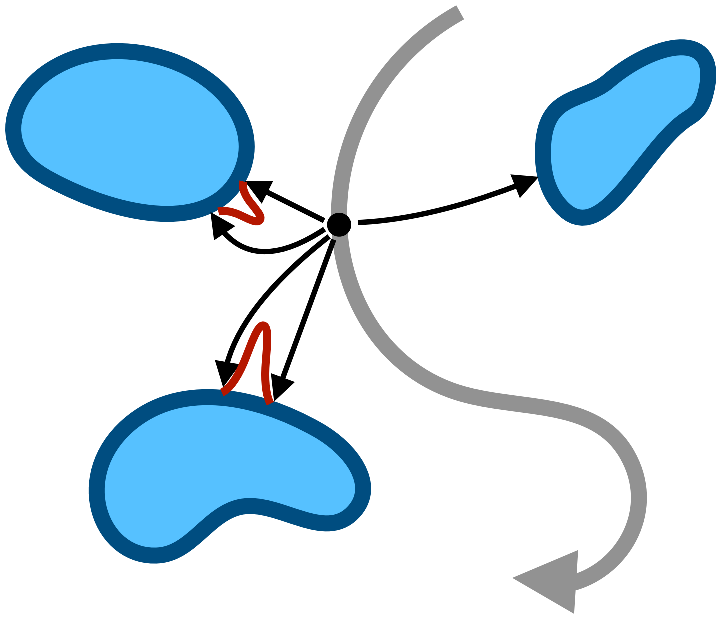
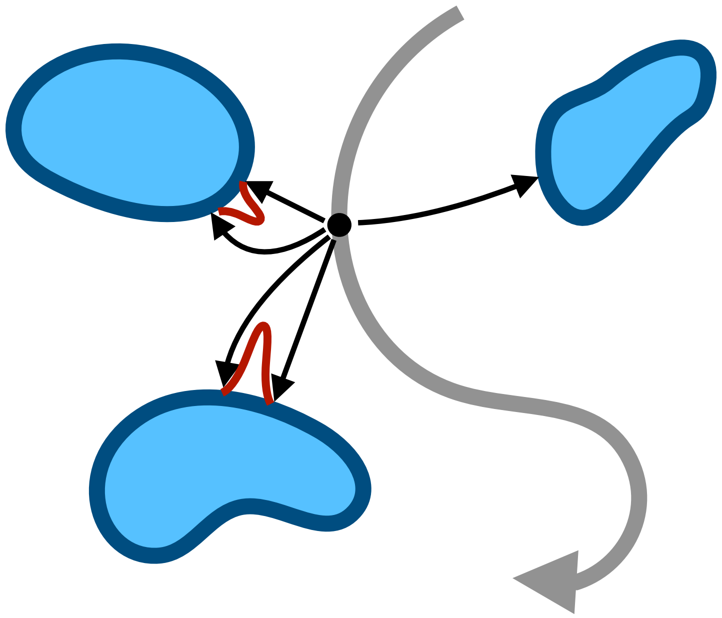
# Ramo Shockley theory



**Philipp Windischhofer**
*University of Oxford*

**Advanced UK Instrumentation Training**
**May 13, 2022**

# How to compute the electrical signal induced in your detector



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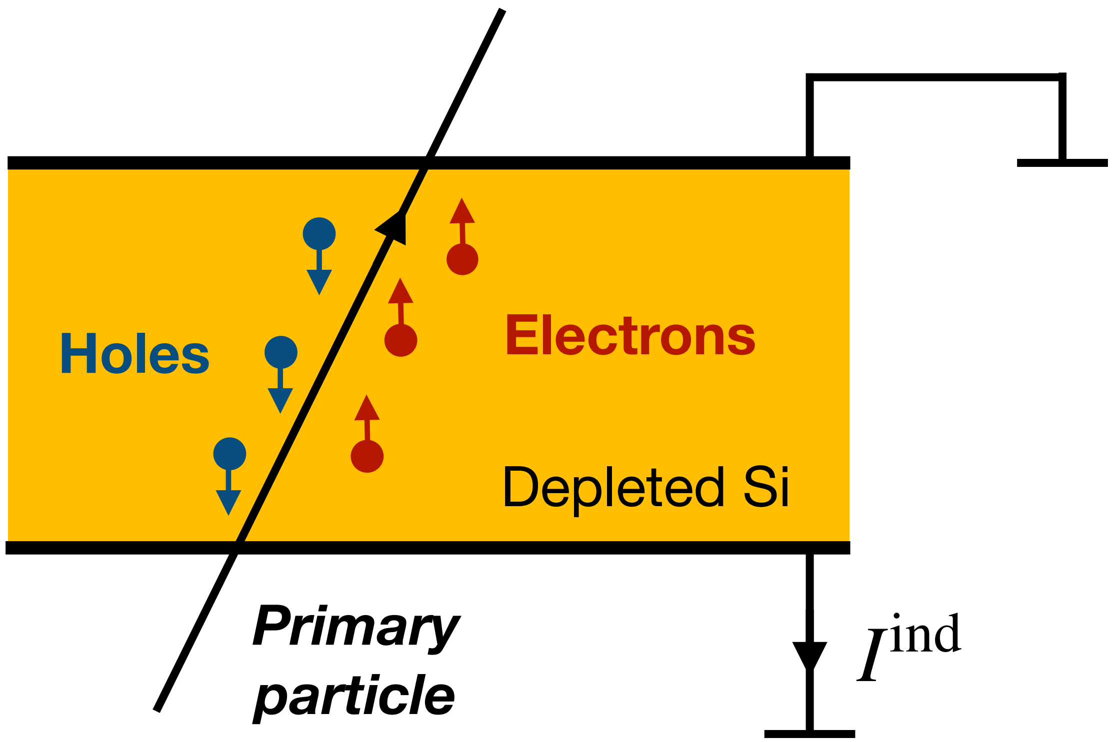
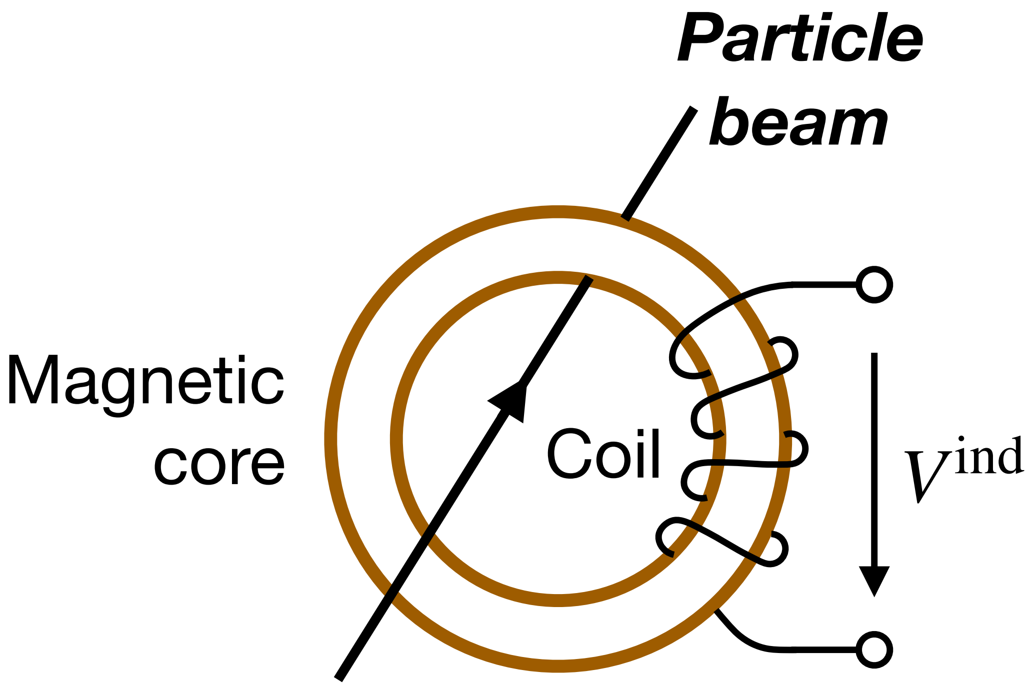
# From moving charges to electrical signals

Many types of **particle detectors** generate **electrical signals**
when **exposed to moving charged particles**

Silicon detector
*(ATLAS tracker)*



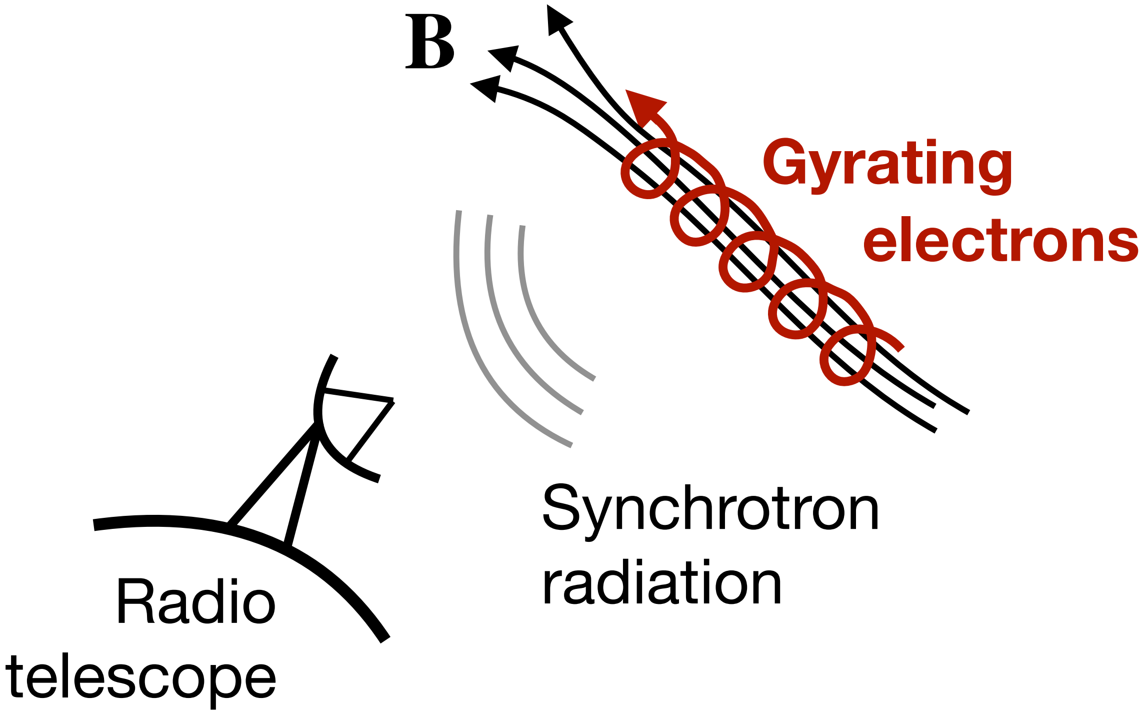
Beam current transformer
*(LHC instrumentation)*



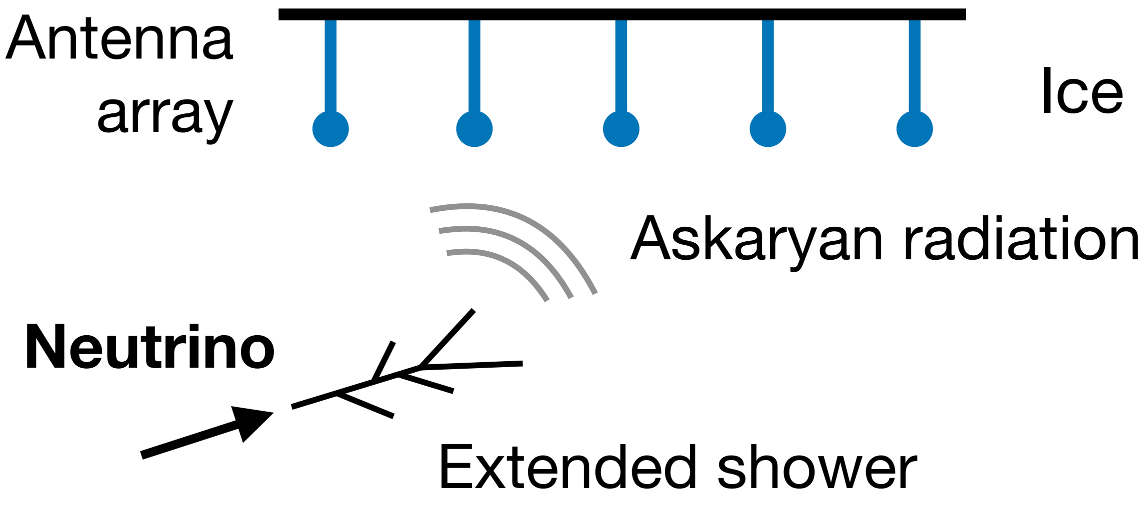
# From moving charges to electrical signals

Many types of **particle detectors** generate **electrical signals** when **exposed to moving charged particles**

