Non-Abelian Magnets: Beyond Spin Magnetism

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and a work with J. Lopez-Suarez (now PhD in Stockholm)

Magnetization - A review

Measures how strongly a material responds in a magnetic field B.
In the dipole $\boldsymbol{m}$ approximation the potential energy is

$$U = -\boldsymbol{m} \cdot \boldsymbol{B} \ .$$

Spontaneous Magnetization

The phenomenon where a material develops a non-zero magnetization even if the external magnetic field $\boldsymbol{B} = 0$.

- Occurs only for ferromagnetic materials for $T < T_C$, where T_C is called the Curie temperature.
- It is due to strong spin-spin interactions.
- For $T > T_C$, thermal fluctuations destroy long-range order and magnetization becomes zero.

- Free energy is the key. Any change should lower it, i.e.

$$\Delta F = \Delta E - T\Delta S < 0 \ .$$

  Competition between order driven by energy and disorder driven by entropy.

- The transition from ordered to disorder is a phase transition. In conventional cases it is of second order.

- Spontaneous magnetization is accompanied by a spontaneous symmetry breaking.

  A physical system in a symmetric state spontaneously ends up in an asymmetric state.

- Great physicists, including Nobel Prize laureates, have conducted research on facets of magnetization.

Models for spontaneous magnetization: The Ising model

The Hamiltonian is [Ising 1925]

$$H = -J \sum_{\langle i,j \rangle} s_i s_j \ , \qquad J > 0 \ , \quad s_i = \pm 1 \ ,$$

summing over nearest neighbors. Neighboring spins prefer to align.

- No continuous symmetry, but a Z_2 discrete one, $s_i \to -s_i$.
- The s_i can be thought of as a single component of the spin operator $\boldsymbol{s}_i$.
- Lots of the essential Physics is captured by this model.
- The features depend on whether the spins are arranged in $1D, 2D$ or $3D$.

1D Ising model

- The Hamiltonian is

$$H = -J \sum_i s_i s_{i+1} \ ,$$

so that two nearest neighbors.

- Consider a perfect array of N spins up

$$\uparrow\uparrow \cdots \uparrow$$

This and the array with all spins $\downarrow$: ground states at $T = 0$.

- Then flip all spins to the right of some point (a defect)

$$\uparrow\uparrow \cdots \uparrow \downarrow\downarrow \cdots \downarrow$$

Changes in energy and entropy: $\Delta E = 2J > 0$ and $\Delta S \sim \ln N$.

- In the thermodynamic limit the free energy changes as

$$\Delta F \sim 2J - T \ln N < 0 \ , \quad N \gg 1 \ .$$

- Hence, at $T > 0$ the formation of domain walls is favorable

$$\langle m \rangle = 0 \ , \qquad T_C = 0 \ ,$$

  that is zero magnetization and zero critical temperature.

- The model can be solved exactly [Ising 1925].

- Correlation functions of two spins at a distance r fall off as

$$\frac{e^{-r/\xi}}{r^{\#}} \ .$$

  In the 1D Ising model the correlation length diverges as

$$\xi \sim e^{2J/T} \ , \quad \text{as } T \to 0 \ ,$$

  but remains finite for all $T > 0$.

2D Ising model (eg. in a square lattice)

In a very large lattice of spins $\uparrow$, consider a region with all spins $\downarrow$

$$\begin{array}{c}
\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\\
\uparrow\uparrow\uparrow\ \downarrow\downarrow\downarrow\ \uparrow\uparrow\uparrow\\
\uparrow\uparrow\uparrow\ \downarrow\downarrow\downarrow\ \uparrow\uparrow\uparrow\\
\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow
\end{array}$$

The defect is extended, i.e. domain wall, with some length L.

