AUTOMATING TWO-LOOP AMPLITUDE COMPUTATION WITH HELAC2LOOP

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HOCTools-II

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Outline

1. Introduction
2. Amplitude construction with `HELAC2LOOP`
3. OPP reduction at two loops, $\rightarrow$ `CutTools2`
4. Numerically evaluating the amplitude in $d = 4 - 2\epsilon$
5. HELAC2LOOP@work
6. Summary & Outlook

$\rightarrow$ addressing conceptual and practical issues

INTRODUCTION

Higgs

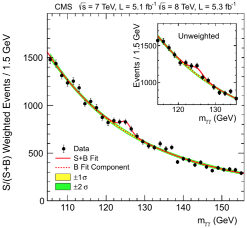


Fig. 3. The diphoton invariant mass distribution with each event weighted by the $S/(S+B)$ value of its category. The lines represent the fitted background and signal, and the coloured bands represent the ±1 and ±2 standard deviation uncertainties in the background estimate. The inset shows the central part of the unweighted invariant mass distribution. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this Letter.)

Gravitational wave

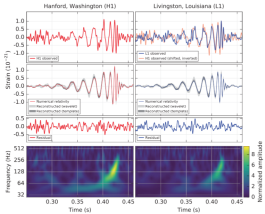


FIG. 1. The gravitational-wave event GW150914 observed by the LIGO Hanford (H1, left column panels) and Livingston (L1, right column panels) detectors. Times are shown relative to September 14, 2015 at 09:50:45 UTC. For visualization, all time series are filtered with a 35–350 Hz bandpass filter to suppress large fluctuations outside the detectors' most sensitive frequency band, and band-reject filters to remove the strong instrumental spectral lines seen in the Fig. 3 spectra. *Top row, left:* H1 strain. *Top row, right:* L1 strain. GW150914 arrived first at L1 and $6.9^{+0.5}_{-0.4}$ ms later at H1; for a visual comparison, the H1 data are also shown, shifted in time by this amount and inverted (to account for the detectors' relative orientations). *Second row:* Gravitational-wave strain projected onto each detector in the 35–350 Hz band. Solid lines show a numerical relativity waveform for a system with parameters consistent with those recovered from GW150914 [37,38] confirmed to 99.9% by an independent calculation based on [15]. Shaded areas show 90% credible regions for two independent waveform reconstructions. One (dark gray) models the signal using binary black hole template waveforms [39]. The other (light gray) does not use an astrophysical model, but instead calculates the strain signal as a linear combination of sine-Gaussian wavelets [40,41]. These reconstructions have a 94% overlap, as shown in [39]. *Third row:* Residuals after subtracting the filtered numerical relativity waveform from the filtered detector time series. *Bottom row:* A time-frequency representation [42] of the strain data, showing the signal frequency increasing over time.

BH Horizon

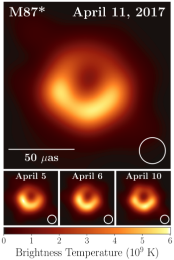
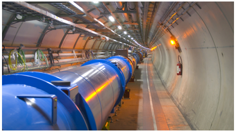
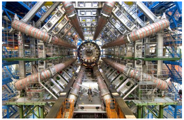


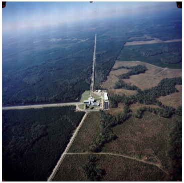
Figure 3. Top: EHT image of M87* from observations on 2017 April 11 as a representative example of the images collected in the 2017 campaign. The image is the average of three different imaging methods after convolving each with a circular Gaussian kernel to give matched resolutions. The largest of the three kernels (20 μas FWHM) is shown in the lower right. The image is shown in units of brightness temperature, $T_b = S\lambda^2/2k_B\Omega$, where S is the flux density, λ is the observing wavelength, k_B is the Boltzmann constant, and Ω is the solid angle of the resolution element. Bottom: similar images taken over different days showing the stability of the basic image structure and the equivalence among different days. North is up and east is to the left.

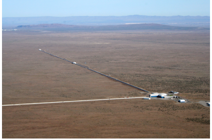
LHC





LIGO





EHT

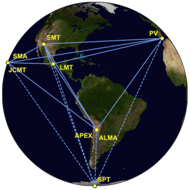
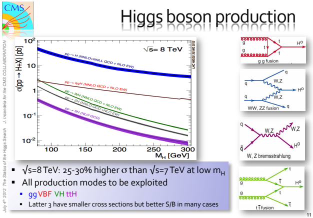
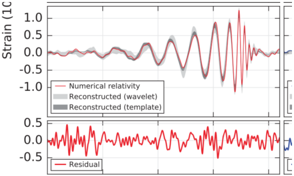


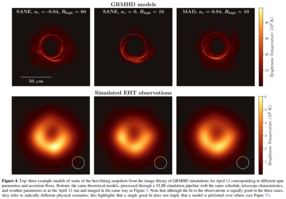
Figure 1. Eight stations of the EHT 2017 campaign over six geographic locations as viewed from the equatorial plane. Solid baselines represent mutual visibility on M87* (+12° declination). The dashed baselines were used for the calibration source 3C279 (see Papers III and IV).

INTRODUCTION

LHC | LIGO | EHT

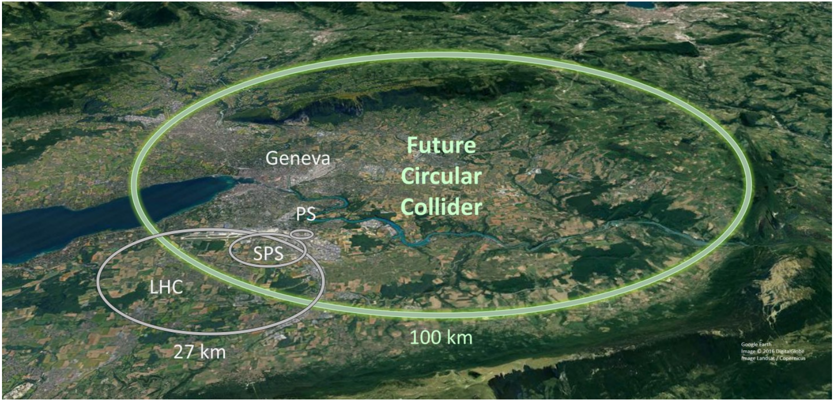






Faint signals; Patience; Theory

FCC

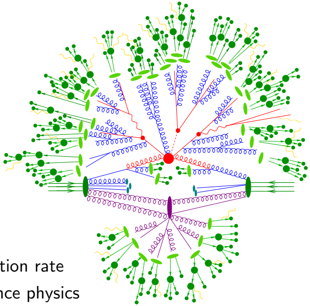


circular accelerators with decelerating pace of expansion!

Factorization

Collins,Soper,Sterman'85–'89

- Calculate
  - Scattering probability
  - Gluon emission probability
- Measure
  - Long distance interactions
  - Particle decay rates



Divide et Impera

- Quantity of interest: Total interaction rate
- Convolution of short & long distance physics

$$\sigma_{p_1 p_2 \to X} = \sum_{i,j \in \{q,g\}} \int \mathrm{d}x_1 \mathrm{d}x_2 \underbrace{f_{p_1,i}(x_1,\mu_F^2) f_{p_2,j}(x_2,\mu_F^2)}_{\text{long distance physics}} \underbrace{\hat{\sigma}_{ij\to X}(x_1 x_2,\mu_F^2)}_{\text{short distance physics}}$$

QCD as a perturbative quantum field theory

Factorization

Collins,Soper,Sterman'85–'89

- Calculate
  - Scattering probability
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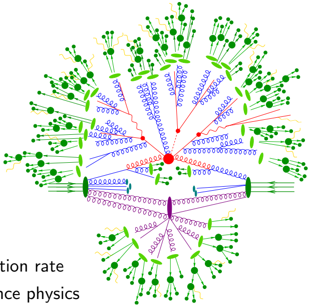

Divide et Impera

- Quantity of interest: Total interaction rate
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$$\sigma_{p_1 p_2 \to X} = \sum_{i,j \in \{q,g\}} \int \mathrm{d}x_1 \mathrm{d}x_2 \underbrace{f_{p_1,i}(x_1,\mu_F^2) f_{p_2,j}(x_2,\mu_F^2)}_{\text{long distance physics}} \underbrace{\hat{\sigma}_{ij\to X}(x_1 x_2,\mu_F^2)}_{\text{short distance physics}}$$

QCD as a perturbative quantum field theory

Lattice QCD results: → C. Alexandrou, *et al.* Phys. Rev. Lett. **121**, 112001 (2018) [arXiv:1803.02685 [hep-lat]].

Fixed-order calculations

DS recursive equations

How to avoid Feynman diagrams

$\rightarrow$ a highly subjective point of view

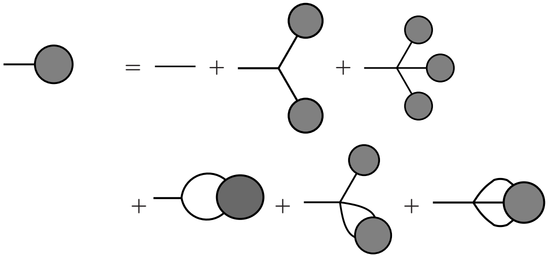
From Feynman Diagrams to recursive equations: taming the n!

- 1999 HELAC: The first code to calculate recursively tree-order amplitudes for (practically) arbitrary number of particles

→A. Kanaki and C. G. Papadopoulos, Comput. Phys. Commun. **132** (2000) 306 [arXiv:hep-ph/0002082].

→F. A. Berends and W. T. Giele, Nucl. Phys. B **306** (1988) 759.

→ F. Caravaglios and M. Moretti, Phys. Lett. B **358** (1995) 332.



Unfortunately not so much on the second line !

→Integrals and Integrands

Taming the beast ...

From Feynman graphs ...

$gg \to ng$	2	3	4	5	6	7	8	9
# FG	4	25	220	2,485	34,300	559,405	10,525,900	224,449,225

to Dyson-Schwinger recursion! `Helac-Phegas`

$gg \to ng$	2	3	4	5	6	7	8	9
#	5	15	35	70	126	210	330	495

NLO

Don't make integrals, make integrands !

What do we need for an NLO calculation ?

$$p_1, p_2 \to p_3, \ldots, p_{m+2}$$

$$\begin{aligned}
\sigma_{NLO} &= \int_m d\Phi_m |M_m^{(0)}|^2 J_m(\Phi) \quad \leftarrow LO \\
&+ \int_m d\Phi_m 2Re(M_m^{(0)*} M_m^{(1)}(\epsilon_{UV}, \epsilon_{IR})) J_m(\Phi) \quad \leftarrow Virtual \\
&+ \int_{m+1} d\Phi_{m+1} |M_{m+1}^{(0)}|^2 J_{m+1}(\Phi) \quad \leftarrow Real
\end{aligned}$$

$J_m(\Phi)$ jet function: Infrared safety $J_{m+1} \to J_m$

What do we need for an NLO calculation ?

$$p_1, p_2 \to p_3, \ldots, p_{m+2}$$

$$\begin{aligned}
\sigma_{NLO} &= \int_m d\Phi_m^{D=4}\left(|M_m^{(0)}|^2 + 2Re\left(M_m^{(0)*} M_m^{(CT)}(\epsilon_{UV})\right)\right) J_m(\Phi) \\
&+ \int_m d\Phi_m^{D=4} 2Re\left(M_m^{(0)*} M_m^{(1)}(\epsilon_{UV}, \epsilon_{IR})\right) J_m(\Phi) \\
&+ \int_{m+1} d\Phi_{m+1}^{D=4-2\epsilon_{IR}} |M_{m+1}^{(0)}|^2 J_{m+1}(\Phi)
\end{aligned}$$

IR and UV divergencies, HV or Four-Dimensional-Helicity scheme; scale dependence μ_R

Perturbative QCD at NLO

What do we need for an NLO calculation ?

$$p_1, p_2 \to p_3, \ldots, p_{m+2}$$

$$\begin{aligned}
\sigma_{NLO} &= \int_m d\Phi_m J_m(\Phi) \\
&+ \int_m d\Phi_m 2Re(M_m^{(0)*} M_m^{(1)}(\epsilon_{UV}, \epsilon_{IR})) J_m(\Phi) \\
&+ \int_{m+1} d\Phi_{m+1} |M_{m+1}^{(0)}|^2 J_{m+1}(\Phi)
\end{aligned}$$

QCD factorization$-\mu_F$ Collinear counter-terms when PDF are involved

The one loop paradigm

basis of scalar integrals:

known already before NLO-R; remember this is not the case for higher orders

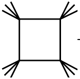
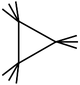
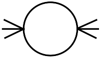
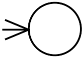
→G. 't Hooft and M. J. G. Veltman, Nucl. Phys. B **153** (1979) 365.

→ Z. Bern, L. J. Dixon and D. A. Kosower, Nucl. Phys. B **412** (1994) 751

→G. Passarino and M. J. G. Veltman, Nucl. Phys. B **160** (1979) 151.

→Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, Nucl. Phys. B **425** (1994) 217.

$$\mathcal{A} \quad = \sum_{I\subset\{0,1,\cdots,m-1\}} \int \frac{\mu^{(4-d)} d^d\bar{q}}{(2\pi)^d} \frac{\bar{N}_I(\bar{q})}{\prod_{i\in I} \bar{D}_i(\bar{q})}$$

$\mathcal{A} = \sum d_{i_1 i_2 i_3 i_4}$  $+ \sum c_{i_1 i_2 i_3}$  $+ \sum b_{i_1 i_2}$  $+ \sum a_{i_1}$  $+ R$

$a, b, c, d \to$ cut-constructible part $\qquad R \to$ rational terms

The old "master" formula

$$
\begin{aligned}
\mathcal{A} \to \int \frac{\bar{N}(\bar{q})}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}} &= \sum_{i_0<i_1<i_2<i_3}^{m-1} d(i_0 i_1 i_2 i_3) \int \frac{1}{\bar{D}_{i0} \bar{D}_{i1} \bar{D}_{i2} \bar{D}_{i2}} \\
&+ \sum_{i_0<i_1<i_2}^{m-1} c(i_0 i_1 i_2) \int \frac{1}{\bar{D}_{i0} \bar{D}_{i1} \bar{D}_{i2}} \\
&+ \sum_{i_0<i_1}^{m-1} b(i_0 i_1) \int \frac{1}{\bar{D}_{i0} \bar{D}_{i1}} \\
&+ \sum_{i_0}^{m-1} a(i_0) \int \frac{1}{\bar{D}_{i0}} \\
&+ \text{rational terms}
\end{aligned}
$$

OPP "MASTER" FORMULA - I

General expression for the 4-dim $N(q)$ at the integrand level in terms of D_i

→ G. Ossola, C. G. Papadopoulos and R. Pittau, [arXiv:hep-ph/0609007 [hep-ph]].

$$
\begin{aligned}
N(q) &= \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} D_i \\
&+ \sum_{i_0<i_1<i_2}^{m-1} \left[c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2) \right] \prod_{i \neq i_0, i_1, i_2}^{m-1} D_i \\
&+ \sum_{i_0<i_1}^{m-1} \left[b(i_0 i_1) + \tilde{b}(q; i_0 i_1) \right] \prod_{i \neq i_0, i_1}^{m-1} D_i \\
&+ \sum_{i_0}^{m-1} \left[a(i_0) + \tilde{a}(q; i_0) \right] \prod_{i \neq i_0}^{m-1} D_i
\end{aligned}
$$

OPP R_1

→G. Ossola, C. G. Papadopoulos and R. Pittau, JHEP **05** (2008), 004 [arXiv:0802.1876 [hep-ph]].

$$\bar{D}_i = (\bar{q} + p_i)^2 - m_i^2 \,, \quad p_0 \neq 0 \,,$$

$$\bar{D}_i = D_i + \tilde{q}^2$$

$$m_i^2 \to m_i^2 - \tilde{q}^2 .$$

$$\begin{aligned} d(ijkl;\tilde{q}^2) &= d(ijkl) + \tilde{q}^2 d^{(2)}(ijkl) + \tilde{q}^4 d^{(4)}(ijkl) \,, \\ c(ijk;\tilde{q}^2) &= c(ijk) + \tilde{q}^2 c^{(2)}(ijk) \,, \\ b(ij;\tilde{q}^2) &= b(ij) + \tilde{q}^2 b^{(2)}(ij) \,. \end{aligned}$$

OPP R_1

$$
\begin{aligned}
d^{(4)}(ijkl) &= \lim_{\tilde{q}^2\to\infty} \frac{d(ijkl;\tilde{q}^2)}{\tilde{q}^4}\,, \\
c^{(2)}(ijk) &= \lim_{\tilde{q}^2\to\infty} \frac{c(ijk;\tilde{q}^2)}{\tilde{q}^2}\,, \\
b^{(2)}(ij) &= \lim_{\tilde{q}^2\to\infty} \frac{b(ij;\tilde{q}^2)}{\tilde{q}^2}\,,
\end{aligned}
$$

$$
\begin{aligned}
d^{(4)}(ijkl) &= \frac{d(ijkl;1)+d(ijkl;-1)-2d(ijkl)}{2}\,, \\
c^{(2)}(ijk) &= c(ijk;1)-c(ijk)\,, \\
b^{(2)}(ij) &= b(ij;1)-b(ij)\,.
\end{aligned}
$$

$$
\begin{aligned}
\int d^n\bar{q} \frac{\tilde{q}^4}{\bar{D}_i \bar{D}_j \bar{D}_k \bar{D}_l} &= -\frac{i\pi^2}{6} + \mathcal{O}(\epsilon)\,, \\
\int d^n\bar{q} \frac{\tilde{q}^2}{\bar{D}_i \bar{D}_j \bar{D}_k} &= -\frac{i\pi^2}{2} + \mathcal{O}(\epsilon)\,, \\
\int d^n\bar{q} \frac{\tilde{q}^2}{\bar{D}_i \bar{D}_j} &= -\frac{i\pi^2}{2} \left[m_i^2 + m_j^2 - \frac{(p_i - p_j)^2}{3} \right] + \mathcal{O}(\epsilon)\,.
\end{aligned}
$$

OPP R_1

$$
\begin{aligned}
\mathrm{R}_1 \quad = & \quad -\frac{i}{96\pi^2} d^{(2m-4)} - \frac{i}{32\pi^2} \sum_{i_0<i_1<i_2}^{m-1} c^{(2)}(i_0 i_1 i_2) \\
- & \quad \frac{i}{32\pi^2} \sum_{i_0<i_1}^{m-1} b^{(2)}(i_0 i_1) \left(m_{i_0}^2 + m_{i_1}^2 - \frac{(p_{i_0} - p_{i_1})^2}{3} \right) .
\end{aligned}
$$

OPP R_2

→P. Draggiotis, M. V. Garzelli, C. G. Papadopoulos and R. Pittau, JHEP **04** (2009), 072 [arXiv:0903.0356 [hep-ph]].

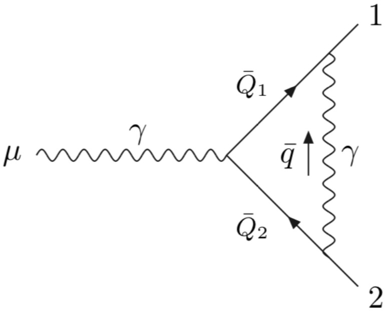
→ M. V. Garzelli, I. Malamos and R. Pittau, JHEP **01** (2010), 040 [erratum: JHEP **10** (2010), 097]

$$\bar{N}(\bar{q}) = N(q) + \tilde{N}(\tilde{q}^2, q, \epsilon)\,.$$

$$\begin{aligned} \bar{q} &= q + \tilde{q}\,, \\ \bar{\gamma}_{\bar{\mu}} &= \gamma_\mu + \tilde{\gamma}_{\tilde{\mu}}\,, \\ \bar{g}^{\bar{\mu}\bar{\nu}} &= g^{\mu\nu} + \tilde{g}^{\tilde{\mu}\tilde{\nu}}\,. \end{aligned}$$

$$\mathrm{R}_2 \equiv \frac{1}{(2\pi)^4}\int d^n\,\bar{q}\,\frac{\tilde{N}(\tilde{q}^2, q, \epsilon)}{\bar{D}_0\bar{D}_1\cdots\bar{D}_{m-1}} \equiv \frac{1}{(2\pi)^4}\int d^n\,\bar{q}\,\mathcal{R}_2\,.$$

OPP R_2



$$\bar{Q}_1 = \bar{q} + p_1 = Q_1 + \tilde{q}$$
$$\bar{Q}_2 = \bar{q} + p_2 = Q_2 + \tilde{q}$$

$$\bar{D}_0 = \bar{q}^2$$

$$\bar{D}_1 = (\bar{q} + p_1)^2$$

$$\bar{D}_2 = (\bar{q} + p_2)^2$$

Figure 1: QED $\gamma e^+ e^-$ diagram in n dimensions.

OPP R_2

ϵ-dimensional γ matrices freely anti-commute with four-dimensional ones:
$\{\gamma_\mu, \tilde{\gamma}_\nu\} = 0$

$$\begin{aligned}
\bar{N}(\bar{q}) &\equiv e^3 \left\{ \bar{\gamma}_{\bar{\beta}} \; (\bar{\not{Q}}_1 + m_e) \; \gamma_\mu \; (\bar{\not{Q}}_2 + m_e) \; \bar{\gamma}^{\bar{\beta}} \right\} \\
&= e^3 \; \{ \gamma_\beta (\not{Q}_1 + m_e) \gamma_\mu (\not{Q}_2 + m_e) \gamma^\beta \\
&- \epsilon (\not{Q}_1 - m_e) \gamma_\mu (\not{Q}_2 - m_e) + \epsilon \tilde{q}^2 \gamma_\mu - \tilde{q}^2 \gamma_\beta \gamma_\mu \gamma^\beta \} \; ,
\end{aligned}$$

$$\begin{aligned}
\int d^n \bar{q} \frac{\tilde{q}^2}{\bar{D}_0 \bar{D}_1 \bar{D}_2} &= -\frac{i\pi^2}{2} + \mathcal{O}(\epsilon) \, , \\
\int d^n \bar{q} \frac{q_\mu q_\nu}{\bar{D}_0 \bar{D}_1 \bar{D}_2} &= -\frac{i\pi^2}{2\epsilon} g_{\mu\nu} + \mathcal{O}(1) \, ,
\end{aligned}$$

gives

$$\mathrm{R}_2 = -\frac{ie^3}{8\pi^2} \gamma_\mu + \mathcal{O}(\epsilon) \, ,$$

OPP R_2

Computing 1PI contributions to $R_2 \rightarrow R_2$ for any 1-loop amplitude

R_2 vertices in full analogy with renormalization CT

1. Determining the on-shell momenta through $D_i = 0$ and computing all coefficients.
2. Determining the on-shell momenta through $D_i = \mu$ and μ dependence of certain coefficients, namely R_1.
3. Using new Feynman rules to compute with tree-like DS the rest of R contribution, namely R_2.

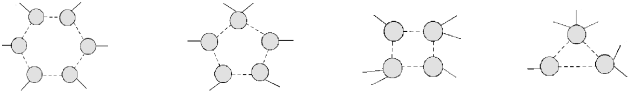
→ G. Ossola, C. G. Papadopoulos and R. Pittau, [arXiv:0802.1876 [hep-ph]].

→ M. V. Garzelli, I. Malamos and R. Pittau, [arXiv:0910.3130 [hep-ph]].

The one-loop calculation in a nutshell

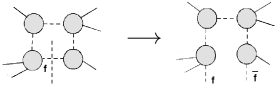
The computation of $pp(p\bar{p}) \to e^+\nu_e\mu^-\bar{\nu}_\mu b\bar{b}$ involves up to six-point functions.
The most generic integrand has therefore the form

$$\mathcal{A}(q) = \sum \underbrace{\frac{N_i^{(6)}(q)}{\bar{D}_{i_0}\bar{D}_{i_1}\cdots\bar{D}_{i_5}}} + \underbrace{\frac{N_i^{(5)}(q)}{\bar{D}_{i_0}\bar{D}_{i_1}\cdots\bar{D}_{i_4}}} + \underbrace{\frac{N_i^{(4)}(q)}{\bar{D}_{i_0}\bar{D}_{i_1}\cdots\bar{D}_{i_3}}} + \underbrace{\frac{N_i^{(3)}(q)}{\bar{D}_{i_0}\bar{D}_{i_1}\bar{D}_{i_2}}} + \cdots$$



In order to apply the OPP reduction, `HELAC` numerically evaluates the numerators $N_i^6(q), N_i^5(q), \ldots.$ with the values of the loop momentum q provided by `CutTools` $+R_1$.

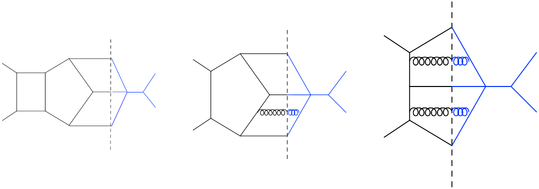
- generates all inequivalent partitions of 6,5,4,3... blobs attached to the loop, and check all possible flavours (and colours) that can be consistently running inside
- hard-cuts the loop (q is fixed) to get a $n+2$ *tree-like* process



The R_2 contributions (rational terms) are calculated in the same way as the tree-order amplitude, taking into account *extra vertices*

What do we need for an NNLO calculation ?

$$p_1, p_2 \to p_3, \ldots, p_{m+2}$$



What do we need for an NNLO calculation ?

$$\sigma_{NNLO} \rightarrow \int_m d\Phi_m \left(2Re(M_m^{(0)*} M_m^{(2)}) + \left|M_m^{(1)}\right|^2\right) J_m(\Phi) \qquad VV$$

$$+ \int_{m+1} d\Phi_{m+1} \left(2Re\left(M_{m+1}^{(0)*} M_{m+1}^{(1)}\right)\right) J_{m+1}(\Phi) \qquad RV$$

$$+ \int_{m+2} d\Phi_{m+2} \left|M_{m+2}^{(0)}\right|^2 J_{m+2}(\Phi) \qquad RR$$

$RV + RR \rightarrow$ antenna-S, colorfull-NNLO, sector-improved residue subtraction, nested soft-collinear, local analytic sector subtraction, projection to born, q_T, N-jetiness

→A. Gehrmann-De Ridder, T. Gehrmann and M. Ritzmann, JHEP **1210** (2012) 047
→ P. Bolzoni, G. Somogyi and Z. Trocsanyi, JHEP **1101** (2011) 059
→ M. Czakon and D. Heymes, Nucl. Phys. B **890** (2014) 152
→S. Catani and M. Grazzini, Phys. Rev. Lett. **98** (2007) 222002
→ R. Boughezal, C. Focke, X. Liu and F. Petriello, Phys. Rev. Lett. **115** (2015) no.6, 062002
→ M. Cacciari, F. A. Dreyer, A. Karlberg, G. P. Salam and G. Zanderighi, Phys. Rev. Lett. **115**, no. 8, 082002 (2015)
→ F. Caola, K. Melnikov and R. Röntsch, Eur. Phys. J. C **77**, no. 4, 248 (2017)
→ L. Magnea, E. Maina, G. Pelliccioli, C. Signorile-Signorile, P. Torrielli and S. Uccirati, arXiv:1806.09570 [hep-ph].