Radio astronomy
*(e.g. ALMA)*



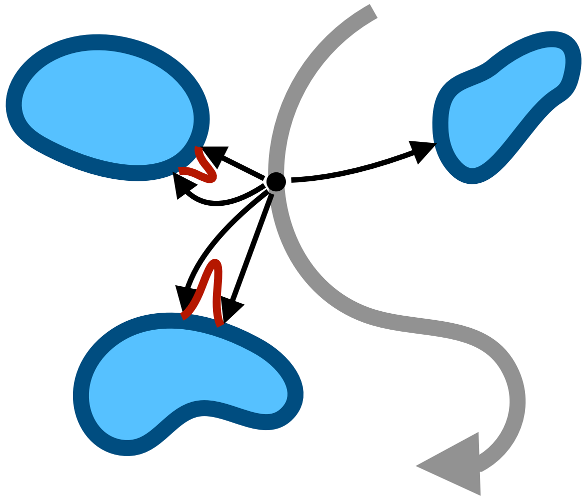
Neutrino observatories
*(e.g. RNO-g)*

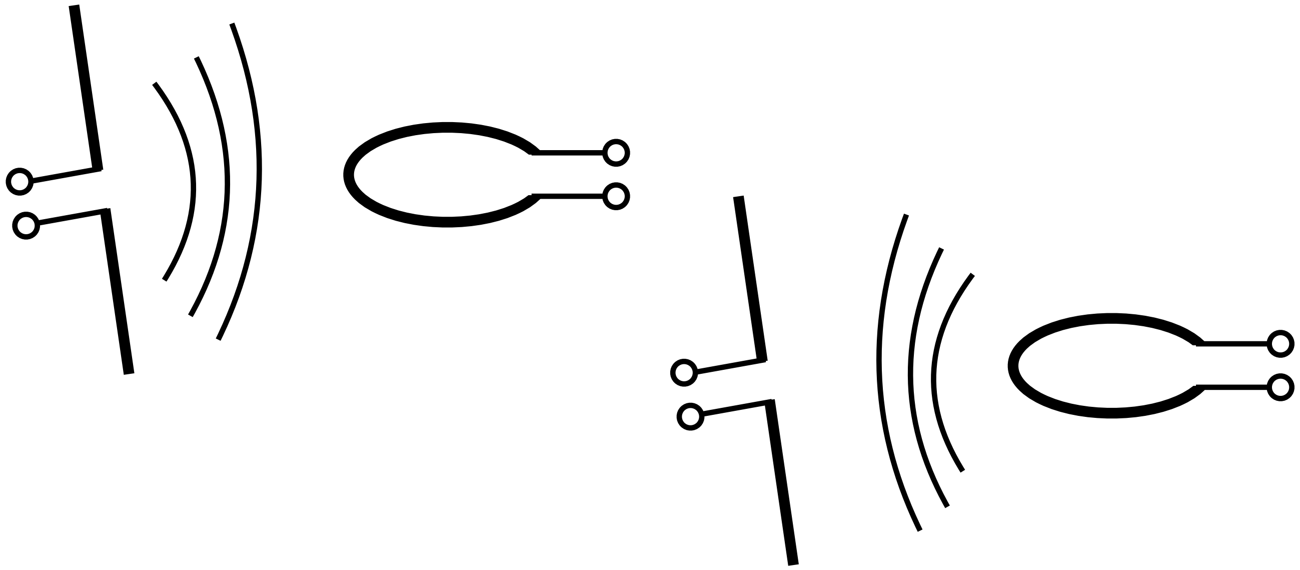


# Goals for today

**The physics / intuition behind signal formation**

*Electrostatics, surface charges, induction, …*



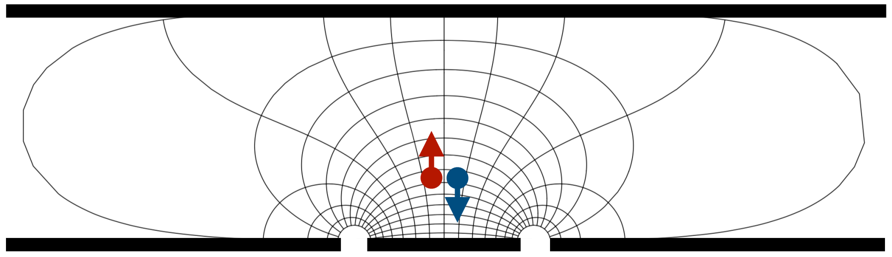


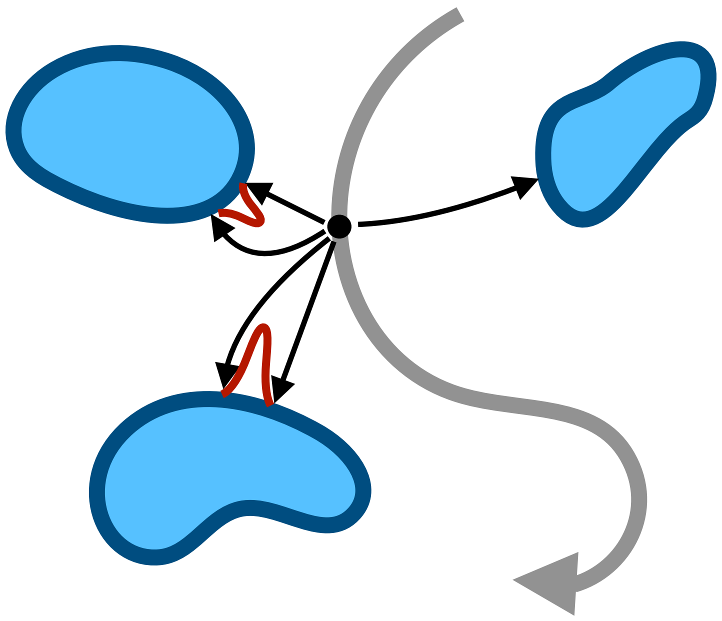
**A general and efficient method**

*Without approximations—*
*they just muddle the water*

**Signals in silicon detectors**

*Pixels, LGADs, …*



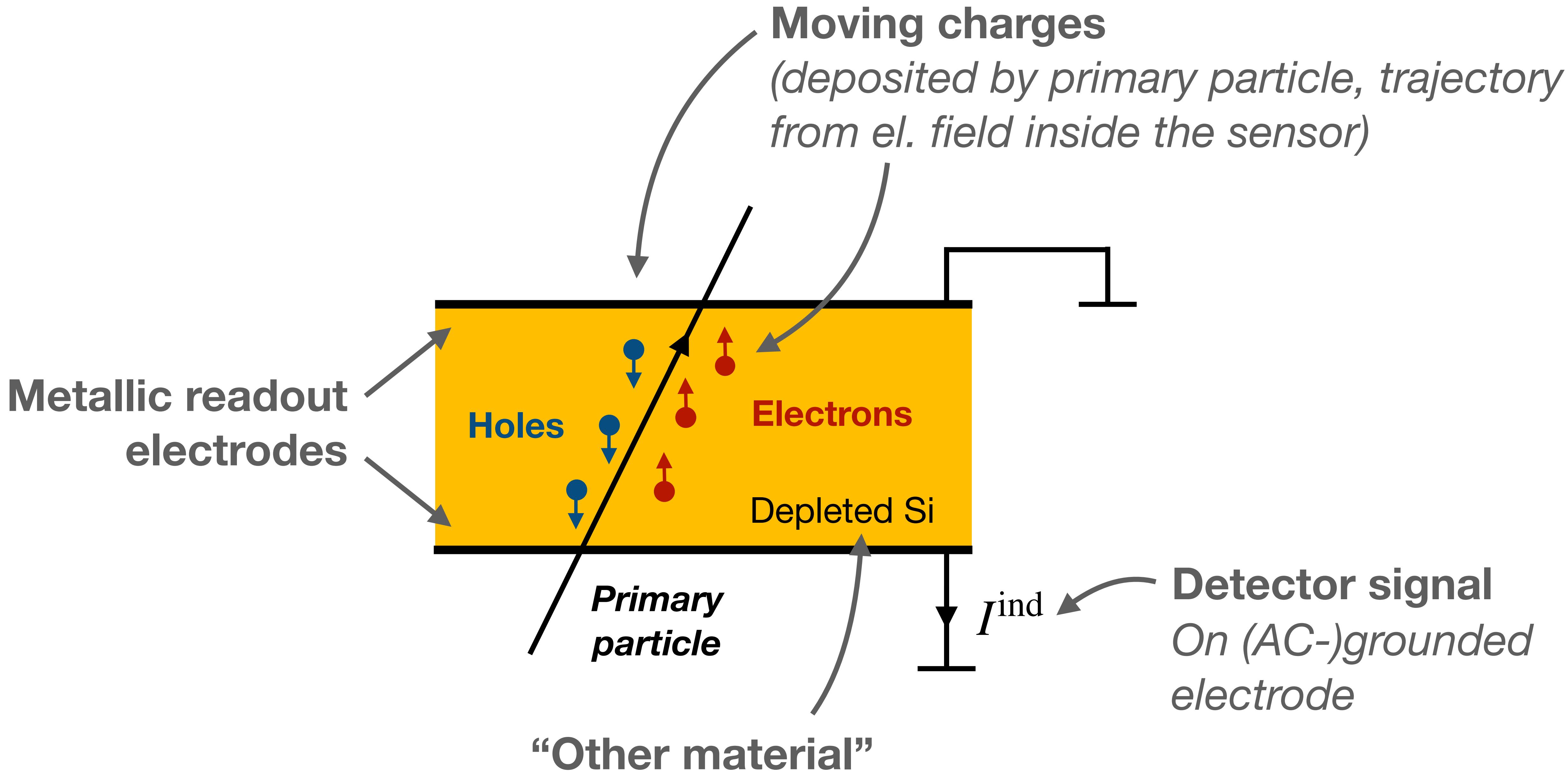


# The physics of signal formation

# Moving charges and conducting electrodes

**For today:**

**Detector** = arrangement of **metallic electrodes** *(and other materials)* through which a **charged particle moves** along a **known trajectory**



**Moving charges**
*(deposited by primary particle, trajectory from el. field inside the sensor)*

**Metallic readout electrodes**

**Detector signal**
*On (AC-)grounded electrode*

**"Other material"**

# Moving charges and conducting electrodes

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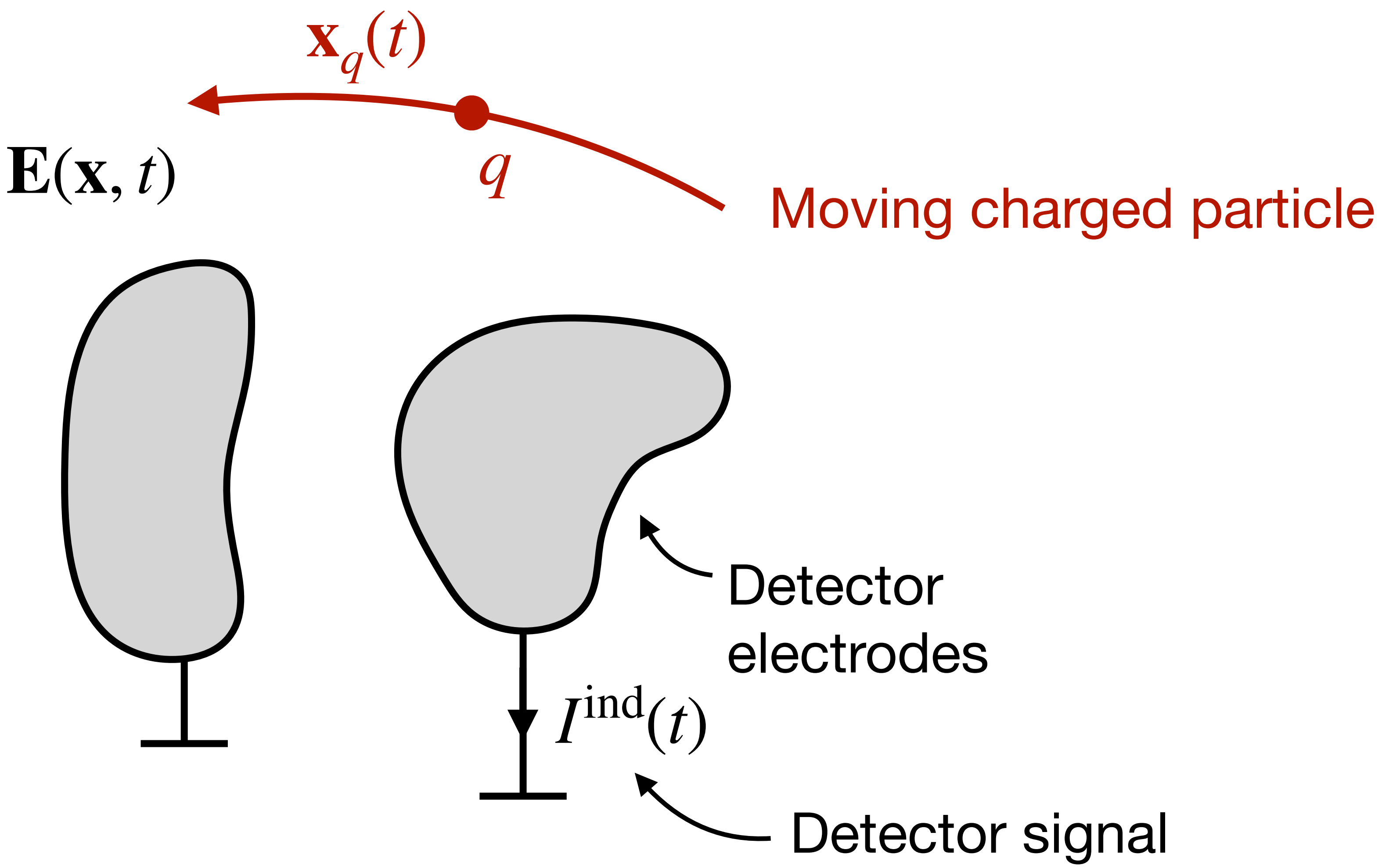


# Simple example: strip electrode

**Charged particle in front of a grounded strip electrode**

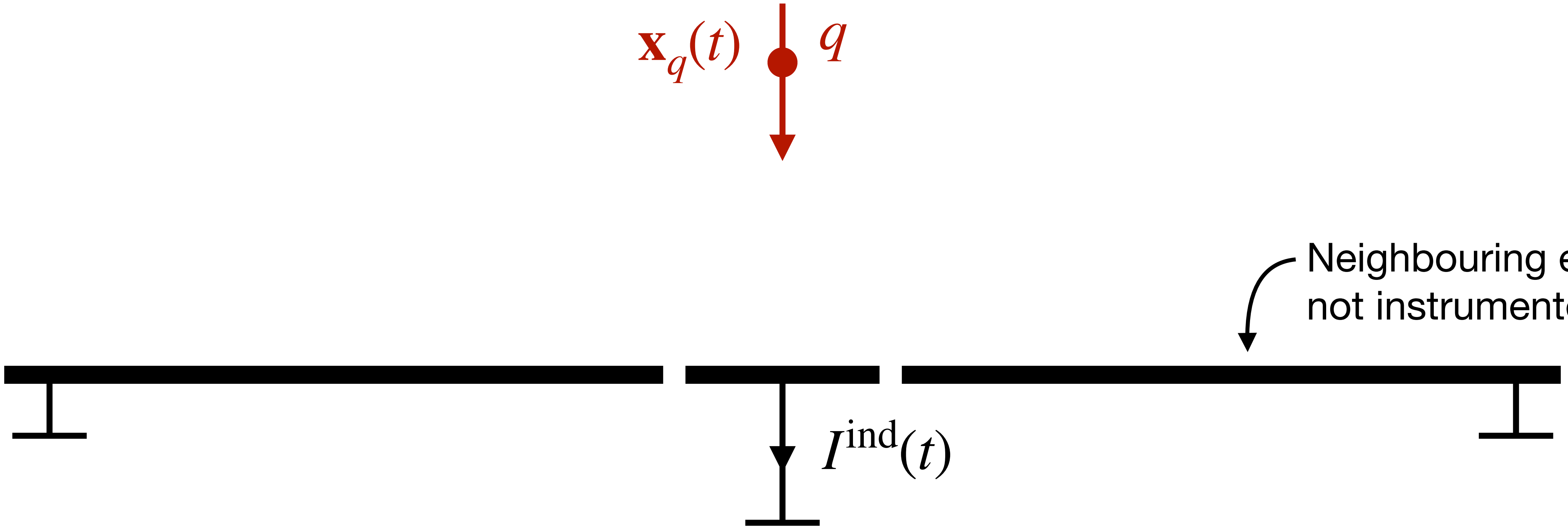
*(Later: signal on electrodes that are* ***not*** *grounded)*

**1.) Charge creates electric field** *(assume charge moves slowly → electrostatics)*

**2.) Electrode builds surface charge density σ** *(need to satisfy* $\boldsymbol{E} = \boldsymbol{0}$ *in a conductor)*

**3.) Total charge Q on strip electrode:** integral over surface charge density

**4.) Total current from electrode:** negative change in time of total charge



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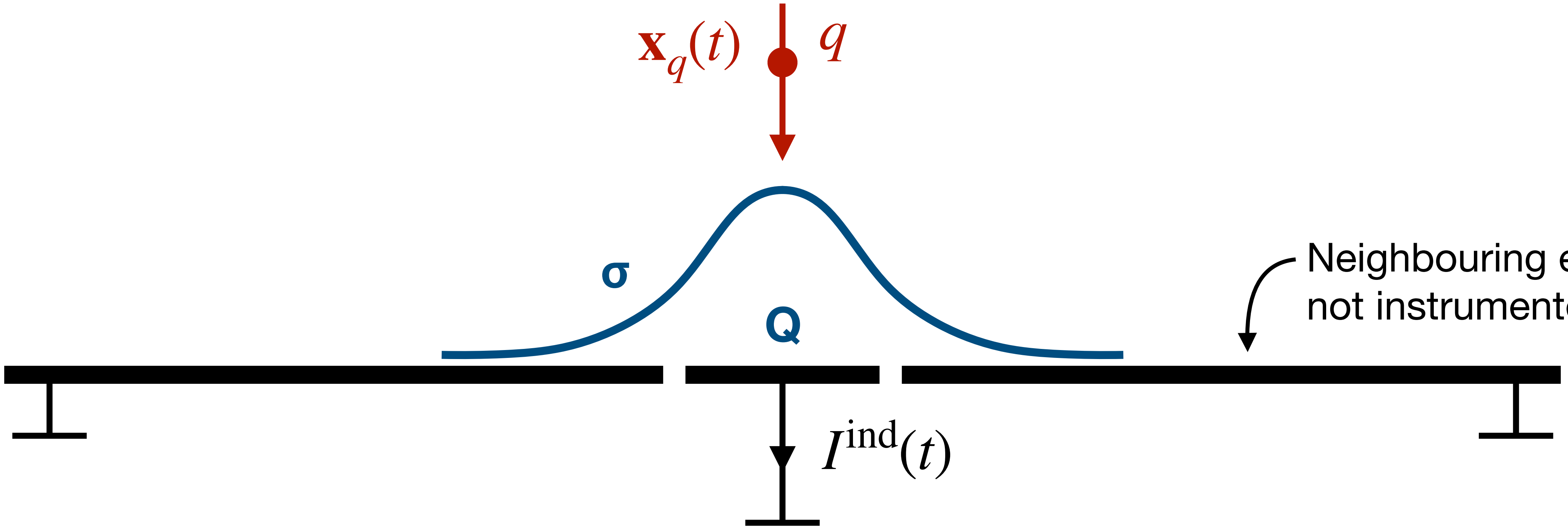


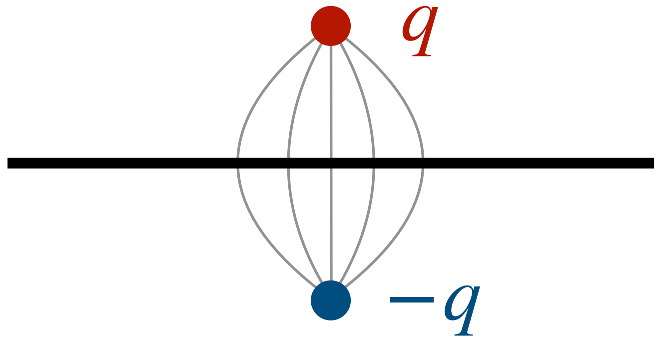
Neighbouring electrode, not instrumented

# Simple example: strip electrode

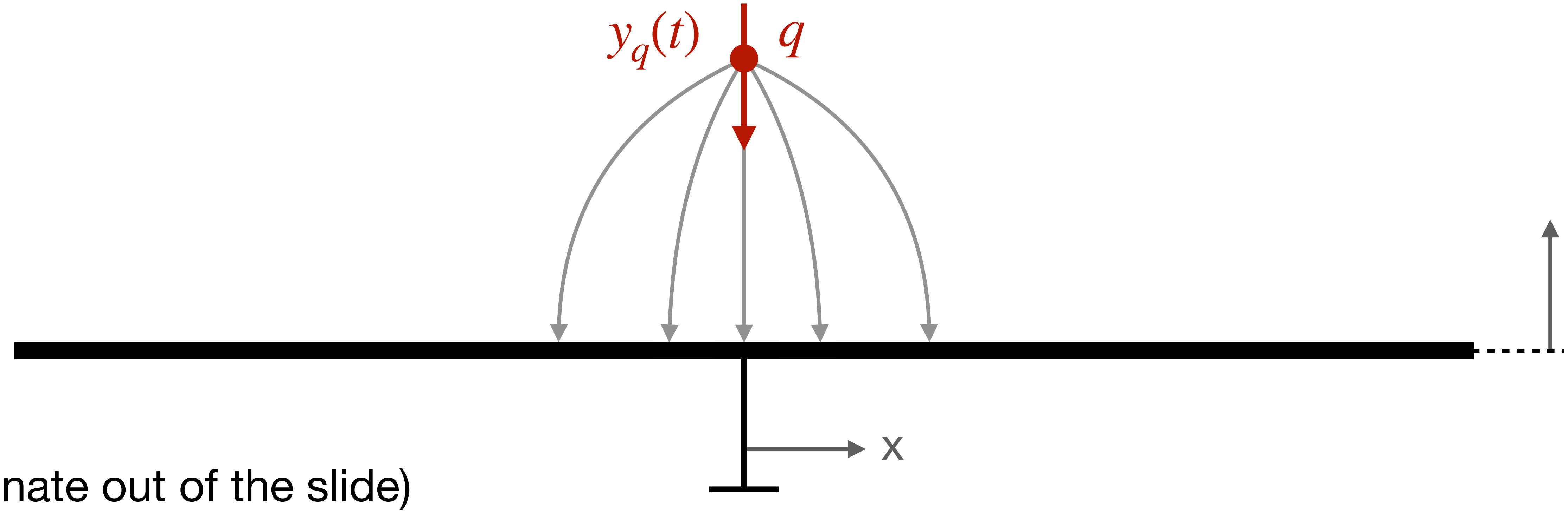
**Electric field produced by charge**

*Narrow gap between strips: electrode appears as infinite* ***grounded plane***

Compute field with method of image charges:



$$E^y(x, z, t) = -\frac{q}{2\pi\epsilon}\frac{y_q(t)}{\left(x^2 + y_q(t)^2 + z^2\right)^{3/2}}$$



(z coordinate out of the slide)