- For spacing a, there are about $n \sim L/a$ segments corresponding to
$$\Delta E \sim Jn \ .$$
- If there are k possibilities to move a segment then
$$\Delta S \sim \ln k^n \sim cn \ .$$
- Therefore the free energy changes
$$\Delta F \sim n(J - cT) \ .$$

- Expect a critical T_C separating order from disorder.
- The model is exactly solvable [Onsager 44] and indeed

$$T_C = \frac{2J}{\ln(1+\sqrt{2})} \ .$$

- The magnetization behaves as

$$m(T) \sim \begin{cases} (T_C - T)^{\beta} \ , & T \lesssim T_C \\ 0, & T \geqslant T_C \end{cases}$$

  and the correlation length diverges near the transition as

$$\xi \sim |T - T_C|^{-\nu} \ ,$$

  where the critical exponents are

$$\beta = \frac{1}{8} \ , \quad \nu = 1 \ .$$

- At $T = T_C$ the theory is described by a CFT of free fermions.

3D Ising model (eg. in a cubic lattice)

- No exact analytical solution is known (one of the central unsolved problems in statistical mechanics)
- As in the 2D case there is a T_C separating order from disorder.
- The critical exponents for the magnetization and the correlation length approximately are

$$\beta \simeq 0.326 \; , \quad \nu \simeq 0.630 \; .$$

Models for spontaneous magnetization: The Heisenberg model

A more realistic model utilizing all components of the spin operator $\boldsymbol{s}_i$ in a rotational invariant way [Heisenberg 28]

$$H = -J \sum_{\langle i,j \rangle} \boldsymbol{s}_i \cdot \boldsymbol{s}_j \ , \qquad J > 0 \ .$$

- At $T = 0$ the ground state is fully polarized

$$|\uparrow\uparrow \cdots \uparrow\rangle \,, \quad S = \frac{N}{2} \,, \quad E_0 = -\frac{JN}{4} \ .$$

  The direction of magnetization can vary continuously.
- As T increases, disorder may appear.
  Defect/domain wall type arguments are not suitable.
- Thermal fluctuations cause spin-waves which are configurations where the direction of the magnetization slowly rotates.
- The elementary quantum excitation is the magnon following the Bose - Einstein distribution.

If the fluctuations diverge for long wavelengths (covering macroscopic seizes), order is destroyed at any finite T.

- The magnetization is $\langle S^z \rangle = S - \langle a^\dagger a \rangle$. Then

$$\langle a^\dagger a \rangle = \int \frac{d^d k}{e^{\omega(k)/T} - 1} \ .$$

- Gapless dispersion; for large wavelengths $\omega(k) \simeq Jk^2$, so

$$\langle a^\dagger a \rangle \sim \frac{T}{J} \int dk \, k^{d-3} \ .$$

  (essentially the Mermin-Wagner theorem)

- <u>**In 1D**</u> the integral diverges, similar to the 1D Ising model, so complete disorder.
  - Introducing a cutoff (correlation) length ξ to keep $\langle a^\dagger a \rangle$ finite

$$\xi \sim \frac{J}{T} \ , \quad \text{diverging only for } T \to 0 \ .$$

  - Has been exactly solved by Bethe ansatz techniques [Bethe 31].

- **In 2D** the integral diverges and, unlike the 2D Ising model, complete disorder.
  - The correlation length is
$$\xi \sim e^{J/T} \ , \quad \text{diverging only for } T \to 0 \ .$$
- **In 3D** the integral converges and order may happen below a finite temperature.
  - The critical exponents for the magnetization and the correlation length are now
$$\beta \simeq 0.366 \ , \quad \nu \simeq 0.711 \ .$$
- No exact solution is known for the 2D and 3D Heisenberg model.
- No spin waves for the Ising model: spins can be either $\uparrow$ or $\downarrow$.

Mean Field approximation

With the exception of low dimensions, exact solutions are lacking.

- Need for approximate methods.
- There is no small parameter, hence we should resort to variational like methods (as in QM).

For the Heisenberg model assume the magnetization $\boldsymbol{m} = \langle \boldsymbol{s}_i \rangle, \forall i$.

- Write

$$\boldsymbol{s}_i \cdot \boldsymbol{s}_j = (\boldsymbol{s}_i - \boldsymbol{m})(\boldsymbol{s}_j - \boldsymbol{m}) + \boldsymbol{m} \cdot (\boldsymbol{s}_i - \boldsymbol{m}) + \boldsymbol{m} \cdot (\boldsymbol{s}_j - \boldsymbol{m}) + \boldsymbol{m}^2 \ .$$

- Assume that thermal spin correlations for $i \neq j$ are independent. Then, the first term for $i \neq j$ can be neglected.