Perturbative QCD at NNLO

- Standard approach: QGRAF → symbolic manipulation, dimensionally regularized amplitudes, rational reconstruction → IBP: `FIRE`, `Kira` or numerical `pySecDec`

→ T. Peraro, [arXiv:1608.01902 [hep-ph]], [arXiv:1905.08019 [hep-ph]].

→ S. Badger, D. Chicherin, T. Gehrmann, G. Heinrich, J. M. Henn, T. Peraro, P. Wasser, Y. Zhang and S. Zoia, [arXiv:1905.03733 [hep-ph]].

→ A. V. Smirnov and M. Zeng, [arXiv:2311.02370 [hep-ph]].

→ S. Abreu, D. Chicherin, H. Ita, B. Page, V. Sotnikov, W. Tschernow and S. Zoia, [arXiv:2306.15431 [hep-ph]].

→ G. Heinrich, S. P. Jones, M. Kerner, V. Magerya, A. Olsson and J. Schlenk, [arXiv:2305.19768 [hep-ph]].

→ G. Cullen *et al.* [GoSam], [arXiv:1111.2034 [hep-ph]].

→ V. Shtabovenko, R. Mertig and F. Orellana, [arXiv:2312.14089 [hep-ph]].

- Numerical unitarity → dimensionally regularized amplitudes by gluing tree amplitudes in different integer dimensions → D_s

→ S. Abreu, J. Dormans, F. Febres Cordero, H. Ita, M. Kraus, B. Page, E. Pascual, M. S. Ruf and V. Sotnikov, CPC **267** (2021), 108069

→ V. Sotnikov, doi:10.6094/UNIFR/151540

- OpenLoops → Feynman graph → opening the loops → amplitudes in $d = 4$ → coefficients of tensor integrals

→ S. Pozzorini, N. Schär and M. F. Zoller, [arXiv:2201.11615 [hep-ph]].

→ talks by Gloria Bertolotti, Nicolo Giraudo and Fabian Lange

Amplitude construction

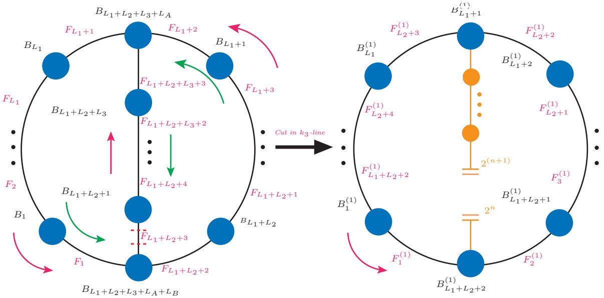
Amplitude construction

- Dyson-Schwinger recursive equations
  $\rightarrow$ QCD color-ordered Berends-Giele

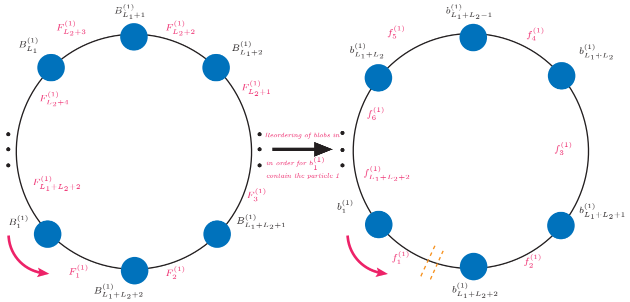
- Color flow representation
  $\rightarrow$ universal simplified treatment of gluons and quarks

- Cutting (or opening) the loop(s)

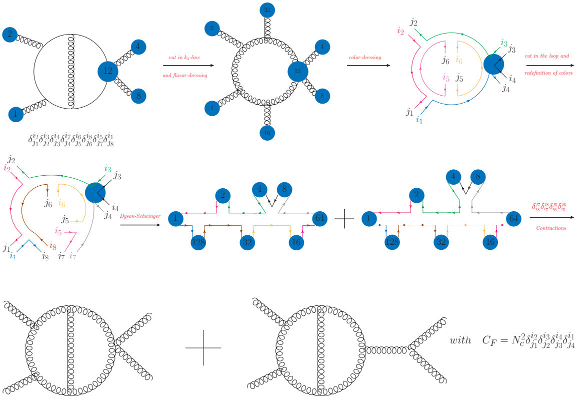
Amplitude construction



Amplitude construction



Amplitude construction

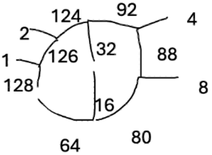


with $C_F = N_c^2 \delta^{i_2}_{j_1}\delta^{i_3}_{j_2}\delta^{i_4}_{j_3}\delta^{i_1}_{j_4}$

Amplitude construction

The "quantum" of HELAC skeleton $\rightarrow$ numerator
$gg \rightarrow gg$, the first out of the six LC flows:

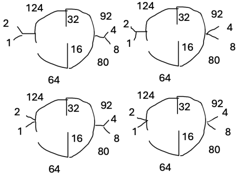
```
 INFO NUM         113  of         580         6
 INFO =====================================
INFO     4    80    35     9     1     1    16    35     5    64    35     7     0     0     0     0     1     2     0     0
INFO     4    88    35    10     1     1     8    35     4    80    35     9     0     0     0     0     1     1     0     0
INFO     4    92    35    11     1     1     4    35     3    88    35    10     0     0     0     0     1     1     0     0
INFO     4   124    35    12     1     1    32    35     6    92    35    11     0     0     0     0     1     2     0     0
INFO     4   126    35    13     1     1     2    35     2   124    35    12     0     0     0     0     1     1     0     0
INFO     4   254    35    14     1     1   128    35     8   126    35    13     0     0     0     0     1     2     0     0
INFO     7     1    36     1     2     4     8    35    35    35    35    35    35    35     0     0     0     0     6     9
```



Each row(s) corresponds to an off-shell current, vector, ghost, fermion

Amplitude construction

```
 INFO NUM         101  of         580          8
 INFO ====================================
INFO     4   80   35    9    1    1   16   35    5   64   35    7    0    0    0    0    1    1    0    0
INFO     4   12   35   10    1    1    4   35    3    8   35    4    0    0    0    0    1    1    0    0
INFO     4   92   35   11    1    2   12   35   10   80   35    9    0    0    0    0    1    2    0    0
INFO     5   92   35   11    2    2    4   35    3    8   35    4   80   35    9    0    1    1    0    0
INFO     4  124   35   12    1    1   32   35    6   92   35   11    0    0    0    0    1    1    0    0
INFO     4  252   35   13    1    1  128   35    8  124   35   12    0    0    0    0    1    1    0    0
INFO     4  254   35   14    1    2    2   35    2  252   35   13    0    0    0    0    1    1    0    0
INFO     5  254   35   14    2    2    2   35    2  128   35    8  124   35   12    0    1    5    0    0
INFO     5    1    6    3   12   35   35   35   35   35    0    0    0    0    0    0    0    0    4    9
```



Remark: Skeleton knows nothing about d: it can be used in $d = 4$ or any other dimension including $d = 4 - 2\epsilon$, and produce numerical and/or analytic(symbolic) results.

A Skeleton Results

Table 1: Summary of the skeleton structures for the QCD corrections to several four-particle SM processes at two loops in full color. The second column specifies the flavors of the particles circulating in the loops. The third column reports the time required to generate the corresponding skeleton. The final column lists the number of individual contributions (numerators) entering the amplitude. All results were obtained using a single CPU core on a personal laptop equipped with an Intel i5 processor and 24GB of RAM.

Process	*Loop-Flavors*	*Time*	*Nums*
$gg \to gg$	$\{g, q, \bar{q}, c, \bar{c}\}$	326 s	74872
$gg \to q\bar{q}$	$\{g, q, \bar{q}, c, \bar{c}\}$	122 s	11336
$gg \to Hg$	$\{g, q, \bar{q}, t, \bar{t}, c, \bar{c}\}$	43 s	3816
$q\bar{q} \to Hg$	$\{g, q, \bar{q}, t, \bar{t}, c, \bar{c}\}$	12 s	392
$gg \to HH$	$\{g, q, \bar{q}, t, \bar{t}, c, \bar{c}\}$	22 s	1424
$q\bar{q} \to HH$	$\{g, q, \bar{q}, t, \bar{t}, c, \bar{c}\}$	4 s	152
$gg \to t\bar{t}$	$\{g, q, \bar{q}, t, \bar{t}, c, \bar{c}\}$	176 s	13120
$q\bar{q} \to t\bar{t}$	$\{g, q, \bar{q}, t, \bar{t}, c, \bar{c}\}$	20 s	1916
$gg \to g\gamma$	$\{g, q, \bar{q}, t, \bar{t}, c, \bar{c}\}$	67 s	7632
$q\bar{q} \to g\gamma$	$\{g, q, \bar{q}, t, \bar{t}, c, \bar{c}\}$	29 s	2932
$gg \to \gamma\gamma$	$\{g, q, \bar{q}, t, \bar{t}, c, \bar{c}\}$	28 s	2848
$q\bar{q} \to \gamma\gamma$	$\{g, q, \bar{q}, t, \bar{t}, c, \bar{c}\}$	8 s	1104
$gg \to H\gamma$	$\{g, q, \bar{q}, t, \bar{t}, c, \bar{c}\}$	30 s	1424
$q\bar{q} \to H\gamma$	$\{g, q, \bar{q}, t, \bar{t}, c, \bar{c}\}$	5 s	176
$gg \to Zg$	$\{g, q, \bar{q}, t, \bar{t}, c, \bar{c}\}$	65 s	7632
$q\bar{q} \to Zg$	$\{g, q, \bar{q}, t, \bar{t}, c, \bar{c}\}$	21 s	2932
$gg \to ZZ$	$\{g, q, \bar{q}, t, \bar{t}, c, \bar{c}\}$	33 s	2848
$q\bar{q} \to ZZ$	$\{g, q, \bar{q}, t, \bar{t}, c, \bar{c}\}$	8 s	1104
$gg \to ZH$	$\{g, q, \bar{q}, t, \bar{t}, c, \bar{c}\}$	24 s	1424
$q\bar{q} \to ZH$	$\{g, q, \bar{q}, t, \bar{t}, c, \bar{c}\}$	5 s	176
$u\bar{d} \to gW^+$	$\{g, u, \bar{u}, d, \bar{d}, t, \bar{t}, c, \bar{c}\}$	19 s	2372
$gg \to W^+W^-$	$\{g, u, \bar{u}, d, \bar{d}, t, \bar{t}, c, \bar{c}\}$	27 s	1424
$u\bar{u} \to W^+W^-$	$\{g, u, \bar{u}, d, \bar{d}, t, \bar{t}, c, \bar{c}\}$	5 s	540
$u\bar{d} \to HW^+$	$\{g, u, \bar{u}, d, \bar{d}, t, \bar{t}, c, \bar{c}\}$	2 s	24
$gg \to Z\gamma$	$\{g, q, \bar{q}, t, \bar{t}, c, \bar{c}\}$	31 s	2848

Continued on next page

Process	*Loop-Flavors*	*Time*	*Nums*
$q\bar{q} \to Z\gamma$	$\{g, q, \bar{q}, t, \bar{t}, c, \bar{c}\}$	8 s	1104
$u\bar{d} \to \gamma W^+$	$\{g, u, \bar{u}, d, \bar{d}, t, \bar{t}, c, \bar{c}\}$	4 s	600
$u\bar{d} \to ZW^+$	$\{g, u, \bar{u}, d, \bar{d}, t, \bar{t}, c, \bar{c}\}$	4 s	600

Table 2: Summary of the skeleton structures for the QCD corrections to some five-particle SM processes at two loops in full color. The structure of the columns is identical to that in table 1, and all results were obtained using the same personal laptop.