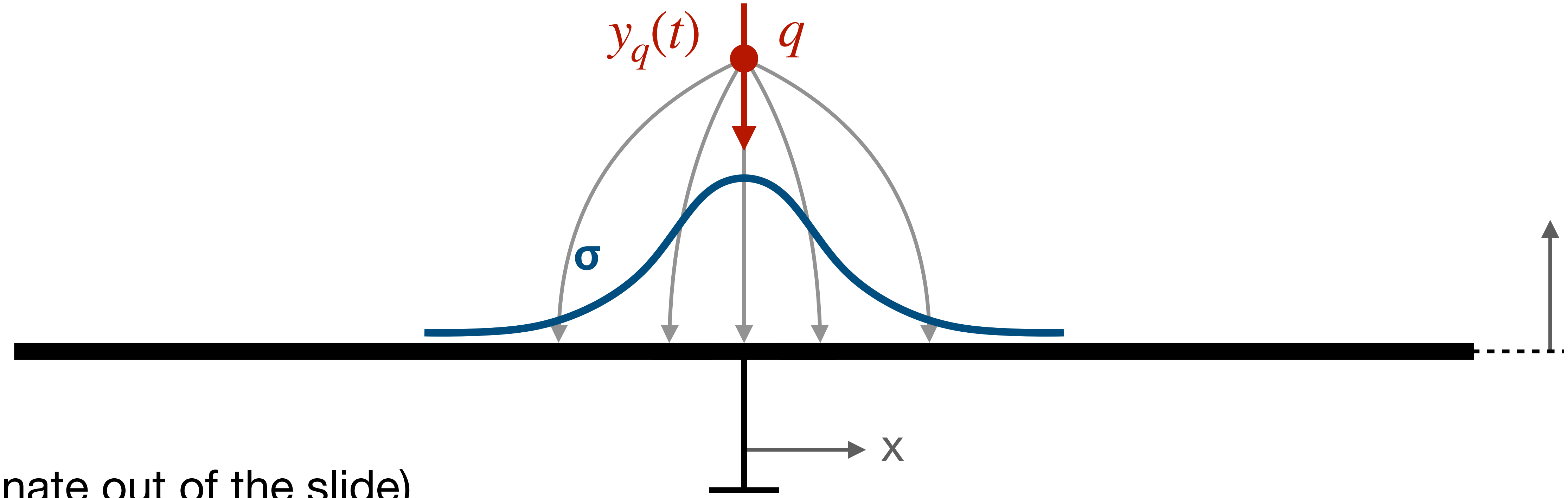
# Simple example: strip electrode

**Surface charge density**

$$\sigma = \epsilon\, \mathbf{E} \cdot \hat{\mathbf{n}}$$

Electric field perpendicular to surface

$$\sigma(x, z, t) = -\frac{q}{2\pi} \frac{y_q(t)}{\left(x^2 + y_q(t)^2 + z^2\right)^{3/2}}$$



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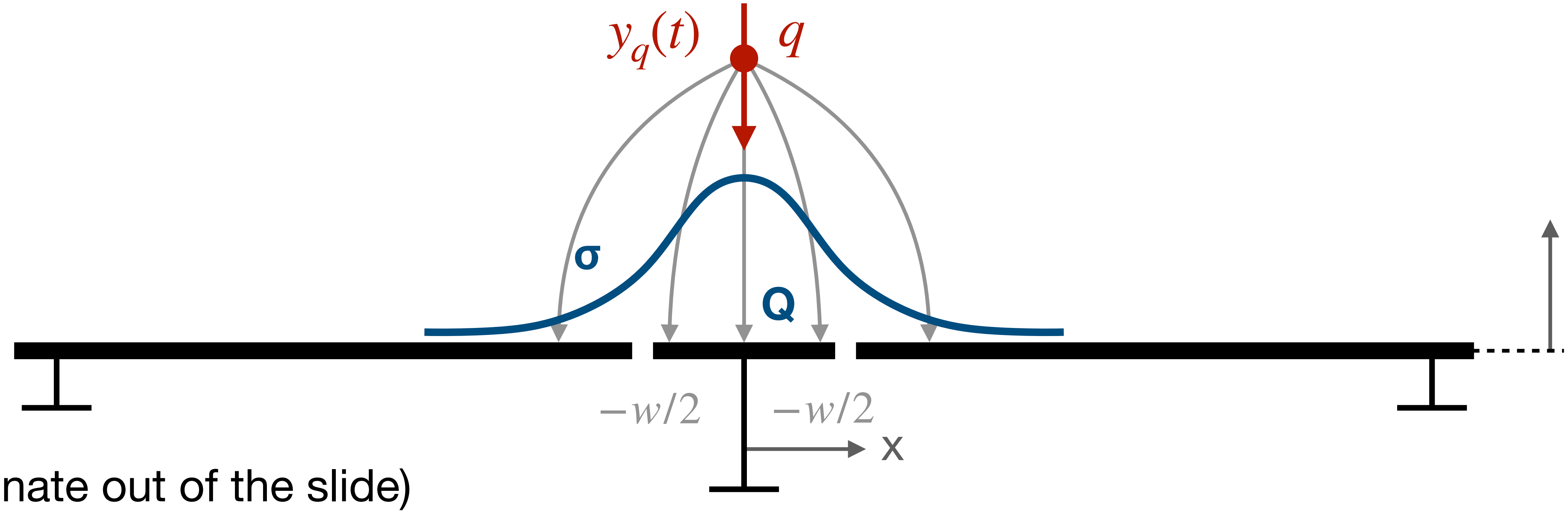
# Simple example: strip electrode

**Total charge on strip electrode**

$$Q(t) = \int_{-w/2}^{w/2} dx \int_{-\infty}^{\infty} dz\, \sigma(x, z, t) = -\, 2\frac{q}{\pi} \arctan\left(\frac{w}{2y_q(t)}\right)$$



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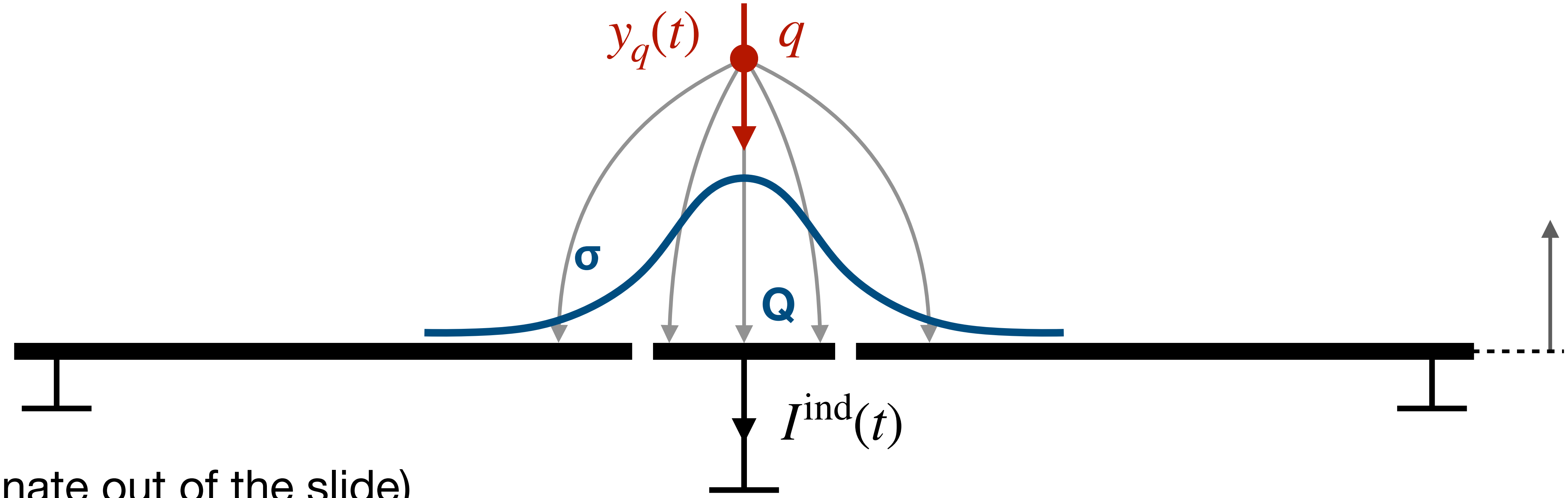
# Simple example: strip electrode

**Total charge on strip electrode**

$$I^{\mathrm{ind}}(t) = -\frac{dQ}{dt} = \frac{4qw}{\pi\left(w^2 + 4y_q(t)^2\right)} \cdot \dot{y}_q(t)$$

**Detector geometry:**
strip width

**Particle trajectory:**
drift velocity



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# Properties of the signal

**Important properties of the induced signal**

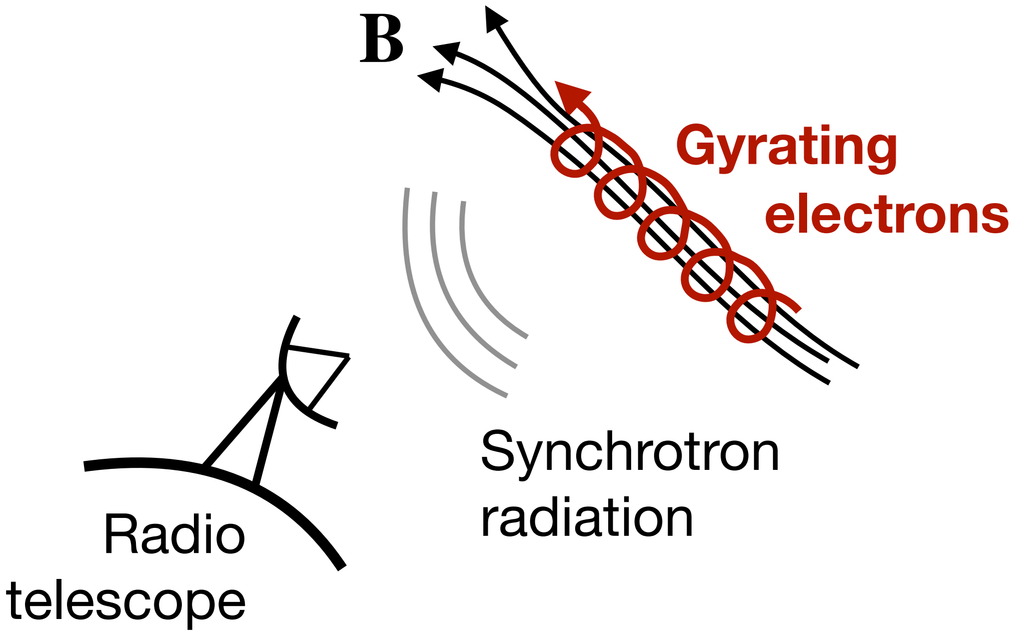
**Produced by moving charges**

*Signal amplitude proportional to number of charges and their velocity*

**Signal ends when charges reach the electrodes**

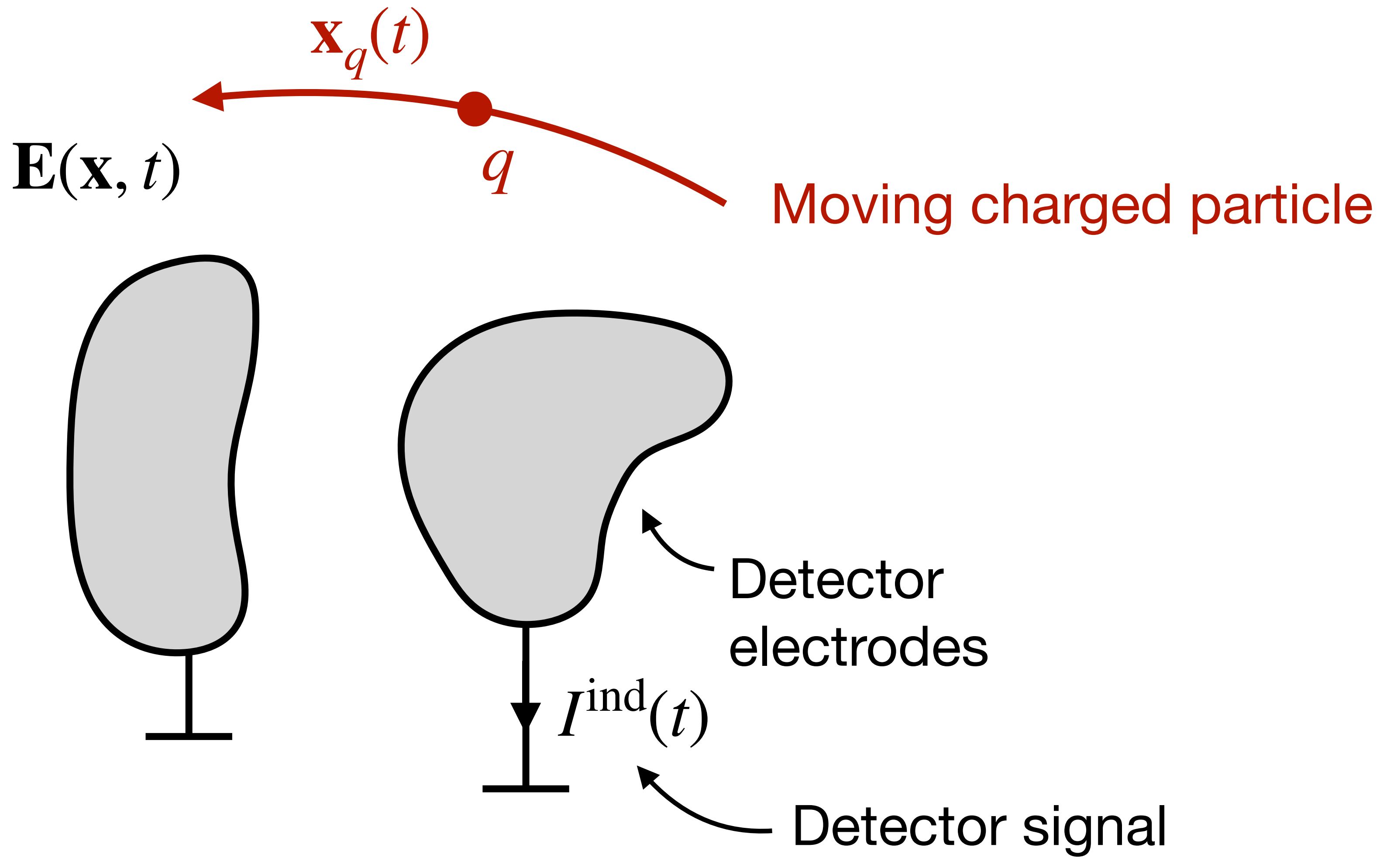
**Signal is not due to charge "collection"**

*Charges can be far away from the electrodes and still induce a signal (radio telescopes!)*



# This is a “particle-centric” calculation

Particle trajectory $\mathbf{x}_q(t)$ → Maxwell’s equations → electric field distribution $\mathbf{E}(\mathbf{x}, t)$ → induced signal $I^{\mathrm{ind}}(t)$



**Very computationally expensive!**

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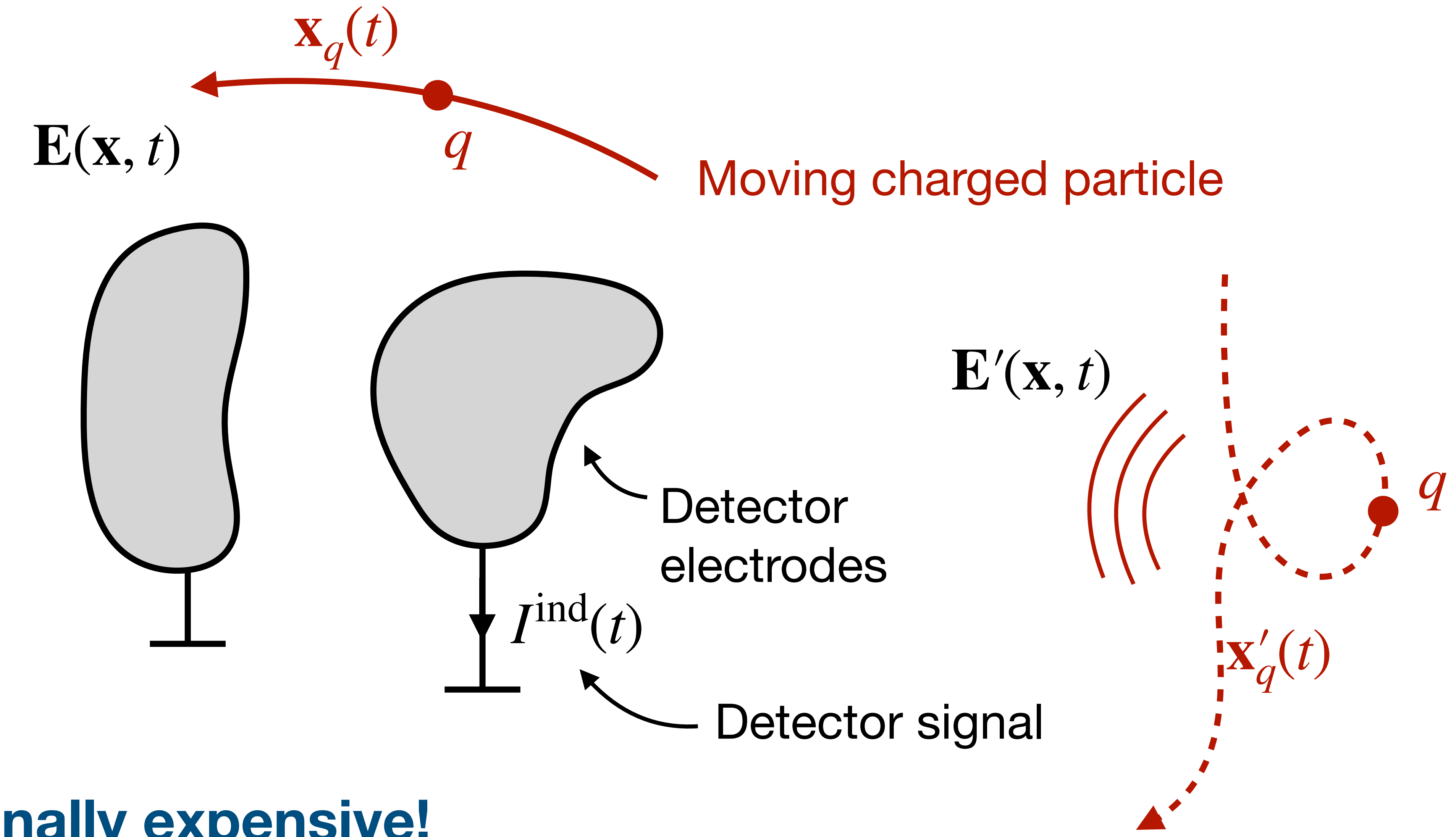
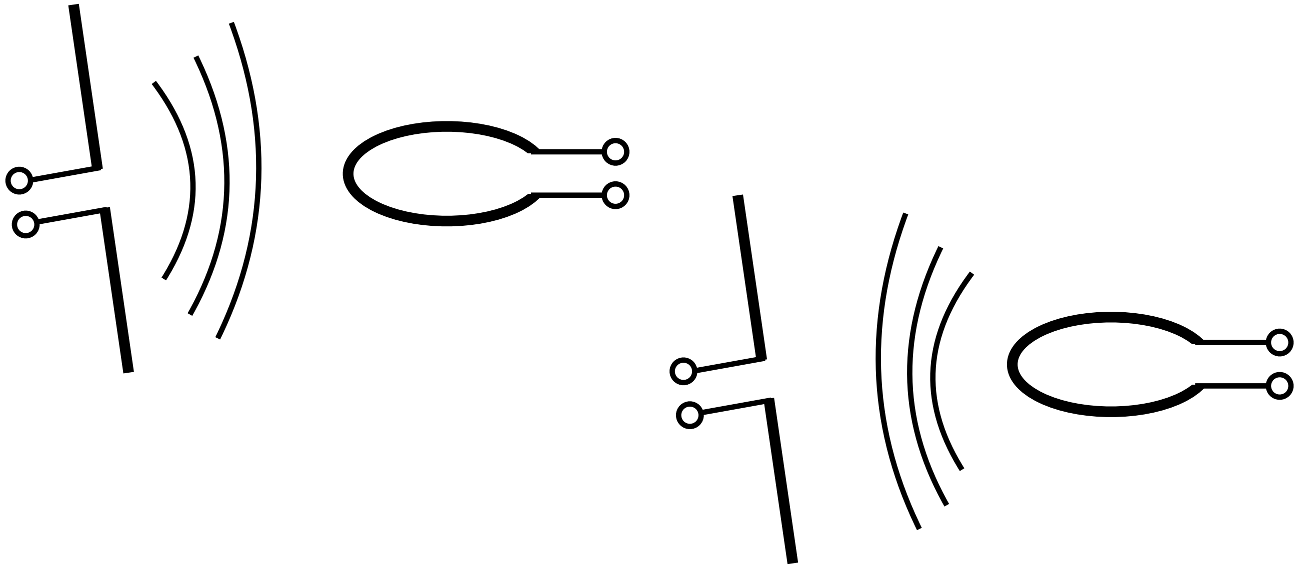