- Then the Hamiltonian is

$$H_{\mathrm{mf}} = -J \sum_{\langle i,j \rangle} \left(\boldsymbol{m} \cdot (\boldsymbol{s}_i + \boldsymbol{s}_j) - \boldsymbol{m}^2 \right) = -\boldsymbol{B}_{\mathrm{eff}} \cdot \sum_i \boldsymbol{s}_i + \mathrm{const.} \ ,$$

  where the effective magnetic field is

$$\boldsymbol{B}_{\mathrm{eff}} = Jq\boldsymbol{m} \ ,$$

  where q is the number of nearest neighbors.
- Take $\boldsymbol{m}$ along the z-axis. Then the magnetization is

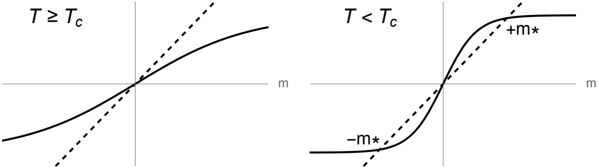
$$m = \frac{\sum_{m_s=-j}^{j} m_s e^{Jqmm_s/T}}{\sum_{m_s=-j}^{j} e^{Jqmm_s/T}} \ .$$

- The sums can be performed but let's take $j = 1/2$. Then

$$\boxed{2m = \tanh\left(\frac{2T_C}{T} m\right)}, \quad T_C = \frac{Jq}{4} \ ,$$

  a transcendental equation for the magnetization.

- $T \geqslant T_C$: the only solution is $m = 0$ (disorder).
  $T < T_C$: a non-trivial solution (and its opposite) (order).



- The magnetization is

$$m \sim \begin{cases} (T_C - T)^{\beta}\,, & T \lesssim T_C \\ 0, & T \geqslant T_C \end{cases}\,, \qquad \beta = \frac{1}{2}\,.$$

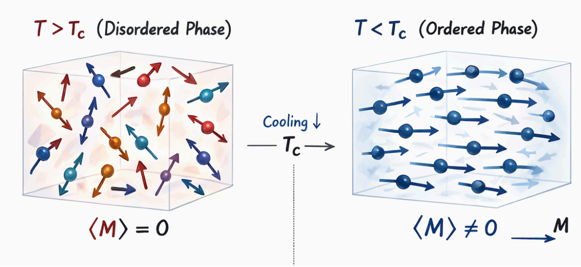
How successful is the mean field approximation?

- Result universal; the same for Ising model as well.
- Is dimension independent.
- Predicts order in cases there is none.
- Predicts wrong values for the critical exponents

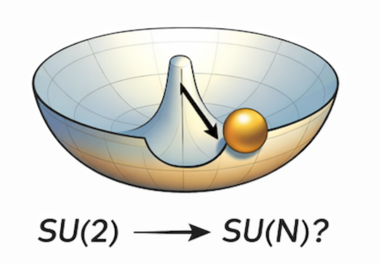
When can mean field approximation be trusted?

- In high dimensions (2D or more).
- In predicting the phase transition and results not near the critical temperature.
- The mean field approximation becomes exact:
  - in models in 4D and higher
  - When the interactions become reasonably long range.

The following quite general picture appears



Schematically: Mexican hat for SU(2)



Expect that the vacuum manifold will become much richer.

General settings and context – Motivation

General motivation: $SU(N)$ symmetry in Physics

- Elementary Particle Physics
  Spin, Isospin, Flavor, Color, Grand Unified Theories...

Extended $SU(N)$ symmetries explore non-perturbative phenomena.