Process	*Loop-Flavors*	*Time*	*Nums*
$gg \to ggg$	$\{g, q, \bar{q}, c, \bar{c}\}$	24494 s	1617600
$gg \to q\bar{q}g$	$\{g, q, \bar{q}, c, \bar{c}\}$	9874 s	236360
$gg \to t\bar{t}g$	$\{g, q, \bar{q}, t, \bar{t}, c, \bar{c}\}$	14375 s	270004
$q\bar{q} \to t\bar{t}g$	$\{g, q, \bar{q}, t, \bar{t}, c, \bar{c}\}$	1184 s	36236
$gg \to ggH$	$\{g, q, \bar{q}, t, \bar{t}, c, \bar{c}\}$	2795 s	61192
$q\bar{q} \to ggH$	$\{g, q, \bar{q}, t, \bar{t}, c, \bar{c}\}$	1054 s	6504
$gg \to t\bar{t}H$	$\{g, q, \bar{q}, t, \bar{t}, c, \bar{c}\}$	1748 s	12936
$q\bar{q} \to t\bar{t}H$	$\{g, q, \bar{q}, t, \bar{t}, c, \bar{c}\}$	214 s	5772
$gg \to HHg$	$\{g, q, \bar{q}, t, \bar{t}, c, \bar{c}\}$	1153 s	20680
$q\bar{q} \to HHg$	$\{g, q, \bar{q}, t, \bar{t}, c, \bar{c}\}$	197 s	1996
$gg \to HHH$	$\{g, q, \bar{q}, t, \bar{t}, c, \bar{c}\}$	196 s	8348
$q\bar{q} \to HHH$	$\{g, q, \bar{q}, t, \bar{t}, c, \bar{c}\}$	63 s	748
$gg \to gg\gamma$	$\{g, q, \bar{q}, t, \bar{t}, c, \bar{c}\}$	3741 s	122384
$q\bar{q} \to gg\gamma$	$\{g, q, \bar{q}, t, \bar{t}, c, \bar{c}\}$	1561 s	54252
$gg \to t\bar{t}\gamma$	$\{g, q, \bar{q}, t, \bar{t}, c, \bar{c}\}$	1758 s	54652
$q\bar{q} \to t\bar{t}\gamma$	$\{g, q, \bar{q}, t, \bar{t}, c, \bar{c}\}$	277 s	11464
$gg \to g\gamma\gamma$	$\{g, q, \bar{q}, t, \bar{t}, c, \bar{c}\}$	1653 s	41360
$q\bar{q} \to g\gamma\gamma$	$\{g, q, \bar{q}, t, \bar{t}, c, \bar{c}\}$	416 s	17384
$gg \to \gamma\gamma\gamma$	$\{g, q, \bar{q}, t, \bar{t}, c, \bar{c}\}$	314 s	16032
$q\bar{q} \to \gamma\gamma\gamma$	$\{g, q, \bar{q}, t, \bar{t}, c, \bar{c}\}$	177 s	7064
$gg \to gH\gamma$	$\{g, q, \bar{q}, t, \bar{t}, c, \bar{c}\}$	1233 s	20680
$q\bar{q} \to gH\gamma$	$\{g, q, \bar{q}, t, \bar{t}, c, \bar{c}\}$	266 s	2628
$gg \to H\gamma\gamma$	$\{g, q, \bar{q}, t, \bar{t}, c, \bar{c}\}$	235 s	8016
$q\bar{q} \to H\gamma\gamma$	$\{g, q, \bar{q}, t, \bar{t}, c, \bar{c}\}$	96 s	1176
$gg \to HH\gamma$	$\{g, q, \bar{q}, t, \bar{t}, c, \bar{c}\}$	193 s	8016
$q\bar{q} \to HH\gamma$	$\{g, q, \bar{q}, t, \bar{t}, c, \bar{c}\}$	71 s	896
$gg \to Zgg$	$\{g, q, \bar{q}, t, \bar{t}, c, \bar{c}\}$	4026 s	122384
$q\bar{q} \to Zgg$	$\{g, q, \bar{q}, t, \bar{t}, c, \bar{c}\}$	1830 s	54252
$gg \to t\bar{t}Z$	$\{g, q, \bar{q}, t, \bar{t}, c, \bar{c}\}$	1803 s	54652
$q\bar{q} \to t\bar{t}Z$	$\{g, q, \bar{q}, t, \bar{t}, c, \bar{c}\}$	244 s	11464

Continued on next page

Amplitude construction

Process	*Loop-Flavors*	*Time*	*Nums*
$gg \to ZZg$	$\{g,q,\bar{q},t,\bar{t},c,\bar{c}\}$	1620 s	41360
$q\bar{q} \to ZZg$	$\{g,q,\bar{q},t,\bar{t},c,\bar{c}\}$	450 s	17384

Table 3: Summary of the skeleton structures for the QCD corrections to a six-particle and some five-particle SM processes at two loops in full color.

Process	*Loop-Flavors*	*Time*	*Nums*
$gg \to ZZZ$	$\{g,q,\bar{q},t,\bar{t},c,\bar{c}\}$	307 s	16032
$q\bar{q} \to ZZZ$	$\{g,q,\bar{q},t,\bar{t},c,\bar{c}\}$	146 s	7064
$gg \to ZHg$	$\{g,q,\bar{q},t,\bar{t},c,\bar{c}\}$	1240 s	20872
$q\bar{q} \to ZHg$	$\{g,q,\bar{q},t,\bar{t},c,\bar{c}\}$	244 s	2652
$gg \to HZZ$	$\{g,q,\bar{q},t,\bar{t},c,\bar{c}\}$	201 s	8016
$q\bar{q} \to HZZ$	$\{g,q,\bar{q},t,\bar{t},c,\bar{c}\}$	92 s	1176
$gg \to HHZ$	$\{g,q,\bar{q},t,\bar{t},c,\bar{c}\}$	186 s	8016
$q\bar{q} \to HHZ$	$\{g,q,\bar{q},t,\bar{t},c,\bar{c}\}$	69 s	896
$u\bar{d} \to ggW^+$	$\{g,u,\bar{u},d,\bar{d},t,\bar{t},c,\bar{c}\}$	1592 s	45508
$u\bar{d} \to t\bar{t}W^+$	$\{g,u,\bar{u},d,\bar{d},t,\bar{t},c,\bar{c}\}$	142 s	5692
$gg \to gW^+W^-$	$\{g,u,\bar{u},d,\bar{d},t,\bar{t},c,\bar{c}\}$	1429 s	20680
$u\bar{u} \to gW^+W^-$	$\{g,u,\bar{u},d,\bar{d},t,\bar{t},c,\bar{c}\}$	294 s	5040
$u\bar{d} \to gHW^+$	$\{g,u,\bar{u},d,\bar{d},t,\bar{t},c,\bar{c}\}$	318 s	8468
$u\bar{d} \to HHW^+$	$\{g,u,\bar{u},d,\bar{d},t,\bar{t},c,\bar{c}\}$	37 s	200
$u\bar{u} \to HW^+W^-$	$\{g,u,\bar{u},d,\bar{d},t,\bar{t},c,\bar{c}\}$	41 s	48
$u\bar{d} \to W^+W^-W^+$	$\{g,u,\bar{u},d,\bar{d},t,\bar{t},c,\bar{c}\}$	81 s	2140
$gg \to Zg\gamma$	$\{g,q,\bar{q},t,\bar{t},c,\bar{c}\}$	1419 s	41360
$q\bar{q} \to Zg\gamma$	$\{g,q,\bar{q},t,\bar{t},c,\bar{c}\}$	390 s	17384
$gg \to ZH\gamma$	$\{g,q,\bar{q},t,\bar{t},c,\bar{c}\}$	204 s	8016
$q\bar{q} \to ZH\gamma$	$\{g,q,\bar{q},t,\bar{t},c,\bar{c}\}$	89 s	1176
$u\bar{d} \to g\gamma W^+$	$\{g,u,\bar{u},d,\bar{d},t,\bar{t},c,\bar{c}\}$	239 s	9860
$u\bar{d} \to \gamma HW^+$	$\{g,u,\bar{u},d,\bar{d},t,\bar{t},c,\bar{c}\}$	45 s	264
$u\bar{d} \to gZW^+$	$\{g,u,\bar{u},d,\bar{d},t,\bar{t},c,\bar{c}\}$	241 s	9860
$u\bar{d} \to ZHW^+$	$\{g,u,\bar{u},d,\bar{d},t,\bar{t},c,\bar{c}\}$	44 s	264
$gg \to Z\gamma\gamma$	$\{g,q,\bar{q},t,\bar{t},c,\bar{c}\}$	288 s	16032
$q\bar{q} \to Z\gamma\gamma$	$\{g,q,\bar{q},t,\bar{t},c,\bar{c}\}$	140 s	7064
$gg \to ZZ\gamma$	$\{g,q,\bar{q},t,\bar{t},c,\bar{c}\}$	297 s	16032
$q\bar{q} \to ZZ\gamma$	$\{g,q,\bar{q},t,\bar{t},c,\bar{c}\}$	146 s	7064
$u\bar{d} \to \gamma\gamma W^+$	$\{g,u,\bar{u},d,\bar{d},t,\bar{t},c,\bar{c}\}$	70 s	3040
$u\bar{d} \to ZZW^+$	$\{g,u,\bar{u},d,\bar{d},t,\bar{t},c,\bar{c}\}$	70 s	3040
$u\bar{d} \to Z\gamma W^+$	$\{g,u,\bar{u},d,\bar{d},t,\bar{t},c,\bar{c}\}$	70 s	3040
$gg \to \gamma W^+W^-$	$\{g,u,\bar{u},d,\bar{d},t,\bar{t},c,\bar{c}\}$	151 s	4054

Continued on next page

Amplitude construction

Process	*Loop-Flavors*	*Time*	*Nums*
$u\bar{u} \to \gamma W^+W^-$	$\{g,u,\bar{u},d,\bar{d},t,\bar{t},c,\bar{c}\}$	71 s	2410
$gg \to ZW^+W^-$	$\{g,u,\bar{u},d,\bar{d},t,\bar{t},c,\bar{c}\}$	167 s	4054
$u\bar{u} \to ZW^+W^-$	$\{g,u,\bar{u},d,\bar{d},t,\bar{t},c,\bar{c}\}$	70 s	2410
$q\bar{q} \to \gamma\gamma\gamma\gamma$	$\{g,q,\bar{q},t,\bar{t},c,\bar{c}\}$	1172 s	40976

Table 4: Summary of the skeleton structures for the QCD corrections to selected SM processes at two loops in the leading-color approximation.

Process	*Loop-Flavors*	*Time*	*Nums*
$q\bar{q} \to ggg$	$\{g,q,\bar{q},c,\bar{c}\}$	612 s	6348
$gg \to ggg$	$\{g,q,\bar{q},c,\bar{c}\}$	6730 s	47280
$q\bar{q} \to t\bar{t}H$	$\{g,q,\bar{q},t,\bar{t},c,\bar{c}\}$	20 s	276
$gg \to t\bar{t}H$	$\{g,q,\bar{q},t,\bar{t},c,\bar{c}\}$	104 s	1198
$q\bar{q} \to \gamma\gamma\gamma\gamma$	$\{g,q,\bar{q},t,\bar{t},c,\bar{c}\}$	239 s	3316
$gg \to t\bar{t}q\bar{q}$	$\{g,q,\bar{q},t,\bar{t},c,\bar{c}\}$	3546 s	10872
$q\bar{q} \to gggg$	$\{g,q,\bar{q},c,\bar{c}\}$	34098 s	77616
$gg \to gggg$	$\{g,q,\bar{q},c,\bar{c}\}$	24783 s	667440

$\to$ Bevilacqua Giuseppe, Canko Dhimiter, Papadopoulos Costas and Spourdalakis Aris, in preparation

Integrand reduction

$\rightarrow$ Bevilacqua Giuseppe, Canko Dhimiter, Papadopoulos Costas and Spourdalakis Aris, hep-ph: 2506.07231

$$\mathcal{A}^{(L)}(\{p\}) = \sum_{g=1}^{G} \left(\int \prod_{i=1}^{L} [dk_i] \, \frac{\mathcal{N}(k_1, \ldots, k_L, \{p\})}{D_1 \, D_2 \, \cdots D_n} \right)_g \, ,$$

Extending the OPP approach to two loops, one can express the numerator $\mathcal{N}$ of a generic loop integrand in the following form:

$$\mathcal{N} = P^{(n)} + \sum_{i=1}^{n} P_i^{(n-1)} D_i + \sum_{i=1}^{n-1} \sum_{j>i}^{n} P_{ij}^{(n-2)} D_i D_j + \ldots + P_{12\ldots n}^{(0)} D_1 D_2 \ldots D_n$$

Focusing on the two-loop case, our goal eventually is to decompose the amplitude, $\mathcal{A}^{(2)}$, in terms of a set of Feynman integrals F_i and coefficients c_i that depend only on the external kinematics,

$$\mathcal{A}^{(2)} = \sum_i c_i(\{p\}) \, F_i \, ,$$

$$F_i \equiv F(a_1, \ldots, a_N) = \int \prod_{i=1}^{2} [dk_i] \frac{1}{D_1^{a_1} \ldots D_N^{a_N}} \, , \qquad a_i \in \mathbb{Z} \, .$$

Notice that $D_1, \ldots, D_N$, $N > n$, define an enlarged set of inverse propagators $\rightarrow$ family.