**Very computationally expensive!**

*(Need to repeat full calculation for every trajectory.)*

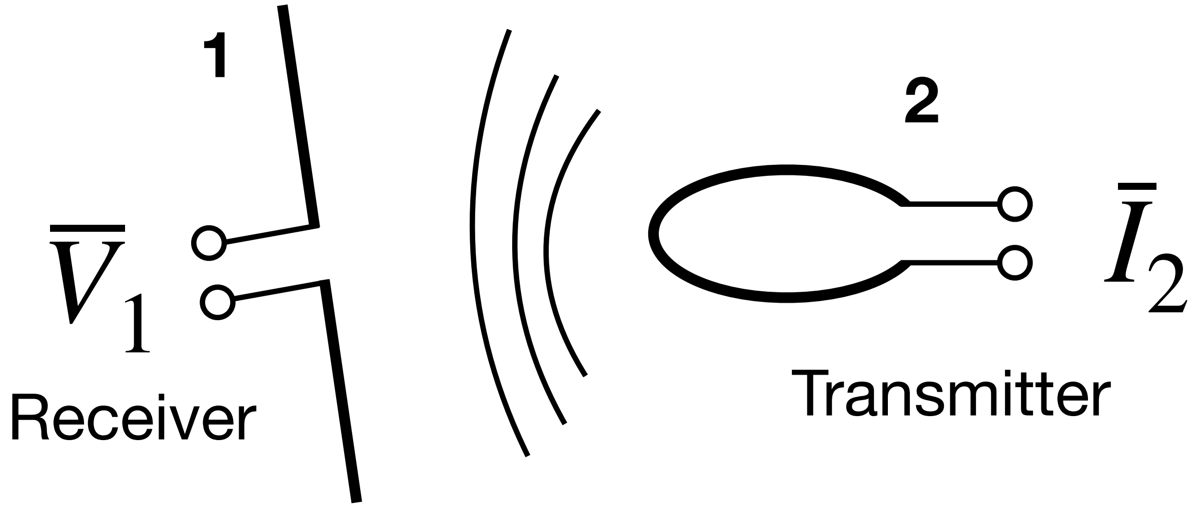


# A general and efficient method to compute the signal

# Towards a “detector-centric” calculation

***Reciprocity:* electrodynamics has a built-in method to relate two different situations** *(with identical geometry)*

**Antenna 2 “sees” antenna 1**

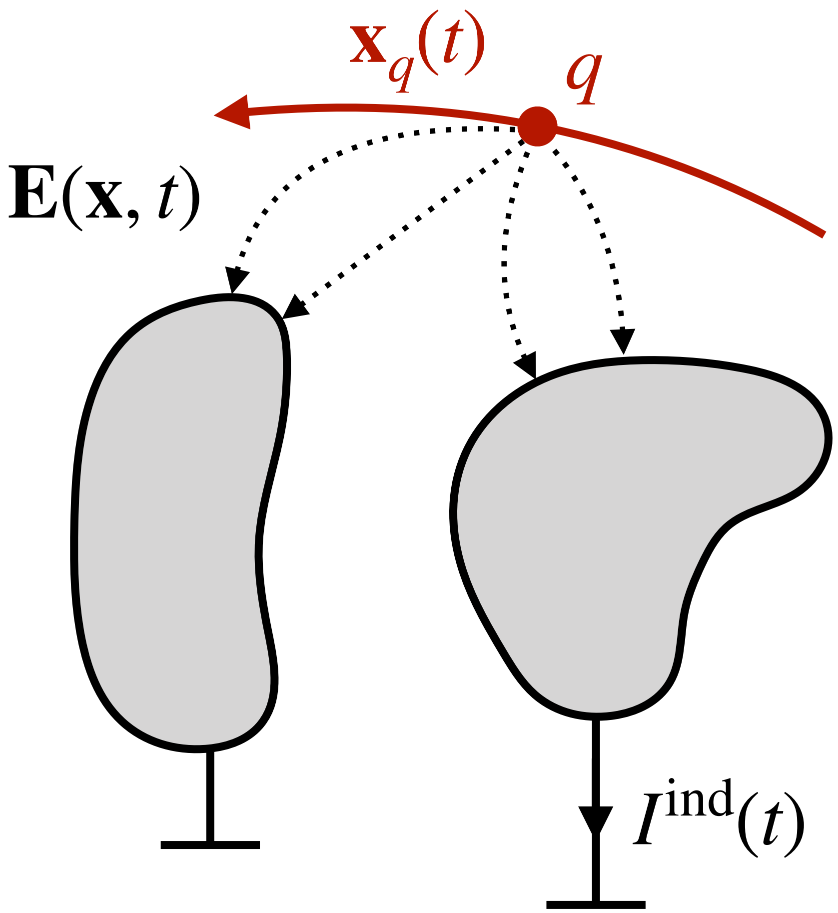


**Antenna 1 “sees” antenna 2**

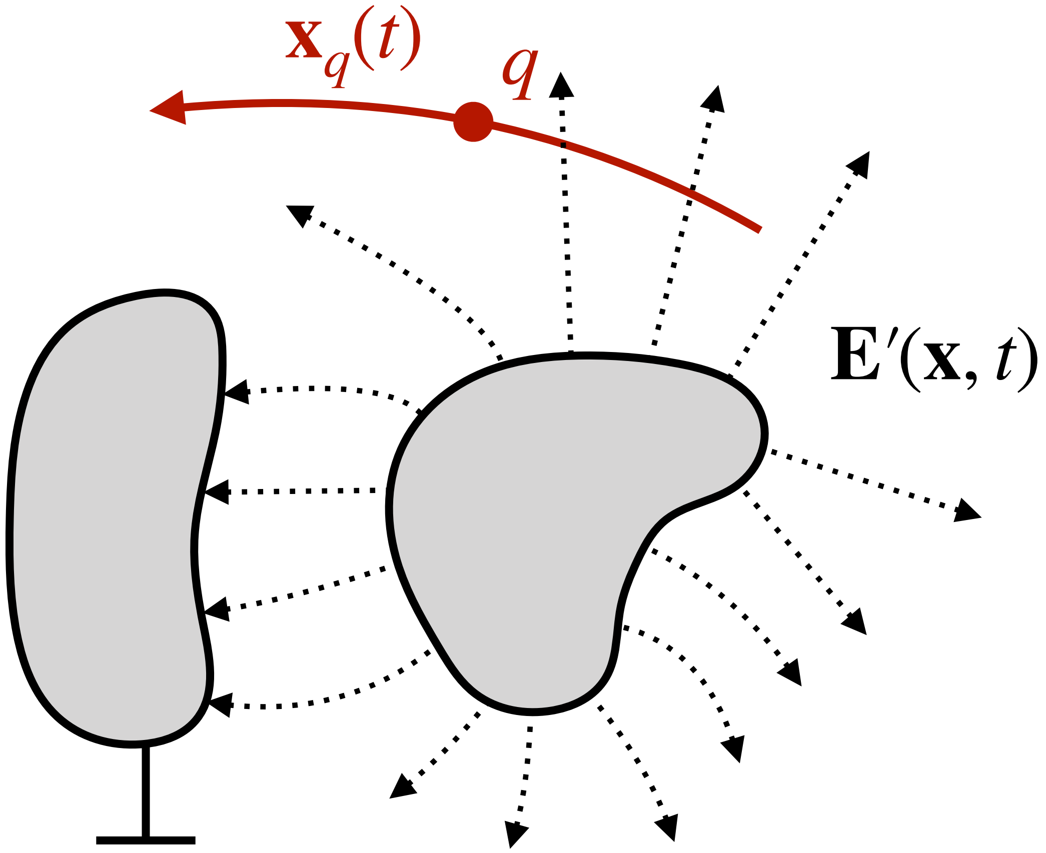
**Reciprocity:**

$$\boxed{V_2/I_1 = \bar{V}_1/\bar{I}_2}$$

# Two views of the same situation



**Detector electrode**
**"sees" particle**



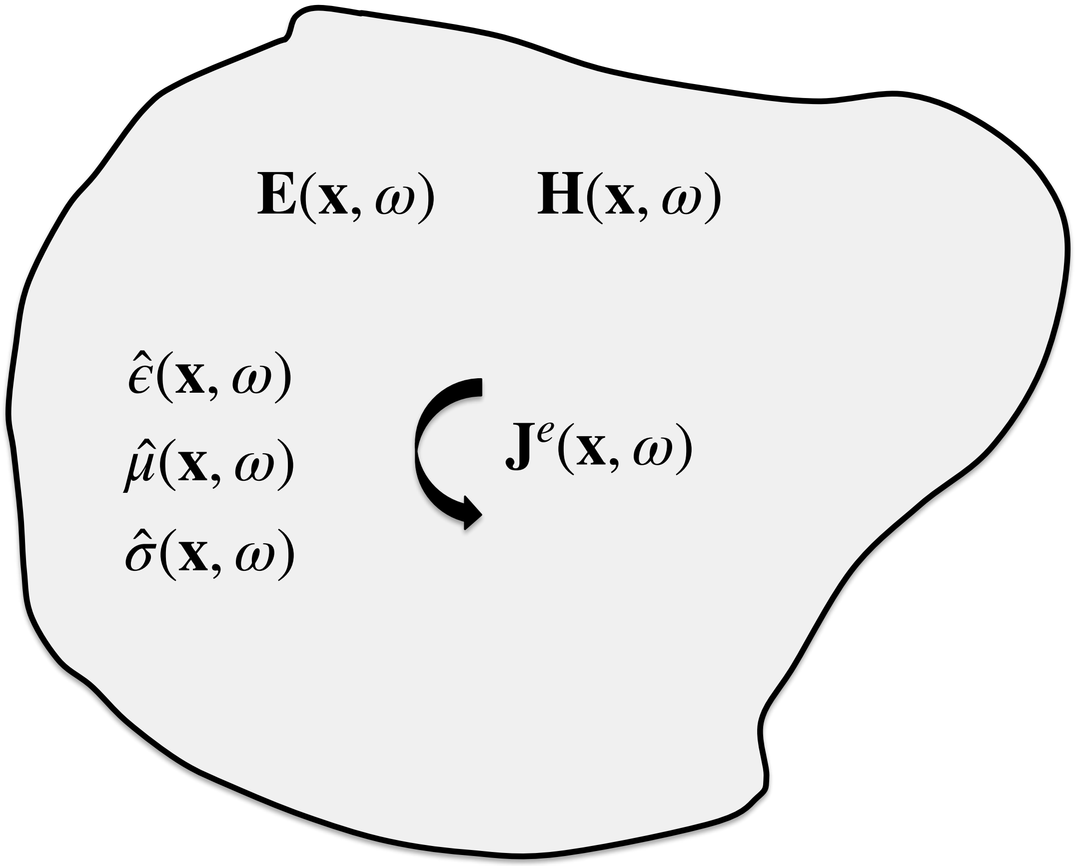
**Particle "sees"**
**detector electrode**

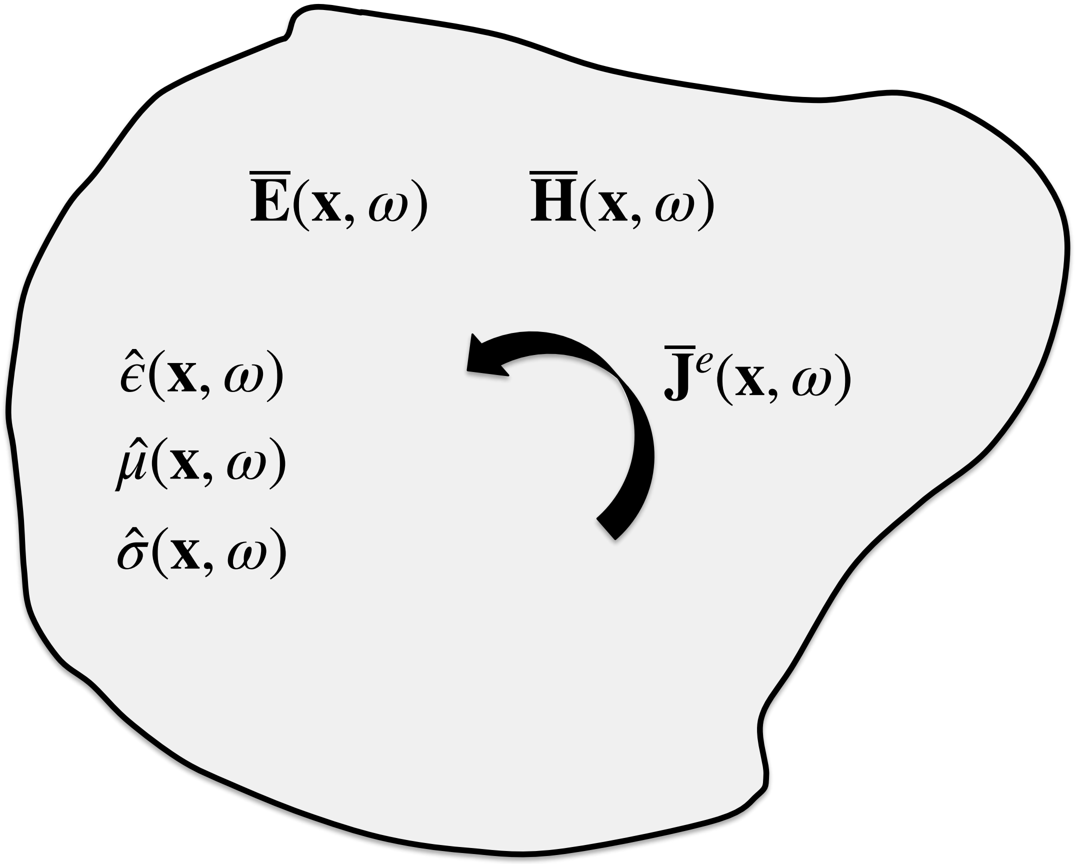
# Towards a "detector-centric" calculation

***Reciprocity:*** **electrodynamics has a built-in method to relate two different situations** *(with identical geometry)*

General, linear material distribution: $\epsilon(\mathbf{x}),\ \mu(\mathbf{x}),\ \sigma(\mathbf{x})$

*(taken to be symmetric for simplicity)*





$$\mathbf{J}^e \xrightarrow{\text{Maxwell's eqns.}} \mathbf{E},\ \mathbf{H}$$

External current distribution

Field distributions

$$\overline{\mathbf{J}}^e \xrightarrow{\text{Maxwell's eqns.}} \overline{\mathbf{E}},\ \overline{\mathbf{H}}$$

External current distribution

Field distributions

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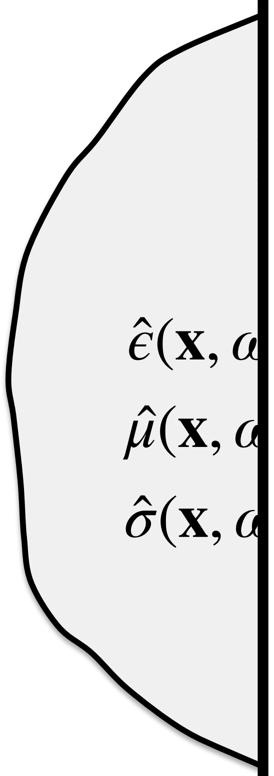
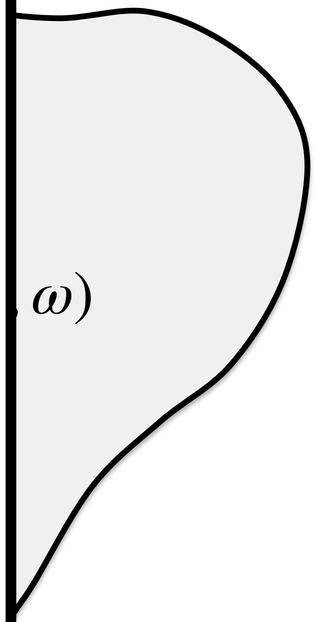




**These field distributions are not independent!**

*For arbitrary* $\mathbf{J}^e,\ \bar{\mathbf{J}}^e$*:*

$$\int_V dV\, \bar{\mathbf{E}}(\mathbf{x},\omega)\cdot\mathbf{J}^e(\mathbf{x},\omega) = \int_V dV\, \mathbf{E}(\mathbf{x},\omega)\cdot\bar{\mathbf{J}}^e(\mathbf{x},\omega)$$

*“Lorentz reciprocity” is a direct consequence of Maxwell’s equations*

*(see backup)*

H.A. Lorentz, Vers. Konig. Akad. Wetensch. 4, 176

$$\mathbf{J}^e \xrightarrow{\text{Maxwell's eqns.}} \mathbf{E},\ \mathbf{H}$$

External current distribution

Field distributions

$$\bar{\mathbf{J}}^e \xrightarrow{\text{Maxwell's eqns.}} \bar{\mathbf{E}},\ \bar{\mathbf{H}}$$