- $SU(N)$-matrix models:
  - To describe non-perturbative aspects in string theory [Gross-Migdal 90, Douglas-Shenker 90]
  - Black holes (thermalization, etc) [Kazakov-Kostov-Kutasov 01]
- Large N-expansion of $SU(N)$ gauge theories:
  - Led to a new understanding of the perturbative expansion by organizing the Feynman diagrams topologically ['t Hooft 74]
  - Eventually to the AdS/CFT correspondence, a breakthrough in our understanding of QFT and Gravity [Maldacena 97]

- Effective symmetry at Atomic Scales:

  In cold atoms (alkaline-earth (-like) atoms, e.g. $^{87}\mathrm{Sr}$) there is virtual independence of atom interactions on the nuclear spin (hyperfine structure) of the atoms s.

  The $N = 2s + 1$ nuclear spin states behave as if they belong in the fundamental representation of $SU(N)$.

  - N could be up to 10 or more.
  - Physics beyond that in regular materials; new exotic phases.
  - Large N limits.

General motivation - Some basic math/phys questions

- $SU(2)$: The total spin of 3 spin-1/2 stetes could be either 1/2 or 3/2 with multiplicities 2 and 1, i.e.

$$2 \otimes 2 \otimes 2 = 2 \oplus 2 \oplus 4\ .$$

What about n spin-1/2 states or n spin-j states?

- $SU(N)$: What is the multiplicity of a general Young Tableau arising in the decomposition of n fundamentals? Schematically:

$$\underbrace{\square \otimes \square \otimes \cdots \otimes \square}_{n \text{ boxes}} = \sum_{\mathbf{k}} d_{n,\mathbf{k}} \ydiagram{5,3,2}$$

or the decomposition of any other irrep of $SU(N)$?

- In there anything interesting for large n and/or N:
If $N = \mathcal{O}(1)$ and $n \gg 1$? We will focus here.
If $N, n \gg 1$ with some ratio kept constant?

Outline

- The $SU(N)$ ferromagnet
- Solution in the thermodynamic limit and finite N.
  Stability and Young tableaux.
  Spontaneous symmetry breaking.
- Phase transitions:
  - $SU(2)$: A single Curie temperature below which spontaneous magnetization occurs; 2nd order phase transition.
  - $SU(N)$, with $N = 3, 4, \ldots$:
    More structure and critical temperatures...stable as well unstable phases. Different phase strucure.
- Focus on the fundamental and the adjoint irreps of $SU(N)$.
- Concluding remarks.

The $SU(N)$ ferromagnet

Consider n atoms on a lattice with two-body interactions.

- Each atom has N degenerate states $|s\rangle$, $s = 1, 2, \ldots, N$.
- Define j_a, $a = 1, \ldots, N^2 - 1$, $SU(N)$ generators in the fundamental $\square$ irrep.
  They form a complete basis on an N-dim space.
- We may model interactions using this basis. Assuming invariance under change of basis, the full Hamiltonian will be

$$H = \sum_{r,s=1}^{n} c_{r,s} \sum_{a=1}^{N^2-1} j_{r,a} \, j_{s,a} \ ,$$

  where $c_{r,s}$ coupling between atoms r and s.
- Hence, $SU(N)$ emerges from such an invariance.

Mean field approximation

- Assume:
  - Long range interactions, so that we are within the validity of the mean field approximation.
  - Ferromagnetic couplings $c_{r,s} < 0$.
- Then, the full Hamiltonian becomes [Polychronakos-KS 23]

$$H = -\frac{c}{n}\sum_{a=1}^{N^2-1} J_a^2 + \text{const.} = -\frac{c}{n} C_2(J) + \text{const.} \; ,$$

where the total $SU(N)$ generators and the coupling are

$$J_a = \sum_{s=1}^{n} j_{s,a} \; , \quad c = -\sum_{r\neq s} c_{r,s} > 0 \; .$$

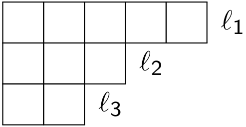
with $C_2(J)$ the quadratic Casimir in the fundamental rep.