OPP@2L WITH HELAC

- In contrast to the one-loop case, $P_{i\ldots}^{(n-\ldots)}$ include monomials from $\{k_i \cdot k_j, k_i \cdot p_j, k_i \cdot \eta\}$, not determined by the cut equations $\rightarrow$ ISP, not all spurious, as in the one-loop case.
- Therefore, at the integral level expecting F_i with $a_i < 0$, $\rightarrow$ stemming from all $\{k_i \cdot k_j, k_i \cdot p_j, k_i \cdot \eta\}$

We can therefore consider the projection of the numerator over the entire family of inverse propagators, allowing a universal treatment of all numerators regardless of the structure of their denominator:

$$\mathcal{N} = P^{(N)} + \underbrace{\sum_{i=1}^{N} \overbrace{P_i^{(N-1)} D_i}^{cuts}}_{level=N-1} + \sum_{i=1}^{N-1} \sum_{j>i}^{N} P_{ij}^{(N-2)} D_i D_j + \ldots + P_{12\ldots N}^{(0)} D_1 D_2 \ldots D_N$$

Just to clarify the terminology, each term represents a level and each term within the level an individual cut.
The algorithm to solve it can be described as follows. For each cut, all the relevant on-shell conditions (cut equations) are solved, for the eight 4-dimensional components of k_1, k_2 and the three μ_{ij} collected in the eleven-dimensional array $\vec{X}$.
Assuming that the set of monomials m_i $(i = 1, \ldots, M)$ parametrizing a given polynomial P is established, $P = \sum_{i=1}^{M} c_i m_i$, then an $M \times M$ matrix, $\mathcal{M}$, is obtained by evaluating the monomials m_i on the solution to the cut equations, by assigning M random values to the free parameters of the solution, obtaining thus M instances of $\vec{X}_j$, $j = 1, \ldots, M$, and then computing the elements of the matrix $\mathcal{M}$, as follows: $\mathcal{M}_{i,j} \equiv m_i(\vec{X}_j)$. $\rightarrow$ Zhang, Yang, hep-ph:1205.5707

The numerator $\mathcal{N}(\vec{X}, d)$ can be cast in the form

$$\mathcal{N} \equiv \mathcal{N}(\vec{X}, d) = \mathcal{N}_0 + \sum_{i \geq 1} \epsilon^i \mathcal{N}_\epsilon^{(i)} ,$$

by expanding in powers of $\epsilon \equiv (4-d)/2$, where both $\mathcal{N}_0 \equiv \mathcal{N}(\vec{X}, d)|_{d=4}$ and $\mathcal{N}_\epsilon^{(i)} \equiv \mathcal{N}_\epsilon^{(i)}(\vec{X})$, are accessible numerically and depending on μ_{ij} through $\vec{X}$.

The master equation is then solved iteratively: first solving for $P^{(N)}$,

$$P^{(N)} = \sum_{i=1}^{M} \left(c_i^{(0)} + \sum \epsilon^j \; c_i^{(j)} \right) m_i$$

using $\vec{X}$ and $\mathcal{M}$ from the top-level cut equations,

$$\vec{c}^{\,(0)} = \mathcal{M}^{-1} \mathcal{B}^{(0)} \qquad \vec{c}^{\,(i)} = \mathcal{M}^{-1} \mathcal{B}^{(i)}$$

with

$$\mathcal{B}_j^{(0)} = \mathcal{N}_0(\vec{X}_j) \quad \mathcal{B}_j^{(i)} = \mathcal{N}_\epsilon^{(i)}(\vec{X}_j).$$

Subsequently, the known polynomial $P^{(N)}$ is subtracted, and the process is repeated to solve for the remaining coefficients $P_i^{(N-1)}$ $(i = 1, \ldots, N)$ and all subsequent terms To ensure the validity of the numerical algebraic reduction, an $N = N$ test is performed at each iteration by comparing the left- and right-hand sides. This is done on values of the parameters on the relevant cut, other than those used in the fitting procedure, at each stage. At the final step of the recursion, the $N = N$ test is performed at completely random values of $\vec{X}$, thus ensuring that the reduction had been fully completed.

Application of the method

→ Bevilacqua Giuseppe, Canko Dhimiter, Papadopoulos Costas and Spourdalakis Aris, hep-ph: 2506.07231

- We have addressed the solution of the integrand master equation for numerators representative of a variety of scattering processes, ranging from four- to six-point kinematics, including massive particles, both analytically and numerically.

- For the generation of the analytic expressions required, we used the `Mathematica` packages `FeynArts` and `FeynCalc`, except for the 6-particle case, which has been generated by `FORM`.

→ T. Hahn, [arXiv:hep-ph/0012260 [hep-ph]].

→ J. Kublbeck, M. Bohm and A. Denner, Comput. Phys. Commun. **60** (1990), 165-180

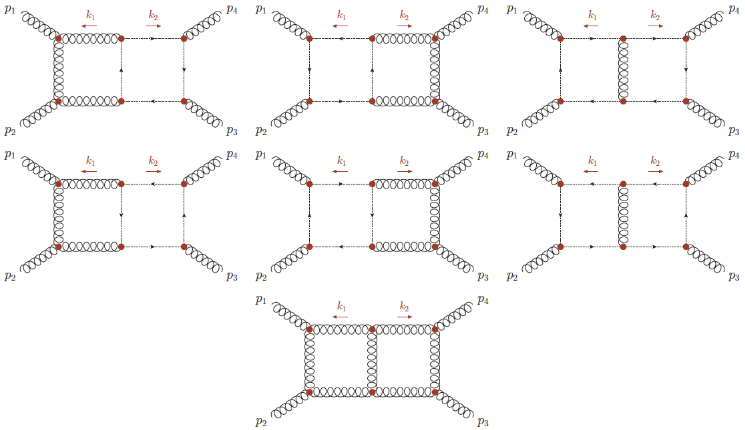
→ V. Shtabovenko, R. Mertig and F. Orellana, [arXiv:2312.14089 [hep-ph]].

→ J. Kuipers, T. Ueda, J. A. M. Vermaseren and J. Vollinga, [arXiv:1203.6543 [cs.SC]].

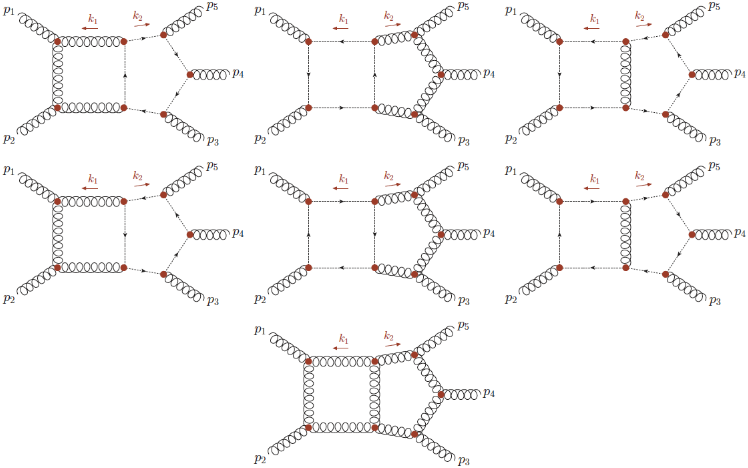
- The procedure is as follows:
  - For each level and cut, solve the cut equations $D_{i1} = D_{i2} = \ldots = 0$.
  - Fit or solve explicitly for the coefficients c_i.

- Perform the analysis in both $d = 4$ and $d = 4 - 2\epsilon$ dimensions.

$gg \to gg$.

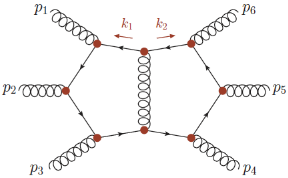


$gg \to ggg$.

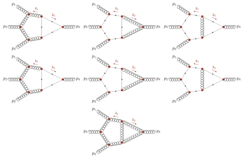
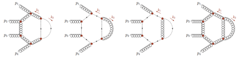
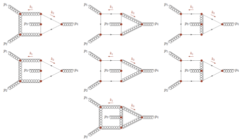
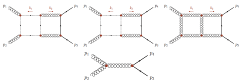
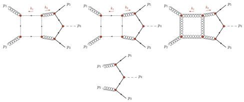


Application of the method

$gg \to gggg$.



Other contributions

Penta-triangle		Hexa-bubble
		
Non-planar double box	$gg \to t\bar{t}$	$gg \to t\bar{t}H$
		

$\rightarrow$ Bevilacqua Giuseppe, Canko Dhimiter, Papadopoulos Costas and Spourdalakis Aris, hep-ph: 2506.07231

Recapitulation

- Solving cut equations
  in $d = 4$, by parametrizing loop momenta in terms of the 8 components of 4-dimensional loop momenta
  in $d = 4 - 2\epsilon$, including also $\mu_{ij} = k_i^{[d-4]} \cdot k_j^{[d-4]}$, i.e. a total of 11 parameters
  $\rightarrow$ find systematically all solutions
- Determine all the coefficients c_i, numerically and/or analytically.
  - In $d = 4 - 2\epsilon$ at each cut solving a full-rank linear system
  - In $d = 4$ linear system solvable but rank-deficient
- Amplitude reduction $d = 4 - 2\epsilon$ seems preferable vs $d = 4 \rightarrow R_1$ & R_2

$\rightarrow$ J. N. Lang, S. Pozzorini, H. Zhang and M. F. Zoller, [arXiv:2107.10288 [hep-ph]].

$\rightarrow$ C. Duhr and P. Thakkar, [arXiv:2310.14551 [hep-ph]].

$\rightarrow$ H. Zhang, PoS **LL2022** (2022), 072

$\rightarrow$ S. Pozzorini, H. Zhang and M. F. Zoller, [arXiv:2001.11388 [hep-ph]].

$\rightarrow$ J. N. Lang, S. Pozzorini, H. Zhang and M. F. Zoller, [arXiv:2007.03713 [hep-ph]].

$$\boxed{\text{The amplitude in } d = 4 - 2\epsilon}$$

$\rightarrow$ Bevilacqua Giuseppe, Canko Dhimiter, Papadopoulos Costas and Spourdalakis Aris, in preparation

Amplitudes in $d = 4 - 2\epsilon$ with HELAC2LOOP

Amplitude reduction in $d = 4 - 2\epsilon$ requires reconstructing numerically the dimensionally regulated numerator $\mathcal{N}$.

- Numerical Unitarity: gluing tree amplitudes in different integer dimensions $\rightarrow D_s > 4$

→ W. T. Giele, Z. Kunszt and K. Melnikov, [arXiv:0801.2237 [hep-ph]].

→ S. Abreu, F. Febres Cordero, H. Ita, M. Jaquier, B. Page and M. Zeng, [arXiv:1703.05273 [hep-ph]].

→ S. Abreu, F. Febres Cordero, H. Ita, B. Page and V. Sotnikov, [arXiv:1809.09067 [hep-ph]].

→ S. Abreu, J. Dormans, F. Febres Cordero, H. Ita, M. Kraus, B. Page, E. Pascual, M. S. Ruf and V. Sotnikov, [arXiv:2009.11957 [hep-ph]].

→ V. Sotnikov, doi:10.6094/UNIFR/151540

- Introducing extra particles and Feynman rules;
  $\rightarrow$ only μ not $d - 4$ terms?

→R. A. Fazio, P. Mastrolia, E. Mirabella and W. J. Torres Bobadilla, [arXiv:1404.4783 [hep-ph]].

→ R. Pittau, [arXiv:1111.4965 [hep-ph]].

Amplitudes in $d = 4 - 2\epsilon$ with HELAC2LOOP

Calculating the dimensionally regulated numerators with `HELAC` in the 't Hooft-Veltman (HV) scheme.

$$\bar{k}_i = k_i + \tilde{k}_i, \quad \bar{\gamma}^\mu = \gamma^\mu + \tilde{\gamma}^\mu, \quad \bar{g}^{\mu\nu} = g^{\mu\nu} + \tilde{g}^{\mu\nu}$$

$$\bar{\varepsilon}_i = \varepsilon_i + \tilde{\varepsilon}_i$$

$$\mu_{ij} = \tilde{k}_i \cdot \tilde{k}_j = \tilde{\not{k}}_i \tilde{\not{k}}_j$$

$$d - 4 = \tilde{g}^{\mu\nu} \tilde{g}_{\mu\nu} = \tilde{\gamma}^\mu \tilde{\gamma}_\mu$$

At one loop $q = k_1$, and for pure YM

`HELAC` aficionados:

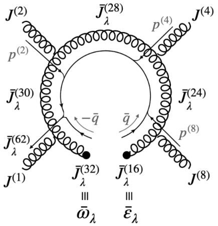


Figure 6: Sketch of recursive computation of the one-loop numerator for a representative $gg \to gg$ topology. The recursion starts from current $\bar{J}^{(16)}$ (particle 2^n) and flows towards $J^{(1)}$. Intermediate currents $\bar{J}_\lambda^{(24)}$, $\bar{J}_\lambda^{(28)}$, $\bar{J}_\lambda^{(30)}$, $\bar{J}_\lambda^{(62)}$ are computed along the recursion. Current $\bar{J}_\lambda^{(16)}$ marks the *loop start point*, while current $\bar{J}_\lambda^{(32)}$ marks the *loop end point*. The numerator is obtained as $\bar{N} = \sum_\lambda \left(\bar{J}_\lambda^{(62)} \cdot J_\lambda^{(1)}\right)$.