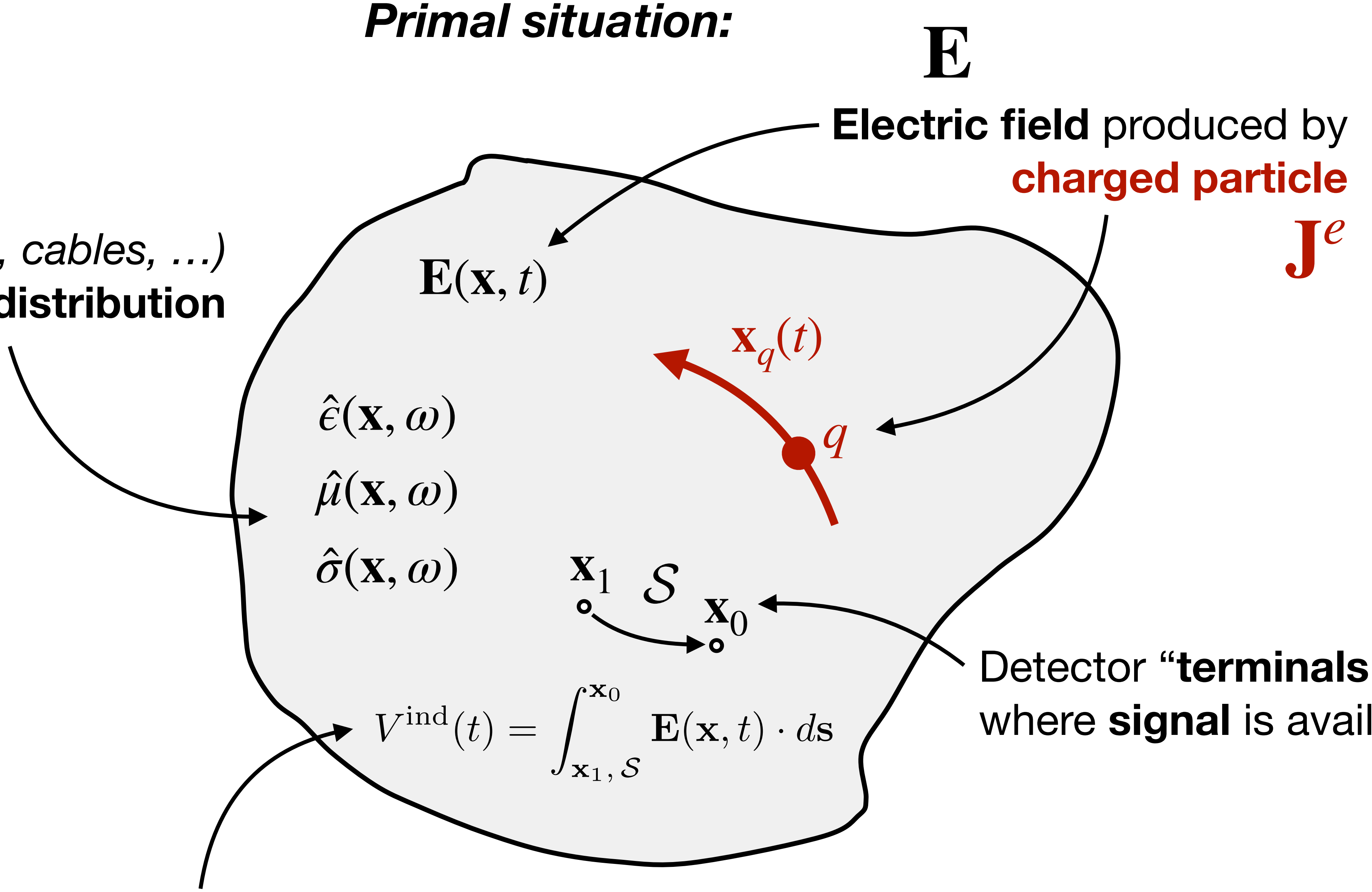
External current distribution

Field distributions

# A detector-centric signal theorem

**Use reciprocity to compute signal induced in detector**

***Primal situation:***



**Electric field** produced by **charged particle**

**Detector** *(antennas, cables, …)* **encoded** in **material distribution**

$$V^{\text{ind}}(t) = \int_{\mathbf{x}_1, \mathcal{S}}^{\mathbf{x}_0} \mathbf{E}(\mathbf{x}, t) \cdot d\mathbf{s}$$

Detector “**terminals**” where **signal** is available

Detector **signal**: energy gain between terminals
*(measured along a specific path $\mathcal{S}$, $\nabla \times \mathbf{E} \neq 0$ in general!)*

# A detector-centric signal theorem

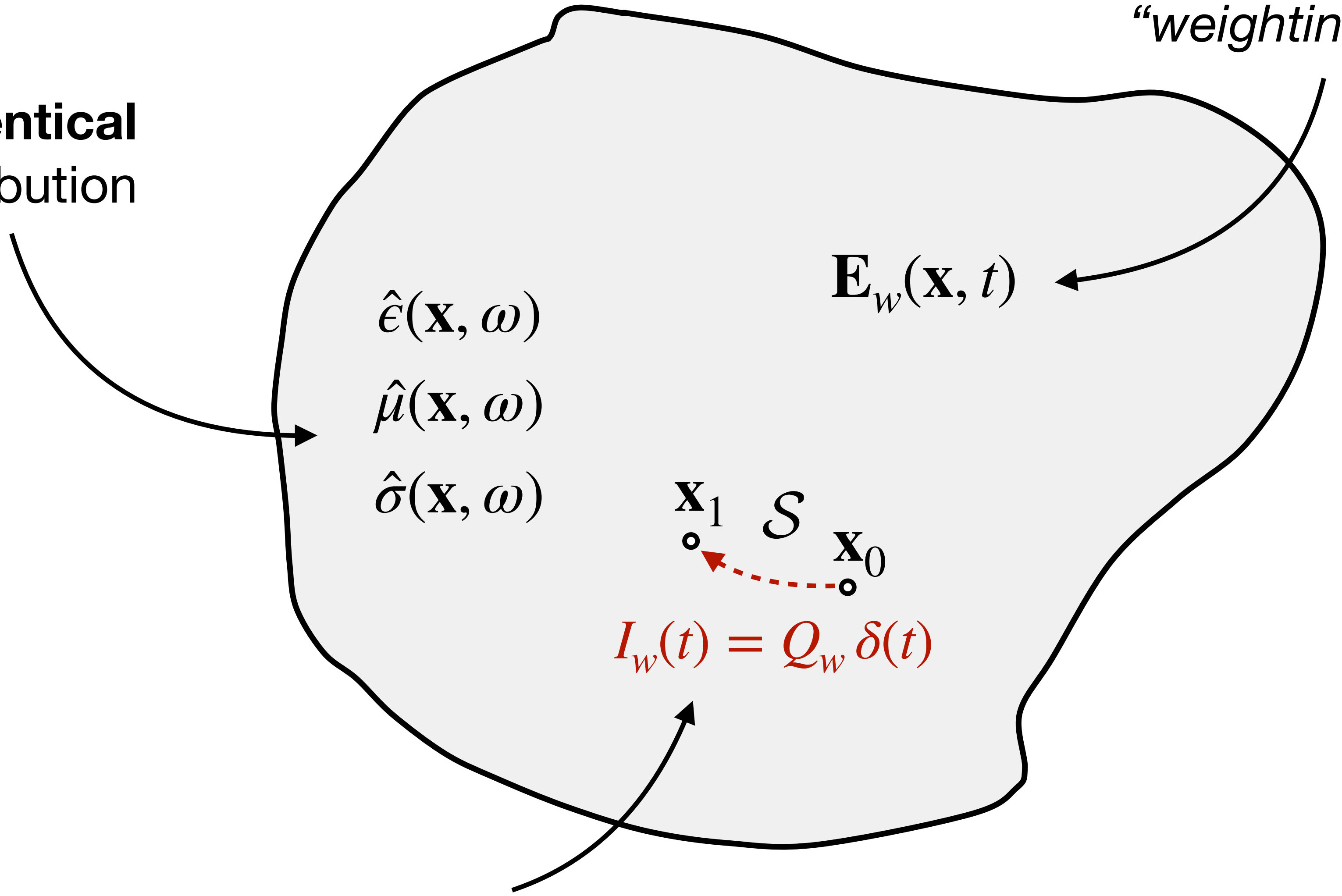
## Use reciprocity to compute signal induced in detector

***The “dual” situation:***

**Detector** acts as **transmitting device**:
*“weighting field distribution”*

$\overline{\mathbf{E}}$

**Identical**
material distribution

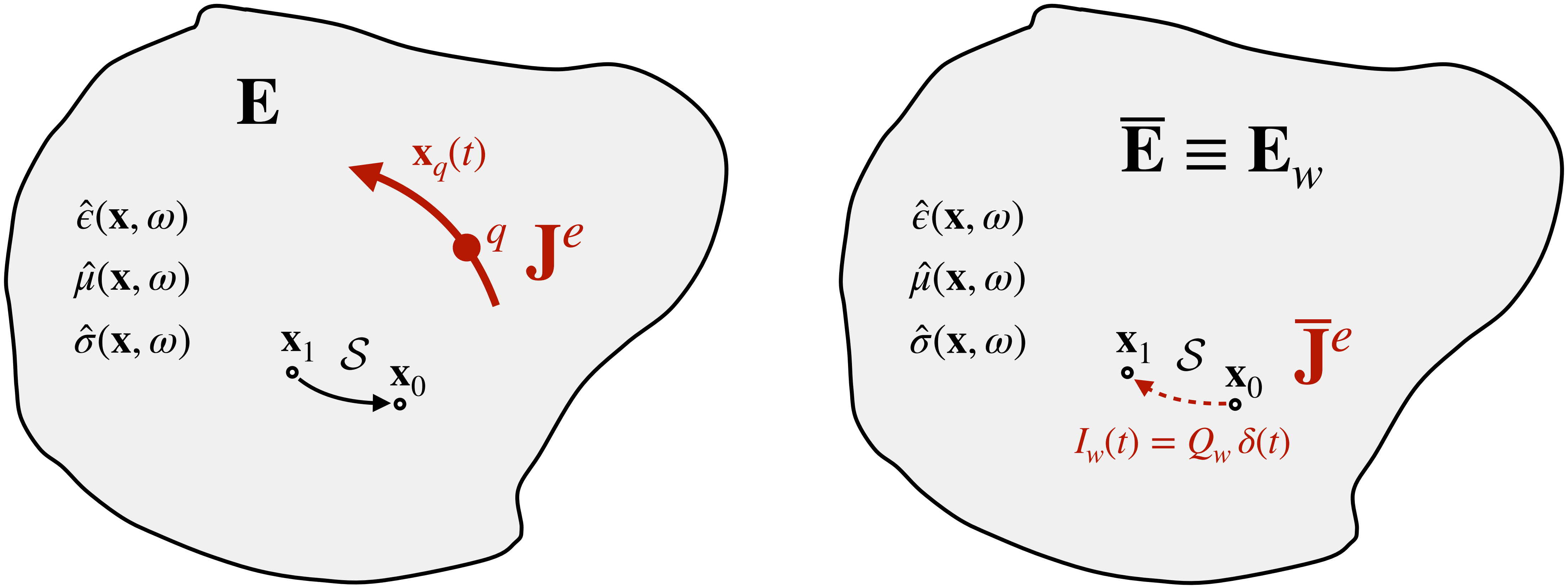


**Current source** attached to **detector terminals**:
*delta-like current along* $\mathcal{S}$ *(convention: in opposite direction)*

$\overline{\mathbf{J}}^e$

# A detector-centric signal theorem

**Use reciprocity to compute signal induced in detector**



*Lorentz reciprocity:*

$$\int_V dV\,\mathbf{E}(\mathbf{x},\omega)\cdot\bar{\mathbf{J}}^e(\mathbf{x},\omega) = \int_V dV\,\bar{\mathbf{E}}(\mathbf{x},\omega)\cdot\mathbf{J}^e(\mathbf{x},\omega)$$

$$\boxed{-Q_w V^{\mathrm{ind}}(\omega) = \int_V dV\,\mathbf{E}_w(\mathbf{x},\omega)\cdot\mathbf{J}^e(\mathbf{x},\omega)}$$

# A detector-centric signal theorem

**In the time domain:**

*“Induced signal = weighting field * particle trajectory”*

$$V^{\mathrm{ind}}(t) = -\frac{q}{Q_w}\int_{-\infty}^{\infty} dt'\,\mathbf{E}_w(\mathbf{x}_q(t'), t - t')\cdot\dot{\mathbf{x}}_q(t')$$

Normalising constant

Weighting field

Particle trajectory

**Weighting field**: Green’s function for detector signal

*Encodes information about detector geometry and environment*

*Reciprocity: compute it by using detector as transmitter (Maxwell solver)*

**Compute once, use for arbitrary particle trajectories**

*(Numerical convolution is cheap!)*

**Fully general, no approximations**

*Holds exactly for all linear, anisotropic materials,*
*approximately for nonlinear, anisotropic materials*