Crash course on $SU(N)$ representation theory

- Irreps of $SU(N)$ are labeled by distinct ordered integers $\{k_i\}$

$$k_1 > k_2 > \cdots > k_N \ .$$

The usual Young tableaux



is labeled by ℓ_i: the # of boxes in the ith row

$$\ell_i = k_i - k_N + i - N \ , \qquad \ell_1 \geqslant \ell_2 \geqslant \cdots \geqslant \ell_{N-1} \geqslant 0 \ .$$

- Invariant under shifts of k_i ($U(1)$ charge); Redundancy fixed by

$$\sum_{i=1}^{N} k_i = n + \frac{N(N-1)}{2} \ .$$

The multiplicity $d_{n,\boldsymbol{k}}$ and how to compute it

What is the multiplicity $d_{n,\mathbf{k}}$ of each irrep $\mathbf{k}$ arising in the decomposition of n irreps of $SU(N)$?

Schematically:

$$\underbrace{\square \otimes \square \otimes \cdots \otimes \square}_{n \text{ boxes}} = \sum_{\mathbf{k}} d_{n,\mathbf{k}} \; \ydiagram{5,3,2}$$

A closed expression can be also obtained.

$$d_{n;\mathbf{k}} = n! \, \frac{\Delta(\mathbf{k})}{\prod_{i=1}^{N} k_i!} \, , \qquad \sum_{i=1}^{N} k_i = n + \frac{N(N-1)}{2} \; .$$

where the Vandermonde determinant is

$$\Delta(\mathbf{k}) = \prod_{j>i=1}^{N} (k_i - k_j) \, .$$

Statistical mechanics of the $SU(N)$ ferromagnet

The partition function & thermodynamic limit

- At temperature $T = \beta^{-1}$ the model is defined as [Poly-KS 23]

$$\boxed{Z = \sum_{\text{states}} e^{-\beta H} = \sum_{\langle \mathbf{k} \rangle} \dim(\mathbf{k})\, d_{n;\mathbf{k}}\, e^{\frac{\beta c}{n} C^{(2)}(\mathbf{k})}}\ ,$$

  where $\langle \mathbf{k} \rangle$ denotes ordered integers. The quadratic Casimir is

$$C^{(2)}(\mathbf{k}) = \frac{1}{2} \sum_{i=1}^{N} k_i^2 + \text{const.}$$

  and the dimension of the irrep

$$\dim(\mathbf{k}) = \frac{\Delta(\mathbf{k})}{\prod_{s=1}^{N-1} s!}\ .$$

- Magnetic fields can be easily included.

- Interested in the thermodynamic limit $n \gg 1$. Then we set

$$k_i = n x_i \ , \quad c = N T_0 \ ,$$

introducing intensive quantities x_i and a temperature scale T_0.
- For finite N the Vandermonde determinant $\Delta(\mathbf{k})$ is a finite degree polynomial and does not contribute.
- Altogether we obtain

$$Z = \sum_{\mathbf{x}} \delta_{x_1+\cdots+x_N,1} e^{-n\beta \mathrm{F}(\mathbf{x}) + \mathcal{O}(n^0)} \ ,$$

where the free energy of the system is

$$F(\mathbf{x}) = \sum_{i=1}^{N} \left(T x_i \ln x_i - \frac{N T_0}{2} x_i^2 \right) \ .$$

- Note that

$$\sum_{i=1}^{N} x_i = 1 \ ,$$

so that the x_i's are constraint.

Ferromagnets in the fundamental [Poly-KS 23]

- Introduce a temperature T_0 (determines stability of singlet)

$$c = NT_0 \ .$$

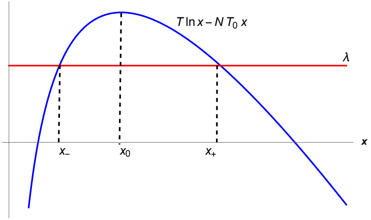
- The saddle point conditions are

$$T \ln x_i - NT_0 x_i = \lambda \ , \quad i = 1, 2, \ldots, N \ ,$$
$$\sum_{i=1}^{N} x_i = 1 \ .$$

- A system of transcendental equations.
- Establishing the phases of the system involves:
  - Solving the above conditions
  - Establishing the local and global stability of the solutions
  - Finding phase transition lines between phases (solutions)