Amplitudes in $d = 4 - 2\epsilon$ with HELAC2LOOP

The problem that we are addressing can be formulated as follows: find a method to reconstruct the $(d-4)$-dimensional part of a recursive current (computed either analytically or numerically) knowing its $d = 4$ component. Schematically:

$$J^{(N)} \rightarrow \overline{J}^{(N)} = J^{(N)} + \mathcal{E}\,[\,J^{(N)}\,]\,,$$

where $\mathcal{E}[\cdot]$ is a formal operator which generates all $d-4$ and $\mu \equiv \tilde{q}^2$ terms associated with an input quantity in $d = 4$.

$$\mathcal{E}\,[\,q^2\,X\,] = \mu\,X\,,$$

$$\mathcal{E}\,[\,\sum_\lambda (\varepsilon_\lambda \cdot \omega_\lambda)\,X\,] = (d-4)\,X\,,$$

$$\mathcal{E}\,[\,\sum_\lambda (\varepsilon_\lambda \cdot q)\,(\omega_\lambda \cdot q)\;X\,] = \mu\,X\,.$$

Amplitudes in $d = 4 - 2\epsilon$ with HELAC2LOOP

$$
\begin{aligned}
\mathcal{E}\,[\,J^{(N)\,\alpha}\,] \quad &\equiv \quad C_q^{(N)}\,\tilde{q}^{\alpha} \,+\, C_{\varepsilon}^{(N)}\,\tilde{\varepsilon}^{\alpha} \,+\, J_{\varepsilon q}^{(N)\,\alpha}\,(\tilde{\varepsilon}\cdot\tilde{q}) \,+\, C_{\varepsilon q,q}^{(N)}\,(\tilde{\varepsilon}\cdot\tilde{q})\,\tilde{q}^{\alpha} \\
&+ \quad J_{\mu}^{(N)\,\alpha} \,+\, (d-4)\,J_{d-4}^{(N)\,\alpha}\,.
\end{aligned}
$$

$$
\begin{aligned}
L^{(n_i)} \quad &= \quad 0 \quad \text{if the current does not belong to a loop propagator}\,; \\
L^{(n_i)} \quad &= \quad 1 \quad \text{if the current belongs to a loop propagator}\,; \\
L^{(n_i)} \quad &= \quad 2 \quad \text{if } n_i = 2^{n+1} \text{ (loop endpoint)} \ \ .
\end{aligned}
$$

Amplitudes in $d = 4 - 2\epsilon$ with HELAC2LOOP

For a 3-gluon vertex:

$$
\begin{aligned}
C_q^{(n_1+n_2)} &= \theta(L^{(n_1)})\; C_q^{(n_1)}\, J^{(n_2)} \cdot \left(2\, p^{(n_1)} + p^{(n_2)}\right) - \theta(L^{(n_2)})\; C_q^{(n_2)}\, J^{(n_1)} \cdot \left(p^{(n_1)} + 2\, p^{(n_2)}\right) \\
&+ \left(\theta(L^{(n_2)}) - \theta(L^{(n_1)})\right)\; J^{(n_1)} \cdot J^{(n_2)} \\
C_\varepsilon^{(n_1+n_2)} &= \theta(L^{(n_1)})\; C_\varepsilon^{(n_1)}\, J^{(n_2)} \cdot \left(2\, p^{(n_1)} + p^{(n_2)}\right) \\
&- \theta(L^{(n_2)})\; C_\varepsilon^{(n_2)}\, J^{(n_1)} \cdot \left(p^{(n_1)} + 2\, p^{(n_2)}\right) \\
J_{\varepsilon q}^{(n_1+n_2)\,\alpha} &= \theta(L^{(n_1)})\; \left(V^\alpha_{g_1 g_2 \to g}\,[J_{\varepsilon q}^{(n_1)}, J^{(n_2)}] \;-\; J^{(n_1)\,\alpha}\left(C_\varepsilon^{(n_1)} + \mu\, C_{\varepsilon q,q}^{(n_1)}\right)\right) \\
&+ \theta(L^{(n_2)})\; \left(V^\alpha_{g_1 g_2 \to g}\,[J^{(n_1)}, J_{\varepsilon q}^{(n_2)}] \;+\; J^{(n_2)\,\alpha}\left(C_\varepsilon^{(n_2)} + \mu\, C_{\varepsilon q,q}^{(n_2)}\right)\right) \\
C_{\varepsilon q,q}^{(n_1+n_2)} &= \theta(L^{(n_1)})\; \left(J_{\varepsilon q}^{(n_1)} \cdot J^{(n_2)} \;+\; C_{\varepsilon q,q}^{(n_1)}\, J^{(n_2)} \cdot \left(2\, p^{(n_1)} + p^{(n_2)}\right)\right) \\
&+ \theta(L^{(n_2)})\; \left(J^{(n_1)} \cdot J_{\varepsilon q}^{(n_2)} \;-\; C_{\varepsilon q,q}^{(n_2)}\, J^{(n_1)} \cdot \left(p^{(n_1)} + 2\, p^{(n_2)}\right)\right) \\
J_\mu^{(n_1+n_2)\,\alpha} &= \theta(L^{(n_1)})\left(-\mu\, C_q^{(n_1)}\, J^{(n_2)\,\alpha}\right) \;+\; \theta(L^{(n_2)})\left(\mu\, C_q^{(n_2)}\, J^{(n_1)\,\alpha}\right) \\
&+ \theta(L^{(n_1)})\,\theta(L^{(n_2)} - 1)\left(2\,\mu\, C_q^{(n_1)}\, J^{(n_2)\,\alpha} +\; \mu\, J_{\varepsilon q}^{(n1)\,\alpha} + \mu\, C_{\varepsilon q,q}^{(n1)}\;\left(p^{(n_2)} - p^{(n_1)}\right)^\alpha\right) \\
&+ \theta(L^{(n_1)} - 1)\,\theta(L^{(n_2)})\left(-2\,\mu\, C_q^{(n_2)}\, J^{(n_1)\,\alpha} -\; \mu\, J_{\varepsilon q}^{(n2)\,\alpha} + \mu\, C_{\varepsilon q,q}^{(n_2)}\;\left(p^{(n_2)} - p^{(n_1)}\right)^\alpha\right) \\
J_{(d-4)}^{(n_1+n_2)\,\alpha} &= \theta(L^{(n_1)})\,\theta(L^{(n_2)} - 1)\left(C_\varepsilon^{(n_1)}\;\left(p^{(n_2)} - p^{(n_1)}\right)^\alpha\right) \\
&+ \theta(L^{(n_1)} - 1)\,\theta(L^{(n_2)})\left(C_\varepsilon^{(n_2)}\;\left(p^{(n_2)} - p^{(n_1)}\right)^\alpha\right)
\end{aligned}
$$

Amplitudes in $d = 4 - 2\epsilon$ with HELAC2LOOP

Similar equations hold for a 4-gluon vertex, whereas the current associated to a ghost can be parametrized

$$\mathcal{E}\,[\,J^{(N)}_{\text{ghost}}\,] \equiv C^{(N)}_{\varepsilon q}\,(\tilde{\varepsilon}\cdot\tilde{q}) + J^{(N)}_{\mu} + (d-4)\,J^{(N)}_{d-4}$$

along with its own recursive equations stemming from gluon-ghost vertices.

- The recursive equations described so far have been implemented in a new version of the one-loop framework, `HELAC1LOOP_DDIM`.
- We have validated the numerical implementation of the recursive equations at one loop against analytic results from FeynCalc and in-house codes based on `Mathematica` and `FORM`.

→ V. Shtabovenko, R. Mertig and F. Orellana, [arXiv:2312.14089 [hep-ph]].

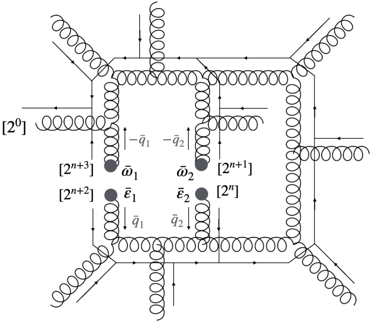
→ J. Kuipers, T. Ueda, J. A. M. Vermaseren and J. Vollinga, [arXiv:1203.6543 [cs.SC]].

- Moreover, we have tested that the results obtained from the new version (no R_1, no R_2 terms), regarding $gg \to gg$ and $gg \to ggg$ one-loop amplitudes, are in complete agreement with those provided by the existing `HELAC-1LOOP`.

→ A. van Hameren, C. G. Papadopoulos and R. Pittau, [arXiv:0903.4665 [hep-ph]].

→ G. Bevilacqua, M. Czakon, M. V. Garzelli, A. van Hameren, A. Kardos, C. G. Papadopoulos, R. Pittau and M. Worek, “HELAC-NLO,” [arXiv:1110.1499 [hep-ph]].

AMPLITUDES IN $d = 4 - 2\epsilon$ WITH HELAC2LOOP



Amplitudes in $d = 4 - 2\epsilon$ with HELAC2LOOP

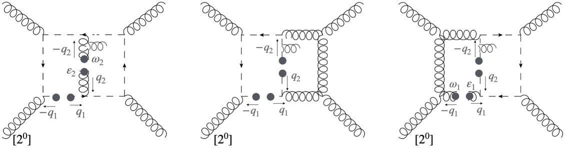
The origin of evanescent terms for the two-loop case can be tracked down to the following relations:

$$
\begin{aligned}
&\mathcal{E}\left[\,(q_i \cdot q_j)\, X\,\right] = \mu_{ij}\, X\,, \\
&\mathcal{E}\left[\,\sum_{\lambda} (q_i \cdot \varepsilon_{k,\,\lambda})\,(q_j \cdot \omega_{k,\,\lambda})\, X\,\right] \;=\; \mu_{ij}\, X\,, \\
&\mathcal{E}\left[\,\sum_{\lambda} (\varepsilon_{i,\,\lambda} \cdot \omega_{i,\,\lambda})\; X\,\right] \;=\; (d-4)\, X\,, \\
&\mathcal{E}\left[\,\sum_{\lambda_1,\lambda_2} (\varepsilon_{1,\lambda_1} \cdot \omega_{1,\lambda_1})\,(\varepsilon_{2,\lambda_2} \cdot \omega_{2,\lambda_2})\; X\,\right] \;=\; (d-4)\, X\,, \\
&\mathcal{E}\left[\,\sum_{\lambda_1,\lambda_2} (\varepsilon_{1,\lambda_1} \cdot \omega_{2,\lambda_2})\,(\varepsilon_{2,\lambda_2} \cdot \omega_{1,\lambda_1})\; X\,\right] \;=\; (d-4)\, X\,, \\
&\mathcal{E}\left[\,\sum_{\lambda_1,\lambda_2} (\varepsilon_{1,\lambda_1} \cdot \varepsilon_{2,\lambda_2})\,(\omega_{1,\lambda_1} \cdot \omega_{2,\lambda_2})\; X\,\right] \;=\; (d-4)\, X
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{E}\,[\,J^{(N)\,\alpha}\,] \quad &\equiv \quad \sum_{i=1}^{2} C^{(N)}_{q_i}\,\tilde{q}_i^{\alpha} + \sum_{i=1}^{2} C^{(N)}_{\varepsilon_i}\,\tilde{\varepsilon}_i^{\alpha} + C^{(N)}_{\omega_2}\,\tilde{\omega}_2^{\alpha} \\
&+ \sum_{i,j=1}^{2} J^{(N)\,\alpha}_{\varepsilon_i q_j}\,(\tilde{\varepsilon}_i\cdot\tilde{q}_j) + J^{(N)\,\alpha}_{\omega_2 q_2}\,(\tilde{\omega}_2\cdot\tilde{q}_2) \\
&+ J^{(N)\,\alpha}_{\varepsilon_1\varepsilon_2}\,(\tilde{\varepsilon}_1\cdot\tilde{\varepsilon}_2) + \sum_{i,j=1}^{2} J^{(N)\,\alpha}_{\varepsilon_1 q_i \varepsilon_2 q_j}\,(\tilde{\varepsilon}_1\cdot\tilde{q}_i)(\tilde{\varepsilon}_2\cdot\tilde{q}_j) \\
&+ \sum_{i,j,k=1}^{2} C^{(N)}_{\varepsilon_i q_j, q_k}(\tilde{\varepsilon}_i\cdot\tilde{q}_j)\,\tilde{q}_k^{\alpha} + \sum_{i,j,k\neq i}^{2} C^{(N)}_{\varepsilon_i q_j,\varepsilon_k}(\tilde{\varepsilon}_i\cdot\tilde{q}_j)\,\tilde{\varepsilon}_k^{\alpha} \\
&+ C^{(N)}_{\omega_2 q_2, q_2}(\tilde{\omega}_2\cdot\tilde{q}_2)\,\tilde{q}_2^{\alpha} + \sum_{i=1}^{2} C^{(N)}_{\varepsilon_1\varepsilon_2, q_i}(\tilde{\varepsilon}_1\cdot\tilde{\varepsilon}_2)\,\tilde{q}_i^{\alpha} \\
&+ \sum_{i,j,k=1}^{2} C^{(N)}_{\varepsilon_i q_j,\varepsilon_k}(\tilde{\varepsilon}_1\cdot\tilde{q}_i)(\tilde{\varepsilon}_2\cdot\tilde{q}_j)\tilde{q}_k^{\alpha} \\
&+ J^{(N)\,\alpha}_{\mu} + \sum_{m}(d-4)^m\,J^{(N)\,\alpha}_{(d-4),m}
\end{aligned}
$$

where $J^{(N)\,\alpha}_{\mu}$ contains all μ_{ij}-dependent terms at $\mathcal{O}((d-4)^0)$, while $J^{(N)\,\alpha}_{(d-4),m}$ denotes $\mathcal{O}((d-4)^m)$ contributions. Let us note that also the latter contributions may contain μ_{ij}-dependent terms.