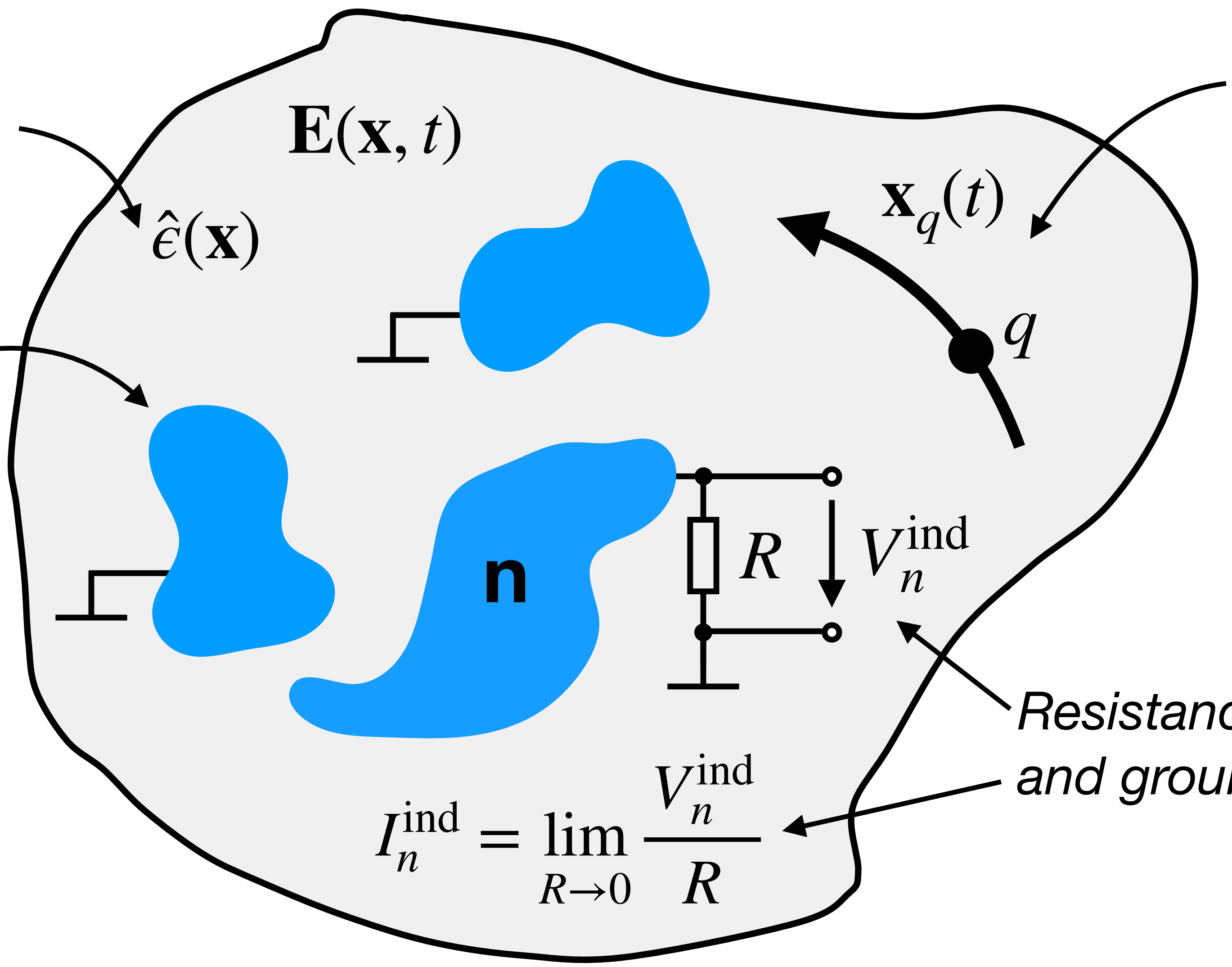
# Nonrelativistic limit

**Charges move nonrelativistically in a typical detector** (gas, silicon)

Quasi-electrostatics (c→∞): *no radiation, no propagation effects, …*

**Also:** want the induced current (on a grounded electrode)

**Primal situation:**



*Electrostatics:*
permeability irrelevant

*Grounded detector electrodes*

*Slowly moving charged particle*

*Resistance b/w electrode and ground measures current*

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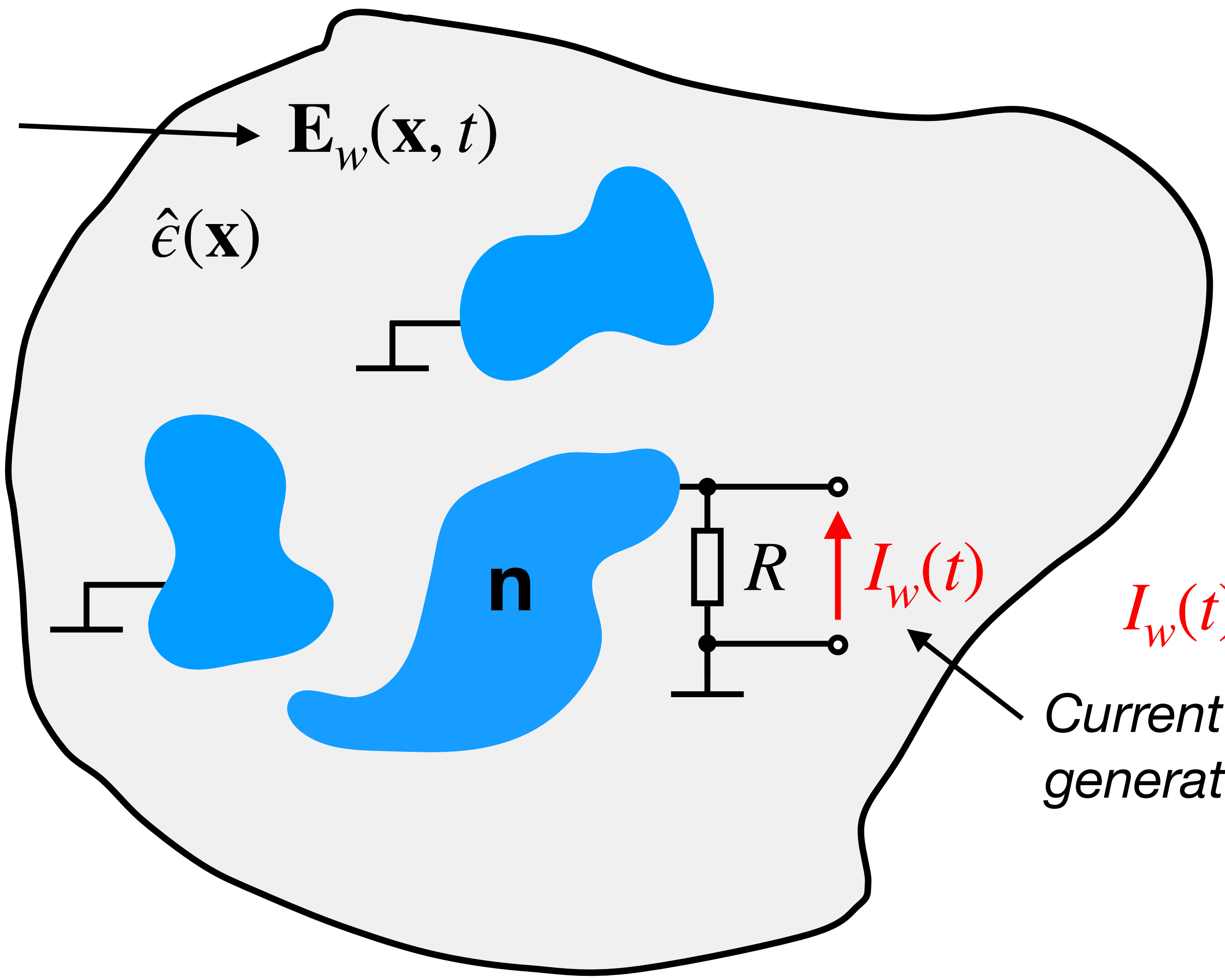
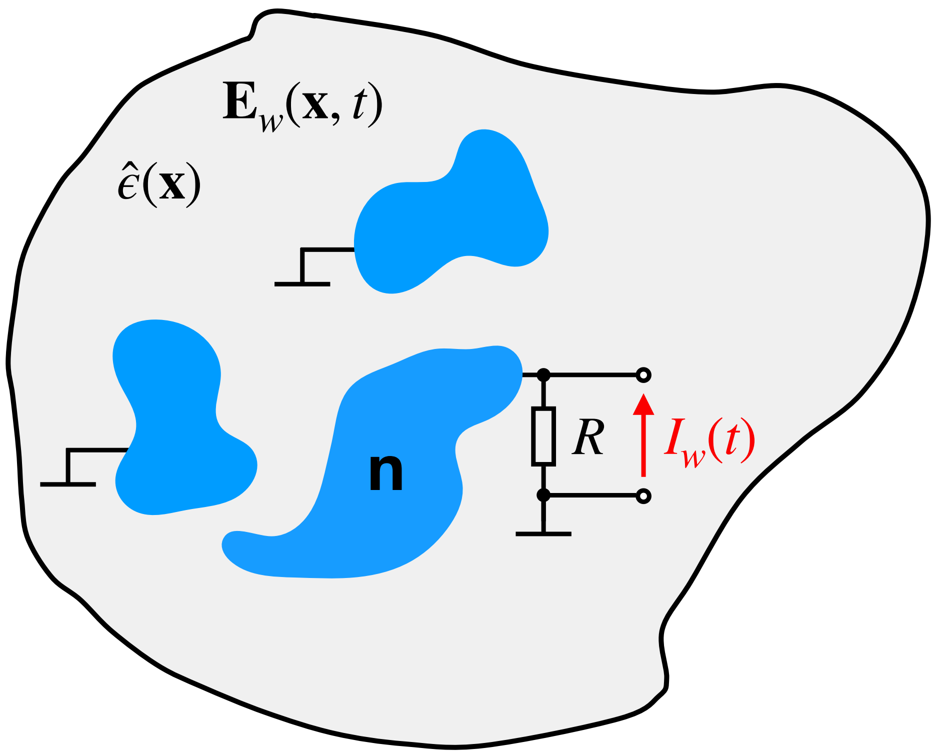


*Weighting field*

$$I_w(t) = Q_w\,\delta(t)$$

*Current across resistor generates weighting field*

# Weighting field



1.) **The current $I_w$ places a charge $Q_w$ on electrode n at t = 0**

$$I_w(t) = Q_w\,\delta(t)$$

2.) **This puts the electrode at potential $V(0)$ and generates the electric field $\mathbf{E}_w(\mathbf{x}, t)$**

3.) **The electrode discharges through the resistor $R$ for t > 0**
→ time-dependent potential $V(t)$

**Quasi-electrostatic limit:** evolution is sequence of electrostatic configurations

If the field is $\mathbf{E}_n(\mathbf{x}; V)$ for electrode *n* at potential V, the weighting field is

$$\mathbf{E}_w(\mathbf{x}, t) = \mathbf{E}_n(\mathbf{x}; V(t)) = \frac{V(t)}{V_w}\mathbf{E}_n(\mathbf{x}; V_w)$$

Arbitrary constant

*(Field scales homogeneously with the potential)*

# Induced voltage

**Weighting field:**

$$\mathbf{E}_w(\mathbf{x}, t) = \frac{V(t)}{V_w}\mathbf{E}_n(\mathbf{x}; V_w)$$

**Signal theorem:**

$$V^{\text{ind}}(t) = -\frac{q}{Q_w}\int_{-\infty}^{\infty} dt'\,\mathbf{E}_w(\mathbf{x}_q(t'), t - t') \cdot \dot{\mathbf{x}}_q(t')$$

**Induced current:**

$$I^{\text{ind}}(t) = \lim_{R\to 0}\frac{V^{\text{ind}}(t)}{R}$$

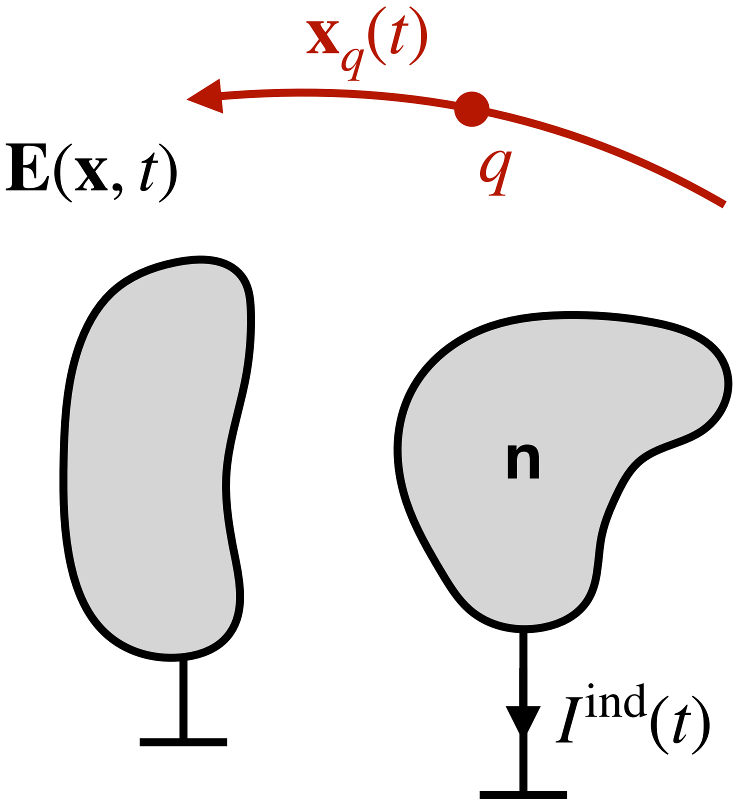
$$\implies I^{\text{ind}}(t) = \lim_{R\to 0} -\frac{q}{Q_w R}\int dt'\,\frac{V_n(t-t')}{V_w}\mathbf{E}_n(\mathbf{x}_q(t'); V_w) \cdot \dot{\mathbf{x}}_q(t')$$

The discharge is very fast
for small resistances

$$\lim_{R\to 0}\frac{V_n(t)}{R} = Q_w\,\delta(t)$$

$$\boxed{\implies I^{\text{ind}}(t) = -\frac{q}{V_w}\mathbf{E}_n(\mathbf{x}_q(t); V_w) \cdot \dot{\mathbf{x}}_q(t)}$$

# Induced voltage

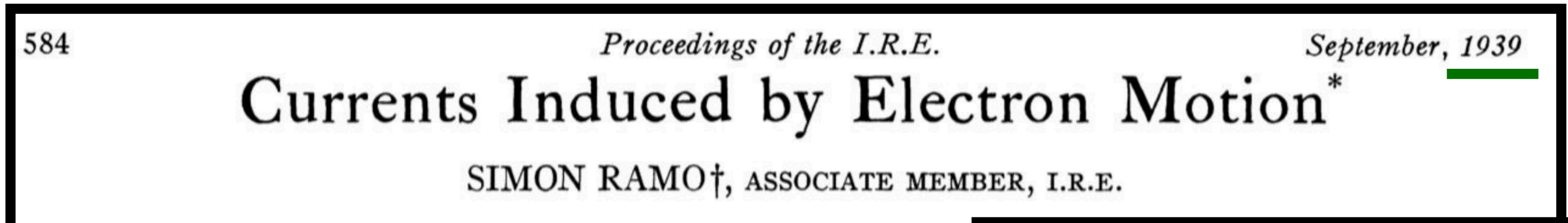


**"Detector-centric" expression for current induced on grounded readout electrodes:**

*(Nonrelativistic limit)*

$$\boxed{I^{\text{ind}}(t) = -\frac{q}{V_w}\mathbf{E}_n(\mathbf{x}_q(t); V_w) \cdot \dot{\mathbf{x}}_q(t)}$$

**"Ramo-Shockley theorem"**



584 *Proceedings of the I.R.E.* *September, 1939*

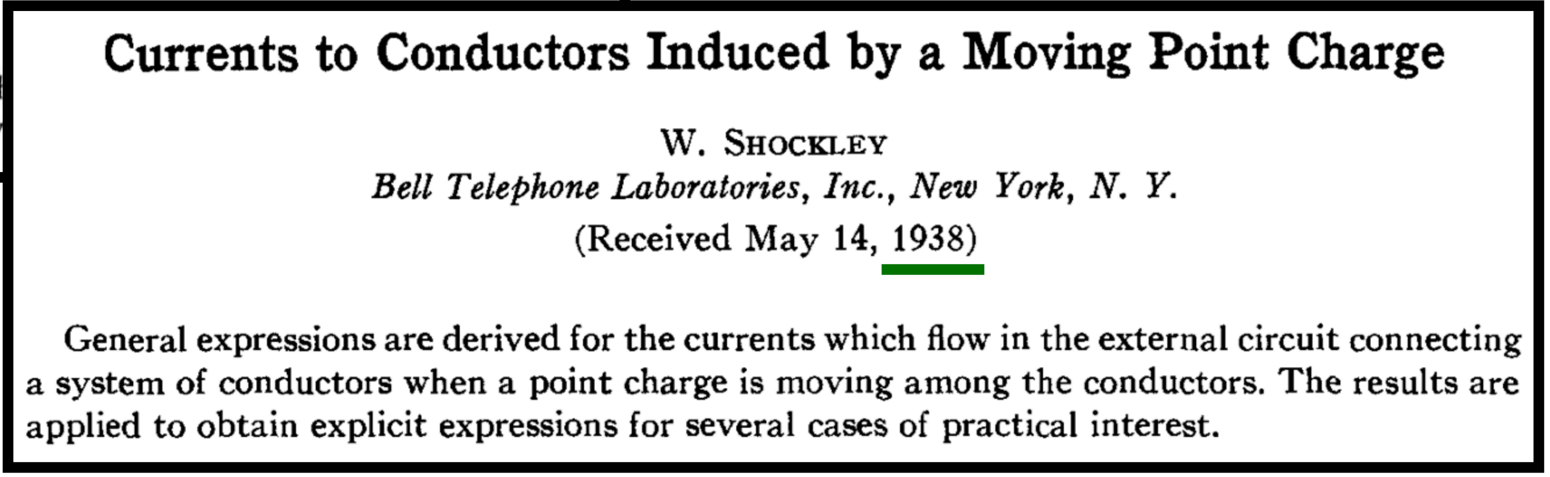
Currents Induced by Electron Motion*

SIMON RAMO†, ASSOCIATE MEMBER, I.R.E.

***Summary***—*A method is given for computing the instantaneous current induced in neighboring conductors by a given specified motion of electrons. The method is based on the repeated use of a simple equation giving the current due to a single electron's movement and is believed to be simpler than methods previously described.*

The meth
whose deriv

Derived in the 1930s by Ramo and Shockley …



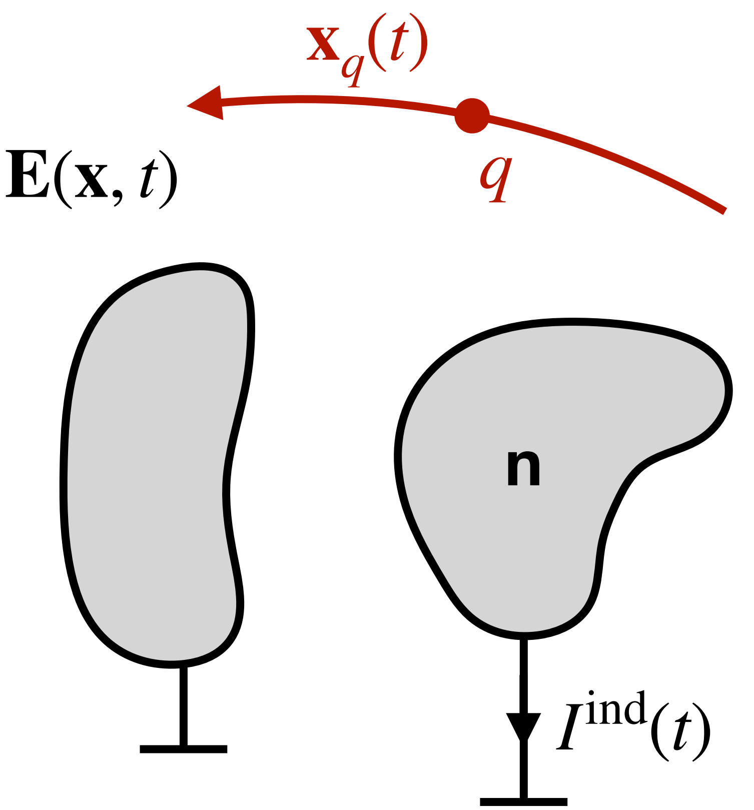
Currents to Conductors Induced by a Moving Point Charge

W. Shockley
*Bell Telephone Laboratories, Inc., New York, N. Y.*
(Received May 14, 1938)

General expressions are derived for the currents which flow in the external circuit connecting a system of conductors when a point charge is moving among the conductors. The results are applied to obtain explicit expressions for several cases of practical interest.

… specifically for the quasi-static case

# Induced voltage



**"Detector-centric" expression for current induced on grounded readout electrodes:**

*(Nonrelativistic limit)*

$$\boxed{I^{\text{ind}}(t) = -\frac{q}{V_w}\mathbf{E}_n(\mathbf{x}_q(t); V_w) \cdot \dot{\mathbf{x}}_q(t)}$$

**"Ramo-Shockley theorem"**

**Weighting field**

*Encodes detector geometry, can be computed once and for all*

**Particle trajectory**

*Nonzero velocity is required to produce a nonzero signal*

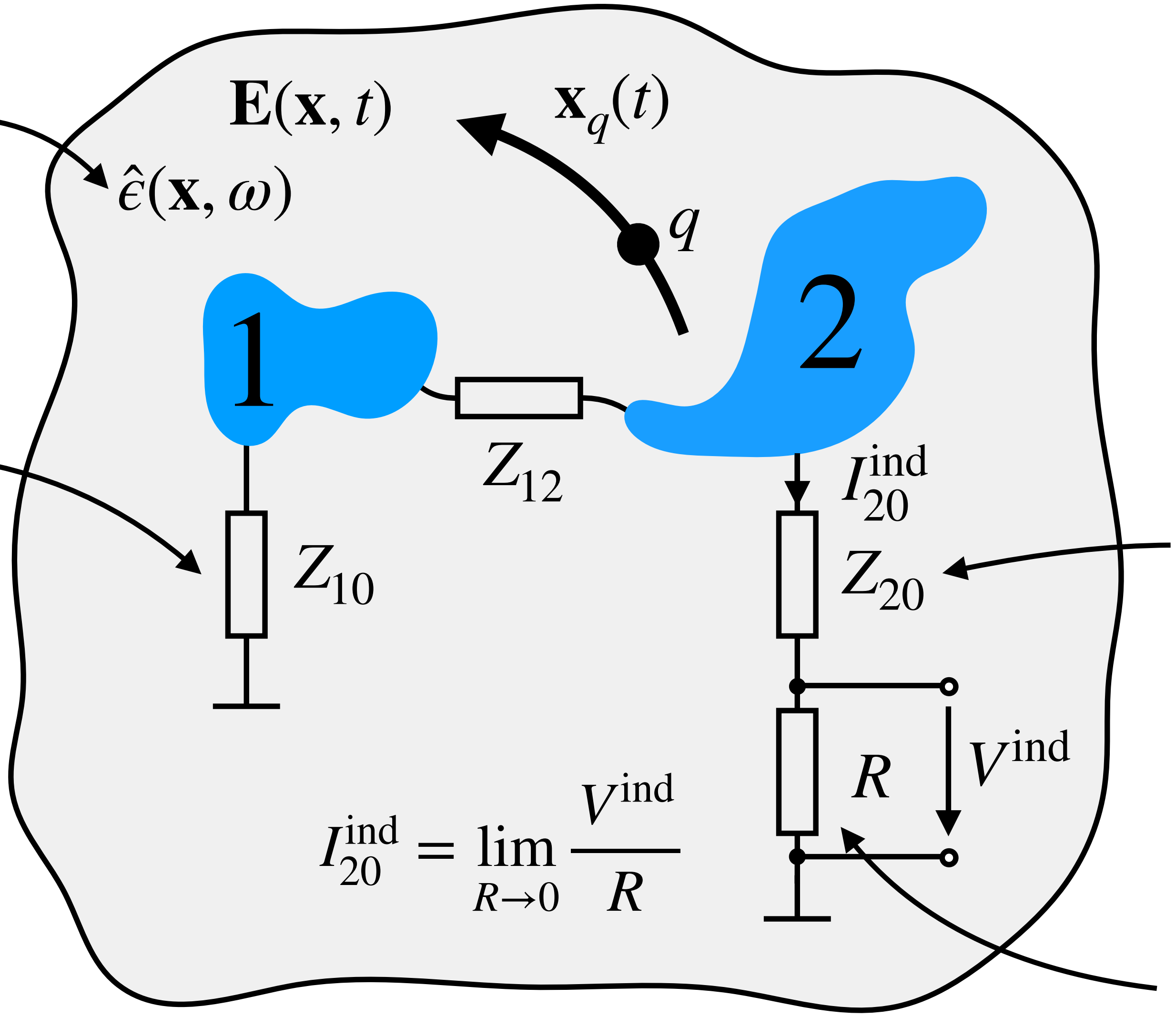
**Very efficient!** Multiplication instead of Poisson equation …