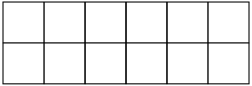
Solutions

The x_i are $N-1$ order parameters; satisfy a common equation



- Only two possible solutions $x_i = x_-$ or $x_i = x_+$
- $x_i = 1/N$ (for all i) is always a solution. If stable, paramagnetic phase with unbroken $SU(N)$
- In general p solutions at x_+ and $N-p$ at $x_- < x_+$.
- This represents order and corresponds to a YT with

p equal rows :



Stability

The only possible stable solutions are

- $p = 0$ ($SU(N)$-singlet, disorder)
- $p = 1$ (fully symmetric irrep, order) One-row YT

$$\underbrace{\boxed{}\boxed{}\boxed{}\boxed{}\cdots\boxed{}\boxed{}}_{\ell_1 \text{ boxes}}, \qquad \ell_1 = \frac{x}{N-1}\, n + \mathcal{O}(1)\ .$$

Both are captured by the single order parameter x

$$x_1 = \frac{1+x}{N}\ , \quad x_i = \frac{1 - x/(N-1)}{N}\ , i = 2, \ldots, N\ ,$$

satisfying

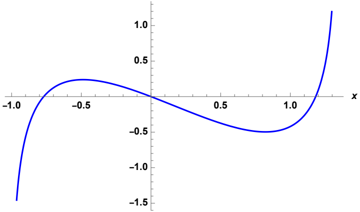
$$(*)\ \boxed{\ T \ln \frac{1+x}{1 - x/(N-1)} - T_0 \frac{N}{N-1}\, x = 0\ }\ .$$

- $x = 0$ (the singlet, no magnetization) is always a solution.

Phase structure

Two temperatures, T_0 and $T_c > T_0$.

- For $T > T_c$: only solution is $x = 0$ (stable); T_c is determined numerically.
- For $T_0 < T < T_c$: $x = 0$ (locally stable) and $0 < x_1 < x_c < x_2$ (locally stable). Global stability?
- For $T < T_0$: $x = 0$ (unstable) and $x_1 < 0 < x_2$ (stable)



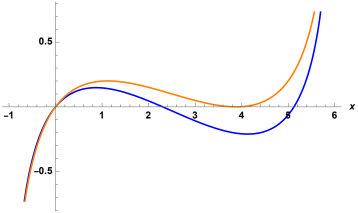


Figure: Plot of the LHS of (*).
Left : $T < T_0$. Right: $T_0 < T < T_c$ (blue) and for $T = T_c$ (yellow).

- Free energy comparison reveals a third intermediate temperature

$$T_m = \frac{T_0}{2}\frac{N(N-2)}{(N-1)\ln(N-1)}\,, \quad T_0 < T_m < T_c\ ,$$

where one of the local minima becomes the global minimum.

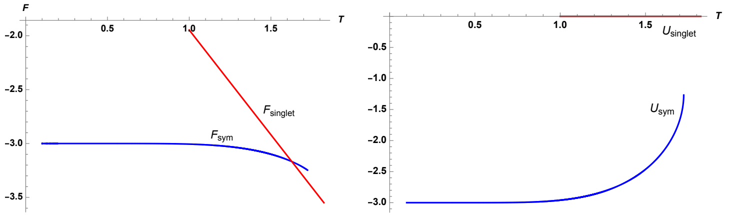
- We get the table for $N \geqslant 3$, Spontaneous magnetization.

state	$T < T_0$	$T_0 < T < T_m$	$T_m < T < T_c$	$T_c < T$
singlet	unstable	metastable	stable	stable
1-row	stable	stable	metastable	$\times$

- Spontaneous magnetization, i.e. symmetry breaking

$$SU(N) \to SU(N-1) \times U(1)\ ,$$

but not with a single Curie temperature



- First order phase transition from a metastable to a stable phase

$$F_{\text{sym}} = -T \ln N + \frac{N-2}{N} \ln(N-1)\left(T - T_m\right) + \ldots \ ,$$
$$F_{\text{singlet}} = -T \ln N \ ,$$

so that the first derivative is discontinuous.