Amplitudes in $d = 4 - 2\epsilon$ with HELAC2LOOP



$$\mathcal{E}\,[\,J^{(N)}_{\mathrm{ghost}}\,] \equiv \sum_{i=1}^{2} C^{(N)}_{\varepsilon_i q_i}\,(\tilde{\varepsilon}_i \cdot \tilde{q}_i) + J^{(N)}_{\mu}$$

Let us note that, at two loops, ghost topologies can accommodate gluon propagators in the loop. This implies that the coefficient $C^{(N)}_{\varepsilon_i q_i}$ can be non-zero, and μ_{ij} terms can be generated along the recursion. Still no $\mathcal{O}(d-4)$ terms can be generated at any stage of the recursion.

Amplitudes in $d = 4 - 2\epsilon$ with HELAC2LOOP

When full QCD is taken into account, the most general $(d-4)$-dimensional structure of a vector current $J^{(N)\,\alpha}$, at one loop, reads:

$$
\begin{aligned}
\mathcal{E}\,[\,J^{(N)\,\alpha}\,] \quad &\equiv \quad C_q^{(N)}\,\tilde{q}^\alpha + C_\varepsilon^{(N)}\,\tilde{\varepsilon}^\alpha + J_{\varepsilon q}^{(N)\,\alpha}\,(\tilde{\varepsilon}\cdot\tilde{q}) + C_{\varepsilon q,q}^{(N)}\,(\tilde{\varepsilon}\cdot\tilde{q})\,\tilde{q}^\alpha \\
&+ \quad C_\gamma \tilde{\gamma}^\alpha \\
&+ \quad J_\mu^{(N)\,\alpha} + (d-4)\,J_{d-4}^{(N)\,\alpha}\,.
\end{aligned}
$$

Amplitudes in $d = 4 - 2\epsilon$ with HELAC2LOOP

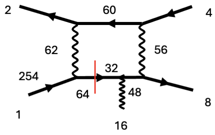
Similarly, for a fermionic current $J^{(N)}$ one has:

$$\begin{aligned}
\mathcal{E}\,[\,J^{(N)}\,] \quad &\equiv \quad \tilde{\not{q}}\,\psi_q^{(N)} + \tilde{\not{\varepsilon}}\,\psi_\varepsilon^{(N)} + \tilde{\not{q}}\tilde{\not{\varepsilon}}\,\psi_{q/\varepsilon}^{(N)} + (\tilde{q}\cdot\tilde{\varepsilon})\,\psi_{\varepsilon q}^{(N)} + (\tilde{q}\cdot\tilde{\varepsilon})\,\tilde{\not{q}}\,\psi_{\varepsilon q,q}^{(N)} \\
&+ \quad \psi_\gamma^{(N)} + \tilde{\not{q}}\,\psi_{\gamma,q}^{(N)} \\
&+ \quad J_\mu^{(N)} + (d-4)\,J_{d-4}^{(N)}\,,
\end{aligned}$$

$$\begin{aligned}
\mathcal{A} &= \left(\overline{J}^{(8)}\gamma_a\left(\not{P}_{48}+m\right)\not{\varepsilon}_{16}\left(\not{P}_{32}+m\right)\gamma_\beta J^{(1)}\right)\left(\overline{J}^{(2)}\gamma^\beta\left(\not{P}_{60}+m\right)\gamma^\alpha J^{(4)}\right) \\
&= \ldots + (d-4)\left(\overline{J}^{(8)}\left(-\not{P}_{48}+m\right)\not{\varepsilon}_{16}\left(-\not{P}_{32}+m\right)J^{(1)}\right)\left(\overline{J}^{(2)}\left(-\not{P}_{60}+m\right)J^{(4)}\right) \\
&\quad + (d-4)\left(-\tilde{q}^2\right)\left(\overline{J}^{(8)}\not{\varepsilon}_{16}J^{(1)}\right)\left(\overline{J}^{(2)}\left(-\not{P}_{60}+m\right)J^{(4)}\right)
\end{aligned}$$

$$J^{(N)\,\alpha}_{\text{vector}} \rightarrow C_q^{(N)}, C_\varepsilon^{(N)}, J_{q\cdot\varepsilon}^{(N)\,\alpha}, C_{q\cdot\varepsilon,q}^{(N)}, C_\gamma^{(N)}$$

$$J^{(N)}_{\text{fermion}} \rightarrow \psi_q^{(N)}, \psi_\varepsilon^{(N)}, \psi_{q\cdot\varepsilon}^{(N)}, \psi_{q\cdot\varepsilon,q}^{(N)}, \psi_\gamma^{(N)}, \psi_{\gamma,q}^{(N)}$$



where the blue quantities enter when a fermion is cut $\rightarrow \tilde{\gamma}$ (possibly in conjunction with $\tilde{\not{q}}$).
$\rightarrow$ implementation for one ($L = 1$) and two loops ($L = 2$) $\rightarrow$ up to $(d-4)^L$ terms

Comment on R_2 terms: integral vs integrand level

$\rightarrow$ G. Ossola, C. G. Papadopoulos and R. Pittau, [arXiv:0802.1876 [hep-ph]].

HELAC2LOOP@WORK: FAMILIES, BASES, SECTORS

Given the process under consideration (i.e. $gg \to gg$), generate the skeleton from `HELAC2LOOP`

- Generate the families

  For instance, for $g(p_1)g(p_2) \to g(p_3)g(p_4)$, p_i all incoming, one of the six LC families is:

$$
\begin{aligned}
&L_1 = k_1, && L_2 = k_1 + p_1, && L_3 = k_1 + p_1 + p_2, \\
&L_4 = k_1 + k_2, && L_5 = k_2, && L_6 = k_2 - p_1 - p_2 - p_3, \\
&L_7 = k_2 - p_1 - p_2, && L_8 = k_2 - p_1, && L_9 = k_1 + p_1 + p_2 + p_3
\end{aligned}
$$

- For each family run `BasisDet` and get the terms appearing in the master integrand equation:

$$\mathcal{N} = P^{(N)} + \underbrace{\sum_{i=1}^{N} \overbrace{P_i^{(N-1)} D_i}^{cuts}}_{level=N-1} + \sum_{i=1}^{N-1} \sum_{j>i}^{N} P_{ij}^{(N-2)} D_i D_j + \dots \qquad N = 9$$

$$\begin{aligned}
P_1^{(8)} =& c_{8,1,1} + c_{8,1,2}(k_1 \cdot p_1) + c_{8,1,3}(k_1 \cdot p_1)^2 + c_{8,1,4}(k_1 \cdot p_1)^3 + c_{8,1,5}(k_1 \cdot p_1)^4 \\
& + (k_1 \cdot \eta)[c_{8,1,6} + c_{8,1,7}(k_1 \cdot p_1) + c_{8,1,8}(k_1 \cdot p_1)^2 + c_{8,1,9}(k_1 \cdot p_1)^3] \\
& + (k_1 \cdot \eta)^2[c_{8,1,10} + c_{8,1,11}(k_1 \cdot p_1) + c_{8,1,12}(k_1 \cdot p_1)^2] \\
& + (k_1 \cdot \eta)^3[c_{8,1,13} + c_{8,1,14}(k_1 \cdot p_1)] + c_{8,1,15}(k_1 \cdot \eta)^4 \\
& + (k_2 \cdot \eta)[c_{8,1,16} + c_{8,1,17}(k_1 \cdot p_1) + c_{8,1,18}(k_1 \cdot p_1)^2 + c_{8,1,19}(k_1 \cdot p_1)^3] \\
& + (k_1 \cdot \eta)(k_2 \cdot \eta)[c_{8,1,20} + c_{8,1,21}(k_1 \cdot p_1) + c_{8,1,22}(k_1 \cdot p_1)^2] + (k_1 \cdot \eta)^2(k_2 \cdot \eta)[c_{8,1,23} + c_{8,1,24}(k_1 \cdot p_1)] \\
& + c_{8,1,25}(k_1 \cdot \eta)^3(k_2 \cdot \eta) + (k_2 \cdot \eta)^2[c_{8,1,26} + c_{8,1,27}(k_1 \cdot p_1) + c_{8,1,28}(k_1 \cdot p_1)^2] \\
& + (k_1 \cdot \eta)(k_2 \cdot \eta)^2[c_{8,1,29} + c_{8,1,30}(k_1 \cdot p_1)] + c_{8,1,31}(k_1 \cdot \eta)^2(k_2 \cdot \eta)^2 \\
& + (k_2 \cdot \eta)^3[c_{8,1,32} + c_{8,1,33}(k_1 \cdot p_1)] + c_{8,1,34}(k_1 \cdot \eta)(k_2 \cdot \eta)^3 + c_{8,1,35}(k_2 \cdot \eta)^4
\end{aligned}$$

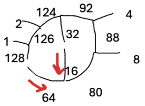
$\rightarrow$ this step is done outside `HELAC2LOOP` and then implemented in `CutTools2(basisdetmonomials.f90)`.

- Run `maptofamily` to get the `k1map`, i.e. the loop momentum running through the cut particle, $q_{2^{n+2}}(=q_{64})$. Notice that $q_{2^n}(=q_{16})$ is always k_1+k_2.

HELAC2LOOP@work: families, bases, sectors

```
 INFO NUM        113  of         580          6
 INFO =====================================
INFO     4   80   35    9    1    1   16   35    5   64   35    7    0    0    0    0    1    2    0    0
INFO     4   88   35   10    1    1    8   35    4   80   35    9    0    0    0    0    1    1    0    0
INFO     4   92   35   11    1    1    4   35    3   88   35   10    0    0    0    0    1    1    0    0
INFO     4  124   35   12    1    1   32   35    6   92   35   11    0    0    0    0    1    2    0    0
INFO     4  126   35   13    1    1    2   35    2  124   35   12    0    0    0    0    1    1    0    0
INFO     4  254   35   14    1    1  128   35    8  126   35   13    0    0    0    0    1    2    0    0
INFO     7    1   36    1    2    4    8   35   35   35   35   35   35   35    0    0    0    0    6    9
```

Numerators: 113, 114, 115, 393, 394, 517, 518



$q_{64} = -k_1 = -L_1, q_{16} = k_1 + k_2 = L_4, q_{80} = k_2 = L_5, q_{88} = k_2 - p_{123} = L_6, q_{92} = k_2 - p_{12} = L_7, q_{126} = k_1 + p_1 = L_2, q_{124} = k_1 + p_{12} = L_3$

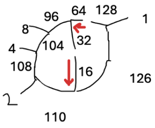
1	**2**	**3**	**4**	**5**	**6**	**7**

$\rightarrow$ sector all numerators

HELAC2LOOP@WORK: FAMILIES, BASES, SECTORS

```
 INFO NUM          168  of          580          6
 INFO =====================================
INFO     4    96    35     9     1     1    32    35     6    64    35     7     0     0     0     0     1     2     0     0
INFO     4   104    35    10     1     1     8    35     4    96    35     9     0     0     0     0     1     1     0     0
INFO     4   108    35    11     1     1     4    35     3   104    35    10     0     0     0     0     1     1     0     0
INFO     4   110    35    12     1     1     2    35     2   108    35    11     0     0     0     0     1     1     0     0
INFO     4   126    35    13     1     1    16    35     5   110    35    12     0     0     0     0     1     2     0     0
INFO     4   254    35    14     1     1   128    35     8   126    35    13     0     0     0     0     1     2     0     0
INFO     7     1    24     2     4     8     1    35    35    35    35    35    35    35     0     0     0     0     6     9
```

Numerators: 168, 169, 170, 415, 416, 539, 540



$q_{110} = k_1 + p_1 = L_2, q_{108} = k_1 + p_{12} = L_3, q_{104} = k_1 + p_{123} = L_9, q_{96} = k_1 = L_1, q_{126} = k_2 - p_1 = L_8, q_{128} = -k_2 = -L_5, q_{64} = k_2 = L_5, q_{16} = k_1 + k_2 = L_4$

1	**2**	**3**	**4**	**5**	**8**	**9**

→ sector all numerators

HELAC2LOOP@WORK: FAMILIES, BASES, SECTORS

- In this way we get all the sectors (the denominator structures) and the numerators belonging to a given sector: $N_{num} = 580$, $N_{sec} = 102$.