# Grounded electrodes?

**Real detector electrodes are embedded in electrical circuits**

*Insulation resistance, frontend electronics …*

**Primal situation:**



$$I_{20}^{\text{ind}} = \lim_{R \to 0} \frac{V^{\text{ind}}}{R}$$

*Electrostatics:* permeability irrelevant

*Detector electrodes connected with arbitrary impedance elements*

*Detector signal:* current flowing through $Z_{20}$ towards ground ("frontend impedance")

Ohmic resistance to measure current (as before)

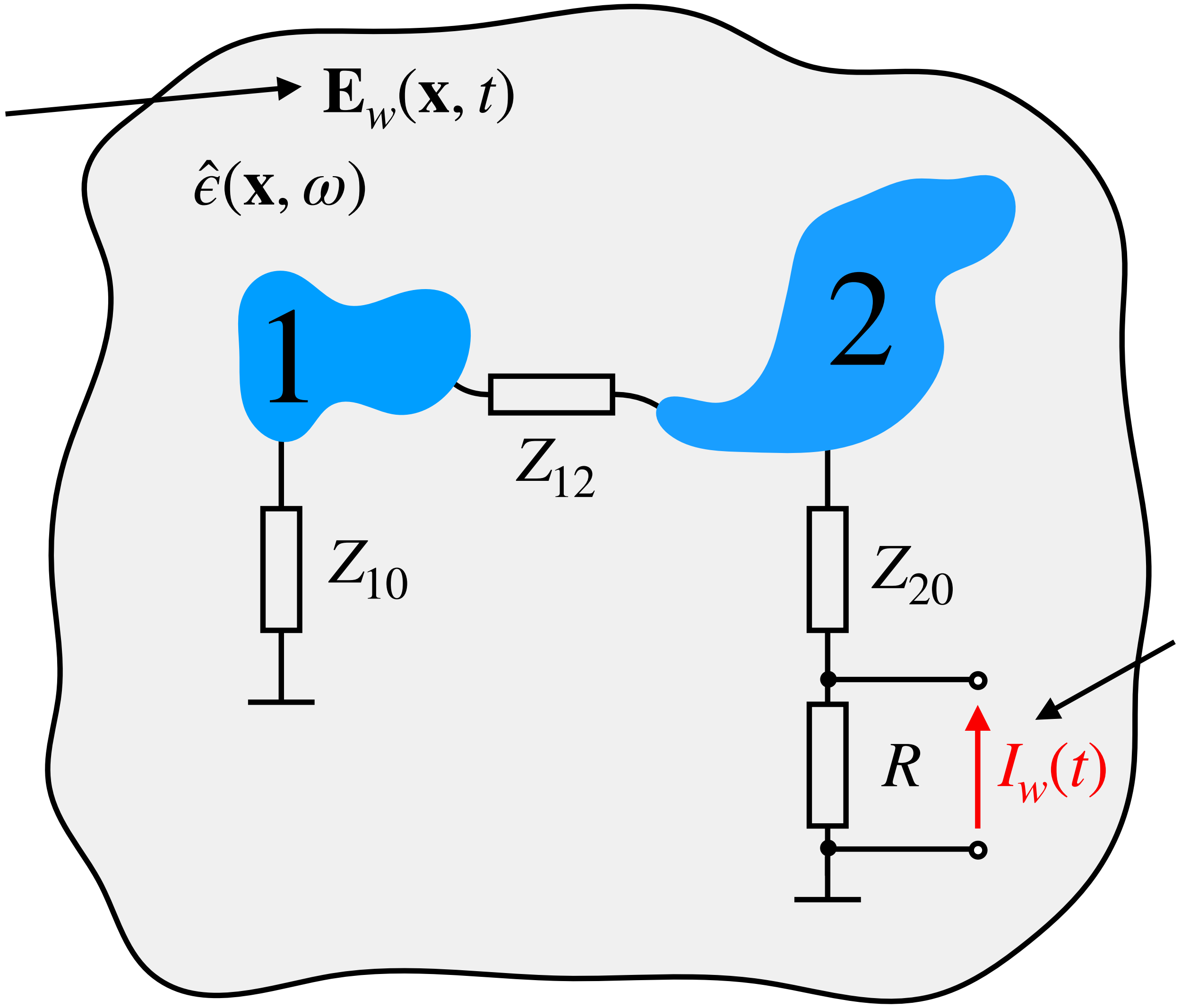
# Grounded electrodes?

**Real detector electrodes are embedded in electrical circuits**

*Insulation resistance, frontend electronics …*

**Dual situation:**

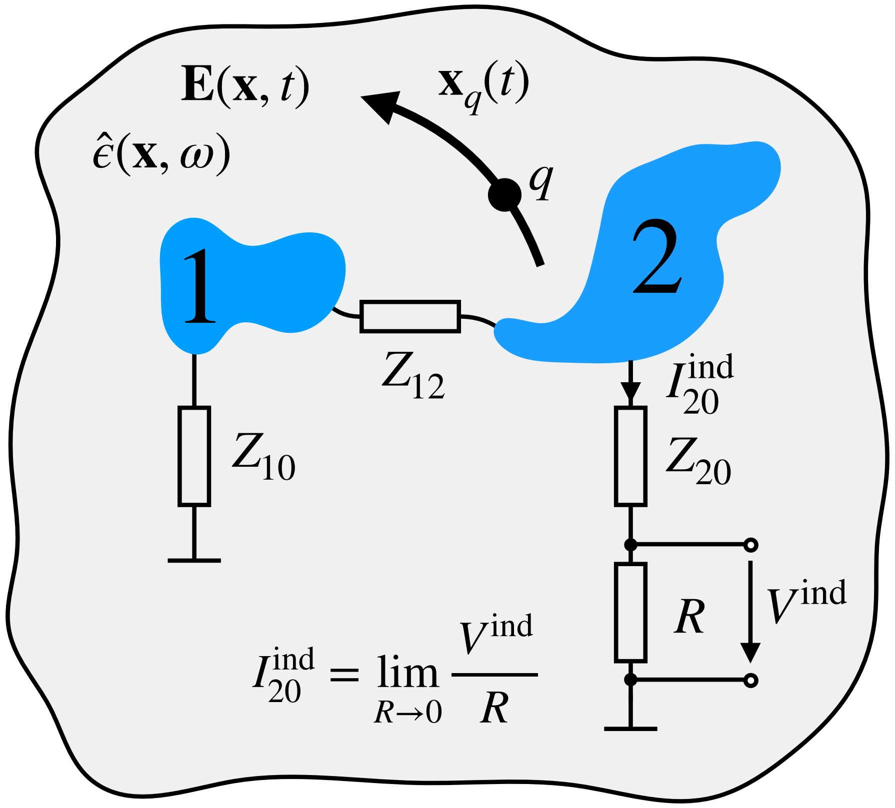
*Weighting field*



*Current across resistor generates weighting field (as before)*

$$I_w(t) = Q_w\,\delta(t)$$

# Induced signal



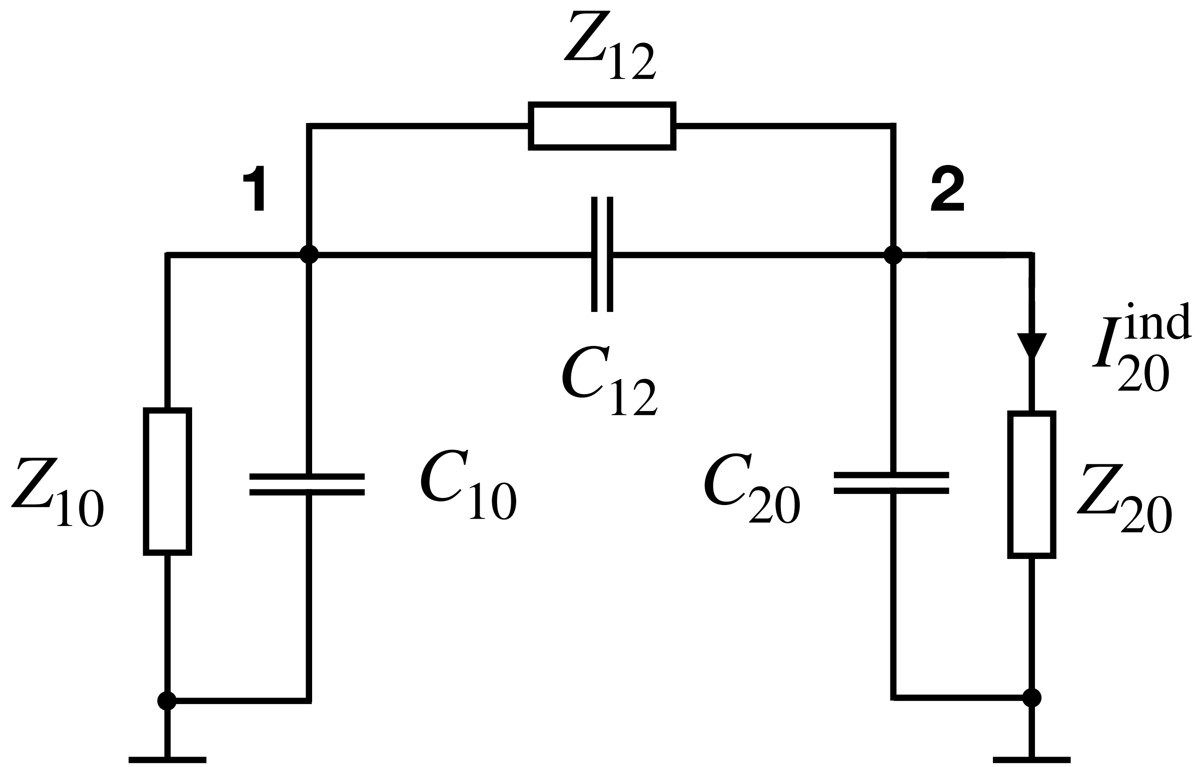
**Derivation works as before:**

$$\mathbf{E}_w(\mathbf{x}, t) = \frac{V_2(t)}{V_w} \mathbf{E}_2(\mathbf{x}; V_w)$$

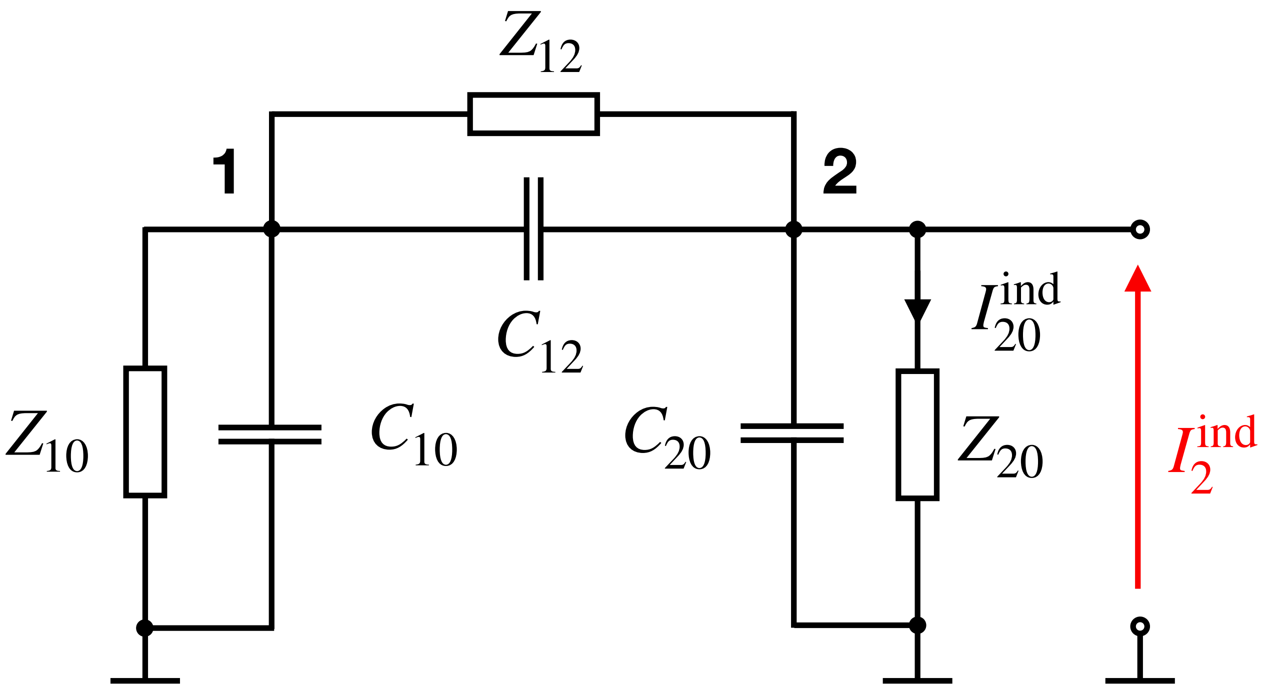
Evolution of $V_2(t)$ now depends on the equivalent circuit of the situation

**Equivalent circuit:**

Explicit representation of circuit including mutual capacitances between electrodes



# Induced signal



**Result of derivation:**

Convolution

$$\lim_{R\to 0} I_{20}^{\mathrm{ind}}(t) = g(t) * I_2^{\mathrm{ind}}(t)$$

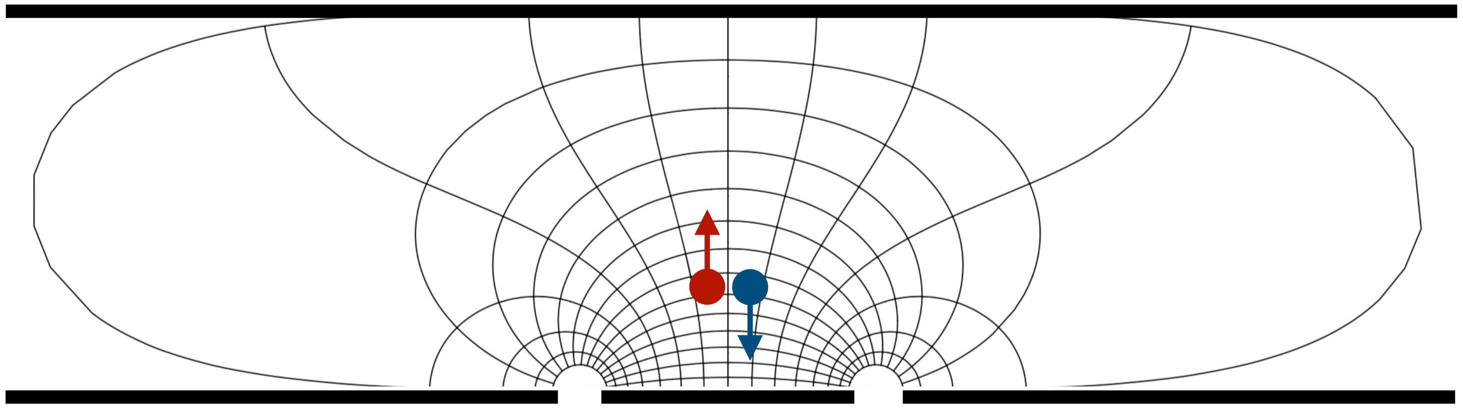
Green's function of equivalent circuit

Current induced on grounded electrode

**Practical workflow:**

1.) Compute current $I_2^{\mathrm{ind}}(t)$ induced on grounded electrode with Ramo-Shockley theorem

2.) Place this current as a source into equivalent circuit and solve with SPICE (or other simulator), read off $I_{20}^{\mathrm{ind}}(t)$



# Signals in silicon detectors

# Signals in silicon detectors

**Shape of signals in silicon detectors determined by**

↓

**Structure of weighting field**

↕

**Geometry of electrodes:**

Pixels, strips, …

↓

**Trajectories of moving charges**

↕

**Electric field in detector**

Drift trajectories,
avalanche multiplication, …

**In the following:** discuss some common situations

# Parallel-plate geometry

*(e.g. silicon pixel detector with large pitch)*

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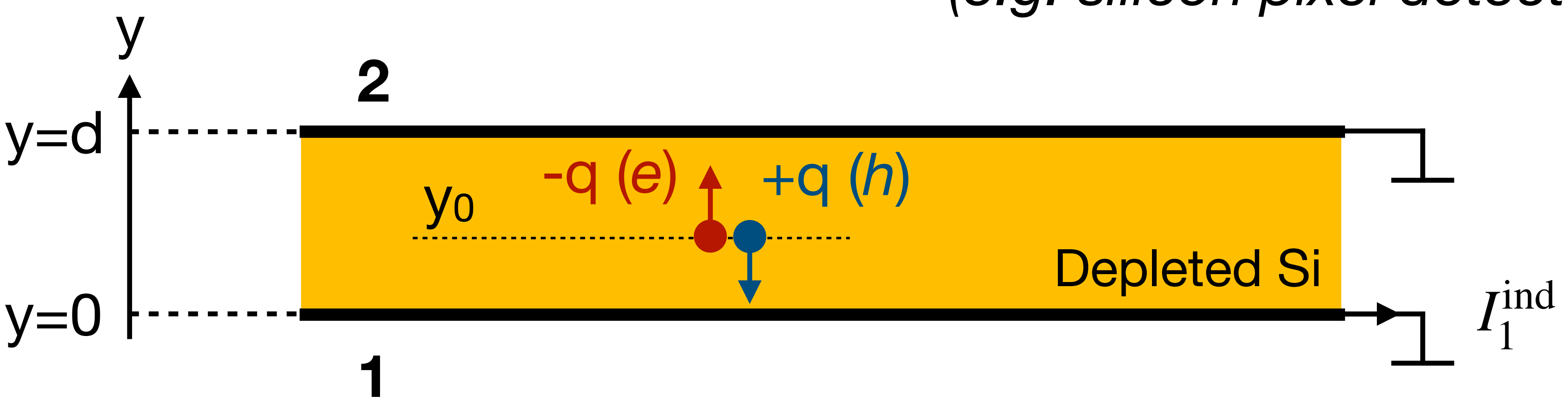
*(e.g. silicon pixel detector with large pitch)*