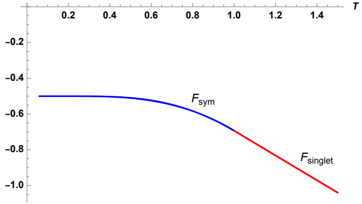
- Although one phase is thermodynamically stable, the other can persist as a metastable state.
The system may remain trapped in the metastable state, leading to hysteresis in temperature, unless it is perturbed, i.e. analogous to that in supercooled water.
Latent heat exchange between phases.

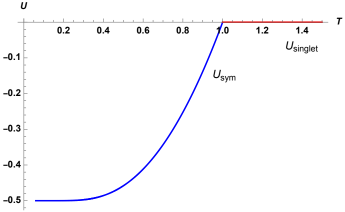
- For $N = 2$, $T_0 = T_m = T_c \implies$ standard ferromagnetism.

state	$T < T_0$	$T > T_0$
singlet	unstable	stable
1-row	stable	$\times$

A single Curie temperature.

- Free and internal energy





- There is a 2nd order phase transition

$$F_{\text{sym}} = -T \ln 2 - \frac{3}{4T_0}\left(T_0 - T\right)^2 + \dots ,$$

$$F_{\text{singlet}} = -T \ln 2 ,$$

so that the second derivative is discontinuous.

Ferromagnets in the adjoint [Lopez-Suarez-Polychronakos-KS 25]

The YT is

$$\text{Adjoint}: \ \boxed{\bullet}\boxed{} \ = \left.\begin{array}{l}\boxed{}\boxed{}\\ \ \vdots \\ \boxed{}\end{array}\right\}(N-1)$$

- The rôles of $\boxed{}$ and their conjugates $\boxed{\bullet}$ are equivalent.
- Hence, expect that stable configurations will respect this.
- However, this is not true, in intermediate temperatures.

Stability

Stable solutions are of two types (a subclass of self-dual YT):

- One row symmetric YT or $N-1$-equals rows

  A: $\boxed{} \cdots \boxed{}$ or $\boxed{\bullet} \cdots \boxed{\bullet}$ (Degenerate, same F)

  $$\text{Symmetry breaking}: \quad SU(N) \to SU(N-1) \times U(1) \ ,$$

- B: $\boxed{\bullet} \cdots \boxed{\bullet}\boxed{} \cdots \boxed{}$ (thinnest selfdual YT)

  $$\text{Symmetry breaking}: \quad SU(N) \to SU(N-2) \times U(1) \times U(1) \ ,$$

- Configuration A breaks Conjugation Symmetry and configuration B preserves it.
- Many other unstable solutions. For instance,

$$\text{AB}: \quad \boxed{\bullet}\boxed{\bullet}\boxed{}\boxed{}\boxed{}\boxed{} \qquad \text{or} \qquad \boxed{\bullet}\boxed{\bullet}\boxed{\bullet}\boxed{\bullet}\boxed{}\boxed{} \ .$$

which interpolates between A and B.

Phase structure

The phase structure depends:

- on the rank N of the group.
- States appear/disappear or/and become stable/unstable depending on the temperature.
  - At high enough T the single is the only solution.
  - At low enough T solution B is the only stable solution.

Keeping only the stable phases

Temperature	$T < T_2$	$T_2 < T < T_4$	$T_4 < T$
irrep	B	A	singlet

Classification

- For $SU(3)$ the picture is simple

irrep	$T < T_0^{\mathrm{adj}}$	$T_0^{\mathrm{adj}} < T$
Singlet	unstable	stable
B	stable	$\times$

  Only solution B appears. At low T there is no symmetry left.

- For $SU(N)$, $N = 4, 5$ the picture gets already complicated

irrep	$T < T_1$	$T_1 < T < T_2$	$T_2 < T < T_3$	$T_3 < T < T_0^{\mathrm{adj}}$	$T_0^{\mathrm{adj}} < T$
Singlet	unstable	unstable	unstable	unstable	stable
A	unstable	metastable	stable	stable	$\times$
B	stable	stable	metastable	unstable	$\times$

  - The temperatures T_i mark the separation of stable and metastable phases where the phase transitions is first order.
  - At $T = T_0^{\mathrm{adj}}$ the transition is second order (for $N = 3, 4, 5$).
- For $N \geqslant 6$ the complete structure is increasingly more complicated.