  For instance, sector 1 is $\{1,1,3,4,5\} \rightarrow L_1^2 L_3 L_4 L_5$ and the numerators belonging to it are: 1,2, 3, 4, 5, 6, 209, 210, 211, 271, 272, 273, 333, 334, 335, 336, 417, 418, 437, 438, 457, 458, 459, 460, 541, 542, 561, 562.
  $\rightarrow$ run `maptofamily` before calling `CutTools2`.

- All the steps above are "structural": performed once and for all, no-dependence on external kinematical data (external momenta and helicities).

- Then generate kinematical data, call `cutfit(CutTools2)` for the sum of all numerators belonging to each sector and update the reduction coefficients `c(level,icut,icoef)`

- This concludes the reduction procedure

HELAC2LOOP@WORK: IBP TABLES

- Using data from monomials basis and sector, identify all integrals F. For instance, the term multiplying $c_{8,1,2}$, $k_1 \cdot p_1$, for the sector $\{1,2,3,4,5,6,7\}$, gives rise to:

$$-\frac{1}{2}F(-1,1,1,1,1,1,1,0,0)+\frac{1}{2}F(0,0,1,1,1,1,1,0,0)$$

- Using an IBP reducer, express all F in terms of Master Integrals (MI), m_i.
- Next step is to formally expand the MI in ϵ. They look like

$$m_i = \sum_{j=-4}^{0} m_{i,j+5}\epsilon^j$$

Notice that an $m_{i,j}$ may contribute to more than one orders in ϵ in the final result, due to the ϵ dependence in the IBP coefficients.

- Expressing the amplitude, $\mathcal{A} = \sum \mathcal{N}_i / \prod \mathcal{D}_j^{(i)}$ in terms of IBP tables, specific to the *family* not to the *process*, requires performing the following summation

$$\sum \overbrace{\left(c_{level,icut,icoef}\right)}^{\text{integrand reduction}} \times \overbrace{\left(m_{sector,level,icut,icoef,mi,mij,ieps}\right)}^{\text{MI coefficients}}$$

so to get for all MI components and orders in ϵ, the numerical value of the amplitude given the color flow, the helicity assignment and the external momenta.

HELAC2LOOP@WORK: IBP TABLES

13	1	-4	(10.8800400234819183325212648550935952,-1.004080562473657018182012682310678560E-0023)
13	1	-3	(2457.96226329490609985808242303176809,-4.039708160319141800953799796252419705E-0023)
13	1	-2	(9587.17187073302728277608462259703866,-1.178357517675036863103977166003989240E-0022)
13	1	-1	(29445.3973800252684601866393856299770,-2.972912475621931786499600278314702155E-0022)
13	1	0	(81416.7376267798433286975379985294713,-6.897191608570867248803844568600558813E-0022)
13	2	-4	(0.00000000000000000000000000000000000,0.00000000000000000000000000000000000)
13	2	-3	(10.8800400234818873124137754649730578,-1.004080562473657020225237409676634164E-0023)
13	2	-2	(2457.96226329490609985808242303176690,-4.039708160319141777563003046095619380E-0023)
13	2	-1	(9587.171870733029222419129125655360787,-1.178357517675036866959112707565753035E-0022)
13	2	0	(29445.3973800252629311594711703652609,-2.972912475621931771481350131521661471E-0022)
13	3	-4	(0.00000000000000000000000000000000000,0.00000000000000000000000000000000000)
13	3	-3	(0.00000000000000000000000000000000000,0.00000000000000000000000000000000000)
13	3	-2	(10.8800400234818873124137754649766847,-1.004080562473657019826452302592779282E-0023)
13	3	-1	(2457.96226329490671813520827469823249,-4.039708160319141794618240644793417592E-0023)
13	3	0	(9587.17187073302739896308724623278965,-1.178357517675036861217275615183064691E-0022)
13	4	-4	(0.00000000000000000000000000000000000,0.00000000000000000000000000000000000)
13	4	-3	(0.00000000000000000000000000000000000,0.00000000000000000000000000000000000)
13	4	-2	(0.00000000000000000000000000000000000,0.00000000000000000000000000000000000)
13	4	-1	(10.8800400234820325082435989712320043,-1.004080562473657020385178834715824950E-0023)
13	4	0	(2457.96226329490618905580304511885628,-4.039708160319141773592219891374753837E-0023)
13	5	-4	(0.00000000000000000000000000000000000,0.00000000000000000000000000000000000)
13	5	-3	(0.00000000000000000000000000000000000,0.00000000000000000000000000000000000)
13	5	-2	(0.00000000000000000000000000000000000,0.00000000000000000000000000000000000)
13	5	-1	(0.00000000000000000000000000000000000,0.00000000000000000000000000000000000)
13	5	0	(10.8800400234819183325212648550920946,-1.004080562473657018183608142127416653E-0023)

- Skeleton generation tested against `Qgraph` and `FeynArts-FeynCalc` → in progress
- Numerator validation either against results from `FeynArts-FeynCalc` or in-house `Mathematica` and `FORM` codes
  skeleton → analytic expression → checking against `HELAC2LOOP` numerical evaluation
- `CutTools2`
  Solving and fitting numerically all reduction coefficients → $N = N$ test, per cut and globally
- In-house `Mathematica` code to generate IBP tables
  basis monomials → Feynman Integrals and through IBP → Master Integral coefficients
- Reading/implementing IBP tables in `HELAC2LOOP`
- `HELAC2LOOP` final step: numerical evaluation of two-loop amplitude expanded in ϵ

  validation against known results, for instance `Caravel` → in progress
  → S. Abreu, J. Dormans, F. Febres Cordero, H. Ita, M. Kraus, B. Page, E. Pascual, M. S. Ruf and V. Sotnikov, [arXiv:2009.11957 [hep-ph]].

Summary & Outlook

Current:

- Integrand construction @2L $\rightarrow$ solved and implemented
- Cut equations @2L: determining on-shell loop momenta $\rightarrow$ solved and implemented
- Integrand basis construction and fitting @2L $\rightarrow$ solved and implemented
- Numerical evaluation of the numerators in $d = 4 - 2\epsilon$
  - solved and implemented for 1 loop, all gluon amplitudes
  - solved and implemented for 2 loop, all gluon amplitudes
  - for 1 and 2 loops, external fermions $\rightarrow$ in progress
- IBP reduction tables, and if available MI numerical evaluation, solved and partially implemented ($2 \rightarrow 2$)

Summary & Outlook

Near future:

- Assessing performance and search for optimizations
- Applications for $2 \to 3$ processes and beyond

Next-to-near future: automated 2-loop amplitude numerical evaluation framework

Thank you for your attention !

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Backup slides

$$(k_1\cdot\eta)^4 \to \frac{3\left[-st(s+t)k_1^2+t^2(k_1\cdot p_1)^2+2tk_1\cdot p_1\left((s+t)k_1\cdot p_2-sk_1\cdot p_3\right)+\left((s+t)k_1\cdot p_2+sk_1\cdot p_3\right)^2\right]^2}{(d-3)(d-1)s^2t^2(s+t)^2}$$

$$(k_2\cdot\eta)^4 \to \frac{3\left[-st(s+t)k_2^2+t^2(k_2\cdot p_1)^2+2tk_2\cdot p_1\left((s+t)k_2\cdot p_2-sk_2\cdot p_3\right)+\left((s+t)k_2\cdot p_2+sk_2\cdot p_3\right)^2\right]^2}{(d-3)(d-1)s^2t^2(s+t)^2}$$

$$\begin{aligned}(k_1\cdot\eta)^2(k_2\cdot\eta)^2 \to &\frac{1}{(d-3)(d-1)s^2t^2(s+t)^2}\Bigg[2\Big(-st(s+t)k_1\cdot k_2+st(k_1\cdot p_2)(k_2\cdot p_1)+t^2(k_1\cdot p_2)(k_2\cdot p_1)\\
&-st(k_1\cdot p_3)(k_2\cdot p_1)+(s+t)^2(k_1\cdot p_2)(k_2\cdot p_2)+s(s+t)(k_1\cdot p_3)(k_2\cdot p_2)\\
&+s(s+t)(k_1\cdot p_2)(k_2\cdot p_3)+s^2(k_1\cdot p_3)(k_2\cdot p_3)\\
&+t(k_1\cdot p_1)(t(k_2\cdot p_1)+(s+t)k_2\cdot p_2-sk_2\cdot p_3)\Big)^2\\
&+\Big(-st(s+t)k_1^2+t^2(k_1\cdot p_1)^2+2tk_1\cdot p_1((s+t)k_1\cdot p_2-sk_1\cdot p_3)\\
&+((s+t)k_1\cdot p_2+sk_1\cdot p_3)^2\Big)\times\Big(-st(s+t)k_2^2+t^2(k_2\cdot p_1)^2\\
&+2tk_2\cdot p_1((s+t)k_2\cdot p_2-sk_2\cdot p_3)+((s+t)k_2\cdot p_2+sk_2\cdot p_3)^2\Big)\Bigg]\end{aligned}$$

Integral relations for ISP

$$(k_1\cdot\eta)^3(k_2\cdot\eta)\to\frac{3\left[-st(s+t)k_1^2+t^2(k_1\cdot p_1)^2+2tk_1\cdot p_1((s+t)k_1\cdot p_2-sk_1\cdot p_3)+((s+t)k_1\cdot p_2+sk_1\cdot p_3)^2\right]}{(d-3)(d-1)s^2t^2(s+t)^2}$$
$$\times\Big(-st(s+t)k_1\cdot k_2+st(k_1\cdot p_2)(k_2\cdot p_1)+t^2(k_1\cdot p_2)(k_2\cdot p_1)-st(k_1\cdot p_3)(k_2\cdot p_1)$$
$$+(s+t)^2(k_1\cdot p_2)(k_2\cdot p_2)+s(s+t)(k_1\cdot p_3)(k_2\cdot p_2)+s(s+t)(k_1\cdot p_2)(k_2\cdot p_3)$$
$$+s^2(k_1\cdot p_3)(k_2\cdot p_3)+t(k_1\cdot p_1)(t(k_2\cdot p_1)+(s+t)k_2\cdot p_2-sk_2\cdot p_3)\Big)$$

$$(k_1\cdot\eta)(k_2\cdot\eta)^3\to\frac{3\left[-st(s+t)k_2^2+t^2(k_2\cdot p_1)^2+2tk_2\cdot p_1((s+t)k_2\cdot p_2-sk_2\cdot p_3)+((s+t)k_2\cdot p_2+sk_2\cdot p_3)^2\right]}{(d-3)(d-1)s^2t^2(s+t)^2}$$
$$\times\Big(-st(s+t)k_1\cdot k_2+st(k_1\cdot p_2)(k_2\cdot p_1)+t^2(k_1\cdot p_2)(k_2\cdot p_1)-st(k_1\cdot p_3)(k_2\cdot p_1)$$
$$+(s+t)^2(k_1\cdot p_2)(k_2\cdot p_2)+s(s+t)(k_1\cdot p_3)(k_2\cdot p_2)+s(s+t)(k_1\cdot p_2)(k_2\cdot p_3)$$
$$+s^2(k_1\cdot p_3)(k_2\cdot p_3)+t(k_1\cdot p_1)(t(k_2\cdot p_1)+(s+t)k_2\cdot p_2-sk_2\cdot p_3)\Big)$$

Colour flow or colour connection representation

$$\mathcal{M}^{a_1,i_2,\ldots,i_k}_{j_2,\ldots,j_k}\, t^{a_1}_{i_1 j_1} \to \mathcal{M}^{i_1,i_2,\ldots,i_k}_{j_1,j_2,\ldots,j_k}$$

$$\mathcal{M}^{i_1,i_2,\ldots,i_k}_{j_1,j_2,\ldots,j_k} = \sum_{\sigma} \delta_{i_{\sigma_1},j_1}\delta_{i_{\sigma_2},j_2}\ldots\delta_{i_{\sigma_k},j_k} A_\sigma \to n!$$

gluons, ghosts $\to (i,j)$, quark $\to (i,0)$, anti-quark $\to (0,j)$, other $\to (0,0)$

$$\sum_{\sigma,\sigma'} A^*_\sigma \mathcal{C}_{\sigma,\sigma'} A_{\sigma'}$$

$$\mathcal{C}_{\sigma,\sigma'} \equiv \sum_{\{i\},\{j\}} \delta_{i_{\sigma_1},j_1}\delta_{i_{\sigma_2},j_2}\ldots\delta_{i_{\sigma_k},j_k}\delta_{i_{\sigma'_1},j_1}\delta_{i_{\sigma'_2},j_2}\ldots\delta_{i_{\sigma'_k},j_k} = N_c^{m(\sigma,\sigma')}$$

Colour-flow Feynman rules