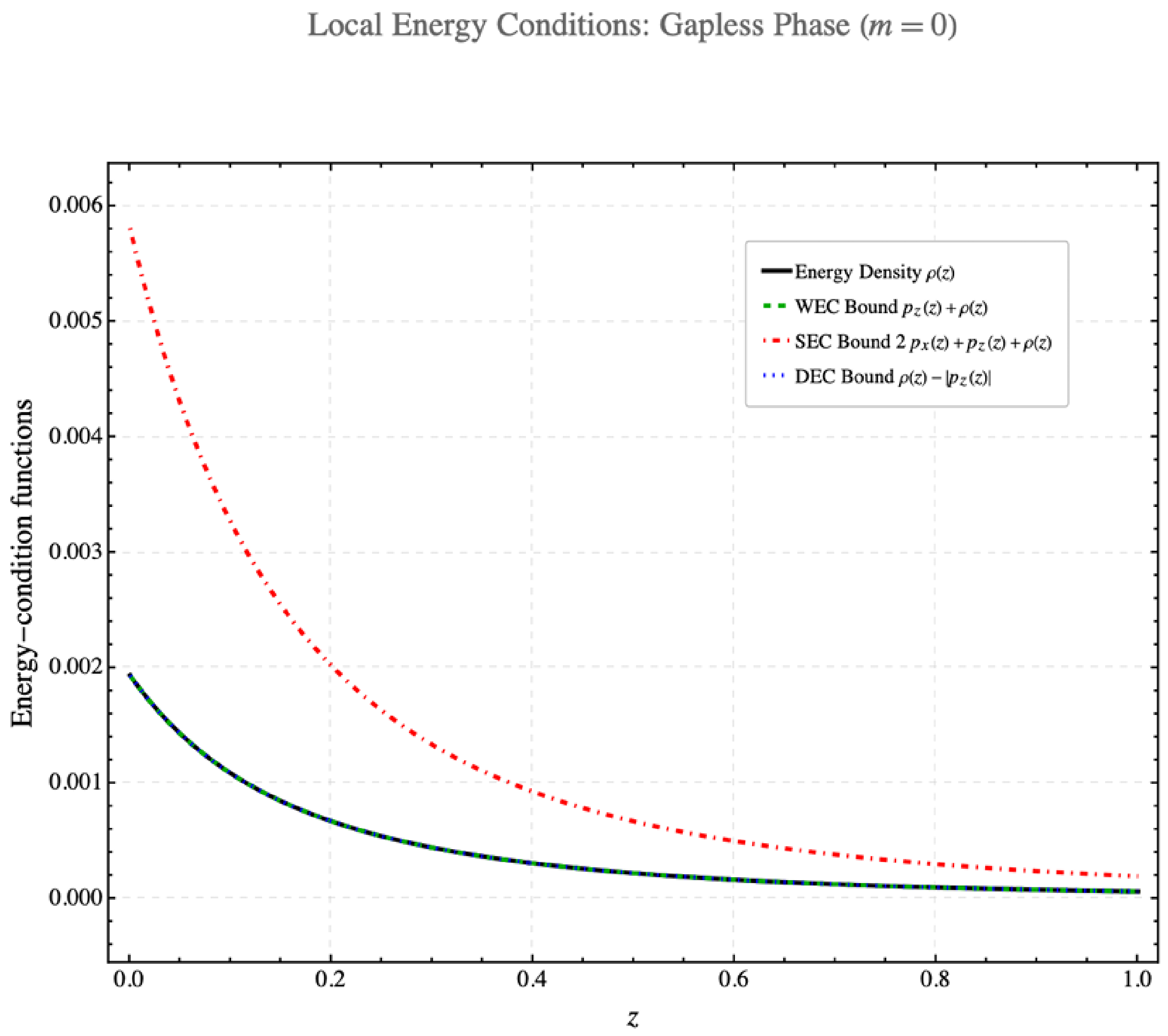


(a) Curvature components for $m = 0$

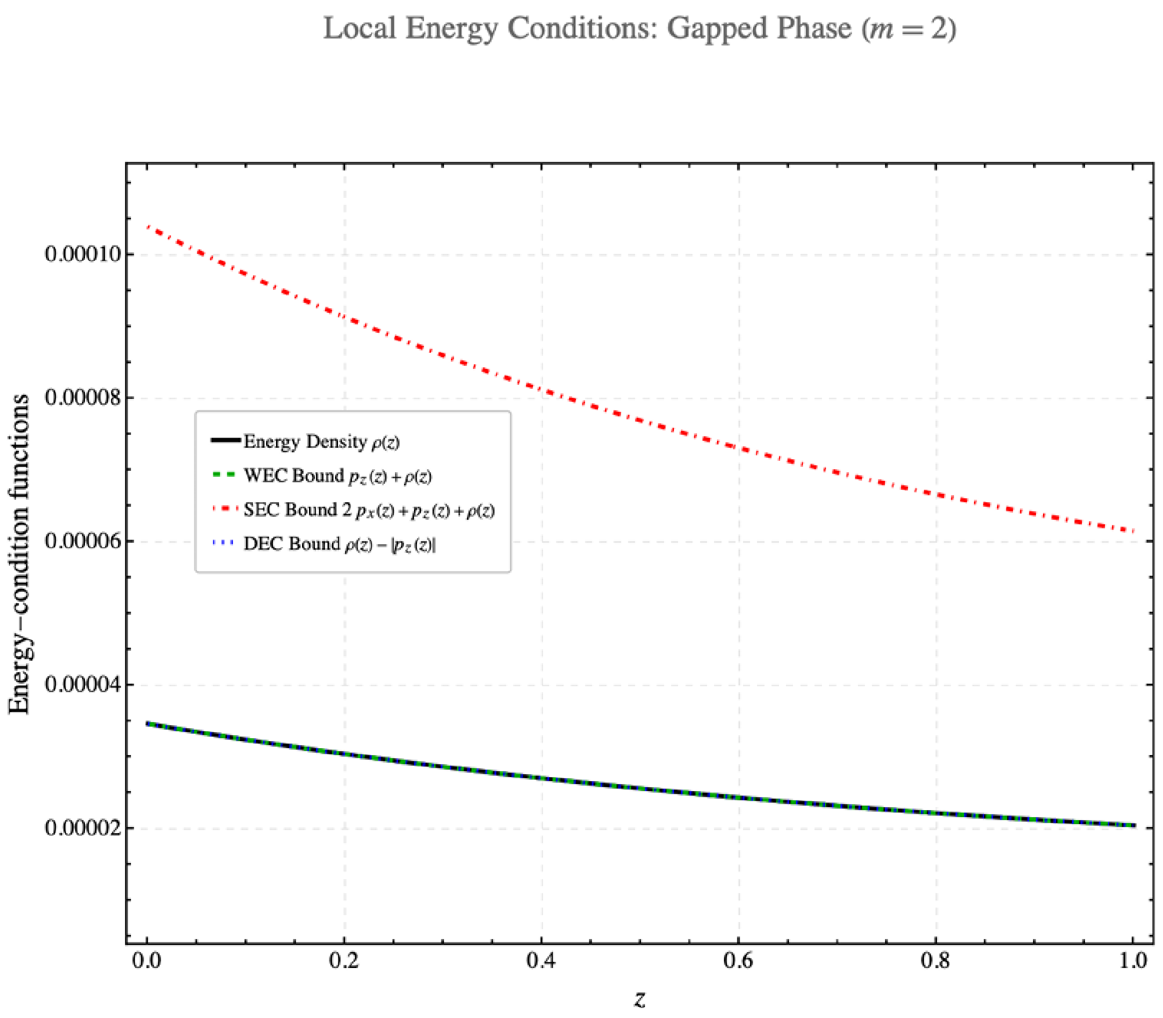


(b) Curvature components for $m = 2$

$\alpha = \beta = u = 1.0$ & $\varepsilon = 0.1$



(a) Energy condition for m=0



(b) Energy condition for m=2

$$\alpha = \beta = u = 1.0 \;\&\; \varepsilon = 0.1$$

Discussion

To properly describe black hole physics and the associated horizon thermodynamics from first principles, it is necessary to solve the full radial evolution without this approximation, fully incorporating the scale-dependent behavior of the interaction propagator $D(x, z)$ defined in Eq. (9). Since the full propagator $D(x, z)$ contains non-linear feedback from the matter-field fluctuations via $G^2(x, z)$, a complete numerical or semi-analytic treatment of the full non-linear system is expected to provide a continuous backreaction that can properly stabilize a thermal horizon. We leave the explicit construction of the fully backreacted finite-temperature geometry for future investigation.

$$-\partial_z \Big\{ D^{-1}(x, z) \Big(\partial_z \varphi(x, z) \Big) \Big\} = i\alpha\beta G(x, z).$$

From multi-particle spectra to black hole geometry