Thermodynamic limit with $N = \mathcal{O}(\sqrt{n})$ [Poly-KS 24]

When both $n, N \gg 1$, then the subleading terms originating from the Vandermonde determinant we have ignored become important.

We reformulate as:

- We think of the k_i as a continuous distribution.
- We define a density $\rho(k)$ and pass to the continuum limit

$$\sum_i \to \int di = \int dk \rho(k) \ .$$

- Perform the scaling $k = Nx$.
- The constraints suggest the scaling $N^2 \sim n$. Thus define

$$\boxed{t = \frac{4n}{N^2} = \text{finite}} \ .$$

How these were computed

- The constraints become

$$\int_0^\infty dx\, \rho(x) = 1\ , \qquad \int_0^\infty dx\, x\, \rho(x) = \frac{t}{4} + \frac{1}{2}\ , \qquad 0 \leqslant \rho(x) \leqslant 1\ .$$

- Minimize the free energy functional

$$\beta F[\rho(x)] = N^2 \Big[- \int_0^\infty dx \int_0^\infty dx'\, \rho(x)\rho(x')\ln|x - x'| + \int_0^\infty dx\, \rho(x)\Big(x(\ln x - 1) - \frac{x^2}{2x_T}\Big)\Big]\ ,$$

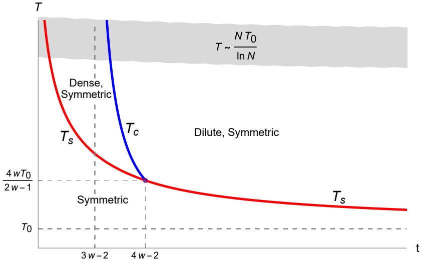
  subject to the constraints.

- This amounts to solving

$$\boxed{2\int_0^\infty dx'\, \frac{\rho(x')}{x - x'} = \ln x - \frac{x}{x_T} + \lambda}\ , \quad x_T = \frac{tT}{4T_0}\ ,$$

  a standard single-cut Cauchy problem.

- The above equation must hold for x such that $0 < \rho(x) < 1$.
- To solve it we define a resolvent etc...
- New phases, as well as higher order phase transitions and a triple point (as for the water) appear.



I skip the details.

- More possibilities when the temperature scales with $\frac{N}{\ln N} T_0$.

Concluding remarks

$SU(N)$ ferromagnets in the thermodynamic limit-new features:

- Focused on the fundamental and adjoint irreps. Stability and YTs. Other irreps possible.
- Spontaneous symmetry breaking, Novel phase transitions, metastable phases, with various patters.
- Admit large-N limit, $N \sim \sqrt{n}$ [Poly-KS, 23,24], Description in terms of density, continuum YT.
- Three body interactions: adds the Cubic Casimir operator in the Hamiltonian [Poly-KS 25]. Quantum transitions at $T = 0$. (not covered)

(Some) future directions:

- Multicomponent $SU(N)$ magnets. Thermodynamics of magnetic chain molecules [Poly-KS to appear]
- Supersymmetric case: atoms with bosonic and fermionic d.o.f.
- Orthogonal groups? YT rules change (slightly). Net effect in the thermodynamic limit?

Exotic directions...

- $SU(N)$, large $N \ldots$, is there a gravity dual?
- Encode (quantum) information with $SU(N)$ spins.
- Infinite order phase transition?: No spontaneous symmetry breaking at finite temperature (in 2-dim). Non-analytic functions, i.e. Berezinskii-Kosterlitz-Thouless type

$$F(T) \sim \begin{cases} e^{-\mathrm{const.}/\sqrt{T - T_{KT}}} \,, & \text{if } T \gtrsim T_{KT} \,, \\ 0 \,, & \text{if } T \lesssim T_{KT} \,. \end{cases}$$

ΕΥΧΑΡΙΣΤΩ