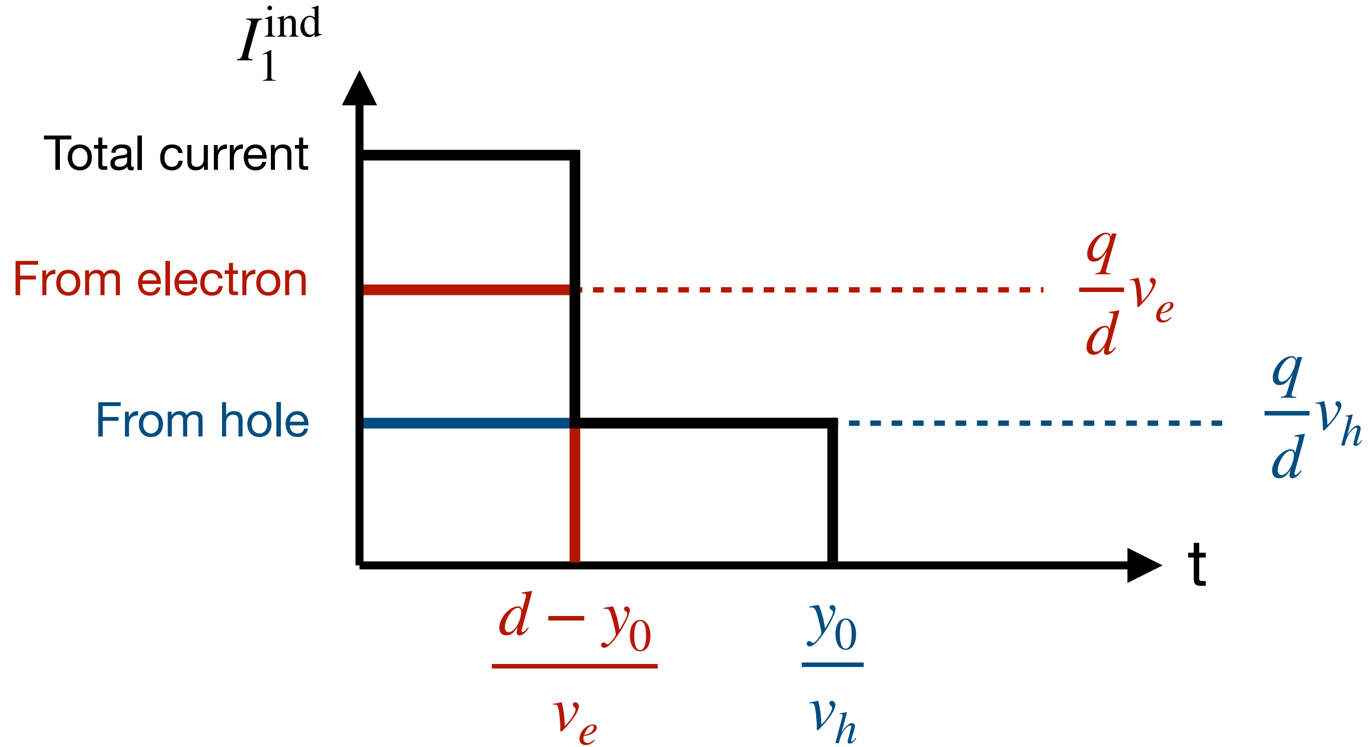
**Weighting field for electrode 1:** $\mathbf{E}_1 = \frac{V_w}{d}\,\hat{\mathbf{y}}$

# Signal induced by electron-hole pair

*(e.g. silicon pixel detector with large pitch)*



**Weighting field for electrode 1:** $\mathbf{E}_1 = \frac{V_w}{d}\hat{\mathbf{y}}$



**Electron-hole pair deposited at y₀, drifting in opposite directions:**

Constant contribution to signal during drift

*(electrons drift faster)*

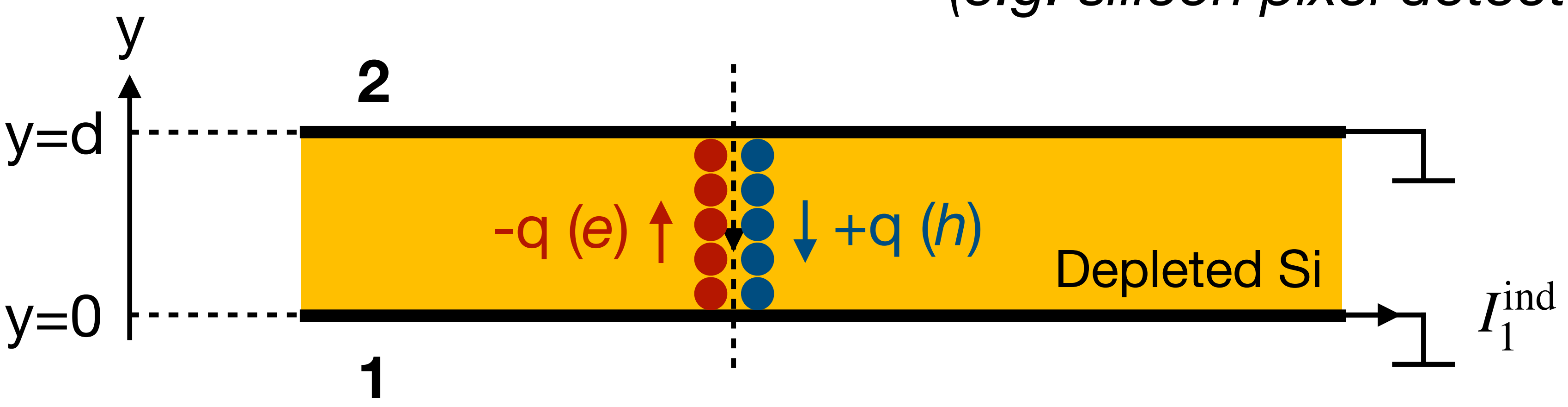
Total charge contained in signal:

$$Q_{\text{tot}} = \frac{q}{d}v_e \cdot \frac{d-y_0}{v_e} + \frac{q}{d}v_h \cdot \frac{y_0}{v_h} = q$$

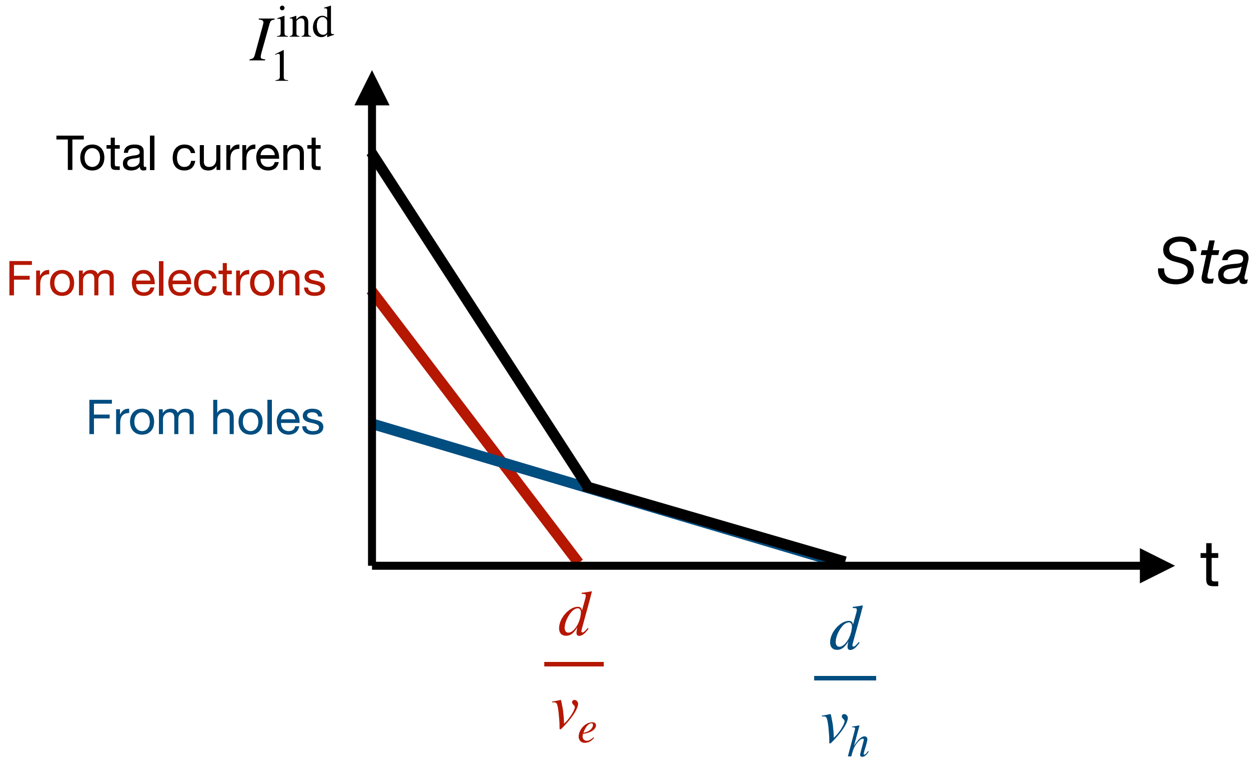
*(fraction carried by electrons / holes depends on $y_0$!)*

# Signal induced by charged particle

*(e.g. silicon pixel detector with large pitch)*



**Weighting field for electrode 1:** $\mathbf{E}_1 = \frac{V_w}{d}\,\hat{\mathbf{y}}$

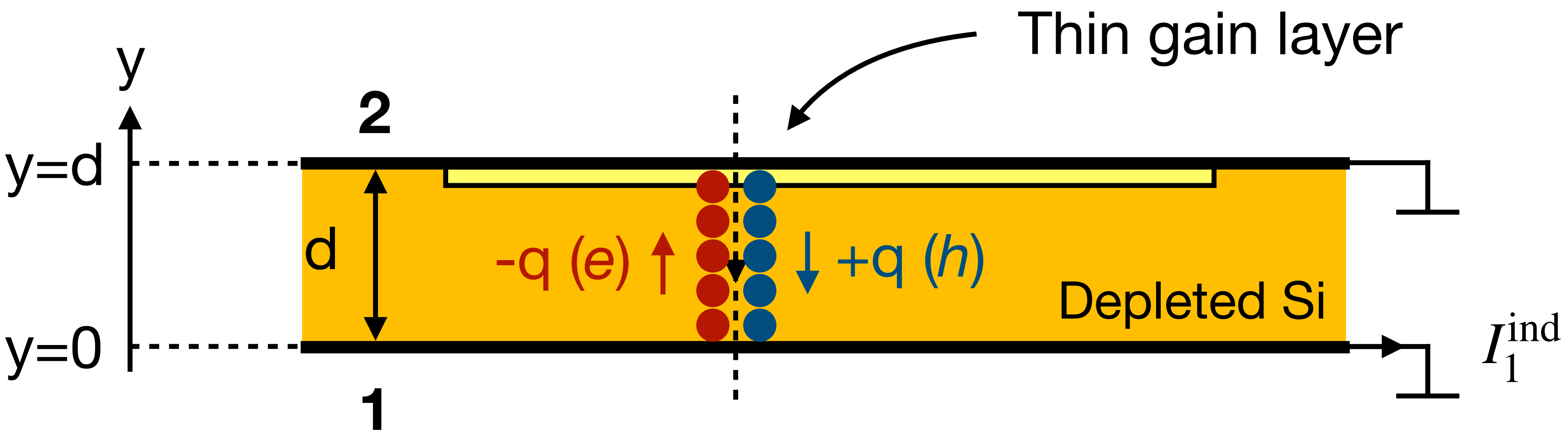


**Primary particle** deposits two **line charges** (clusters of e/h pairs) along its **track**

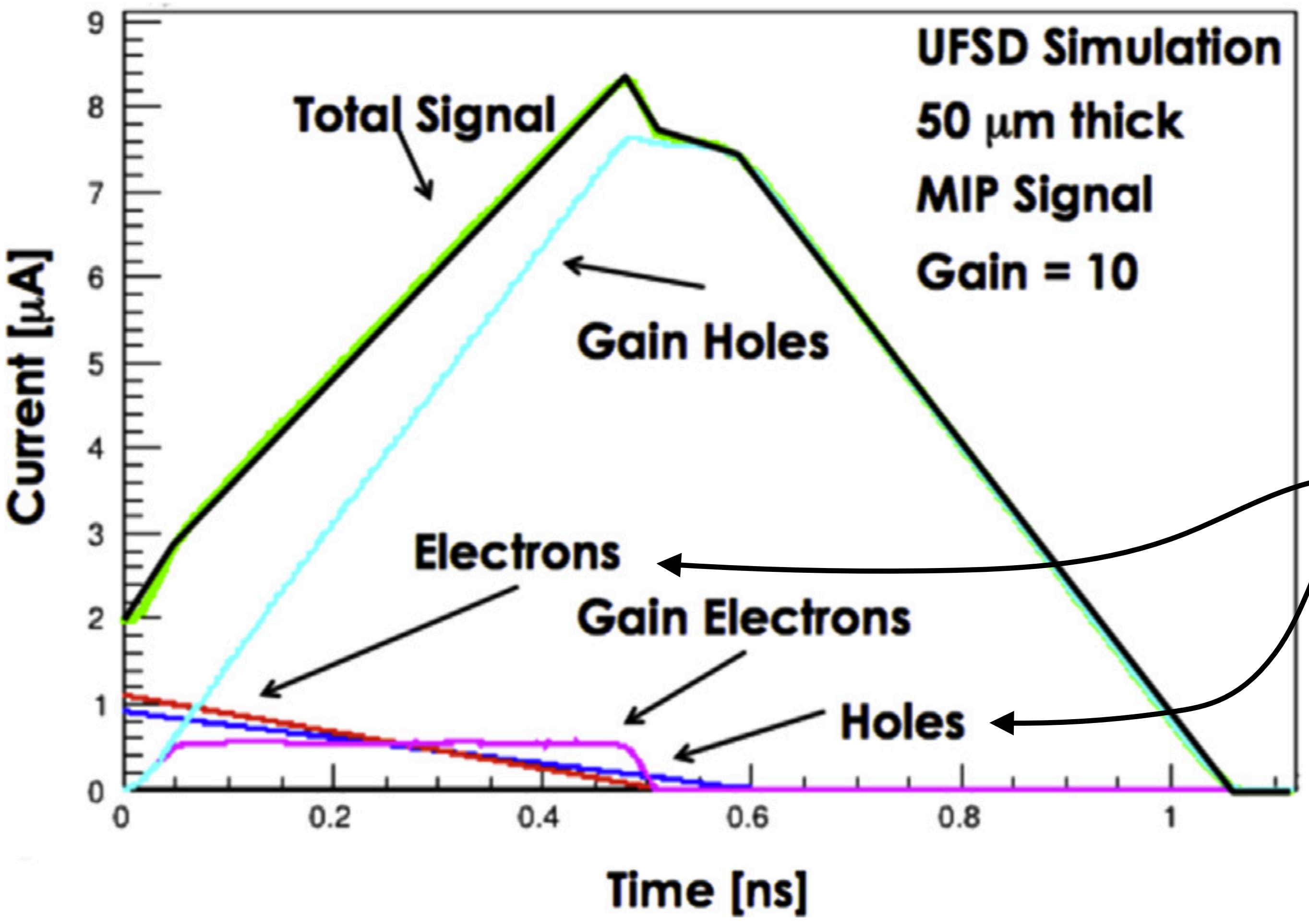
*Statistics of charge deposit:* Philip Allport's lectures

Number of drifting charges reduces linearly over time → triangular signal

# Signals in LGADs



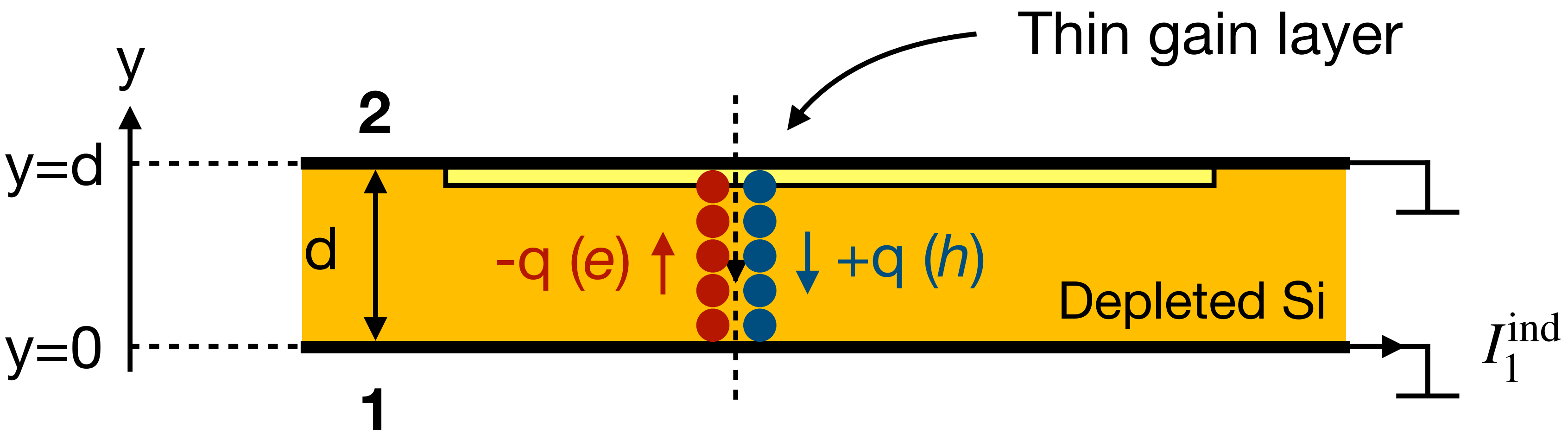
**Cartiglia et al.**, *"Design optimization of ultra-fast silicon detectors"* [link]



**Passing charged particle deposits line-charges, as before**

Primary **electrons** and primary **holes**: *triangular signal contributions,*

# Signals in LGADs



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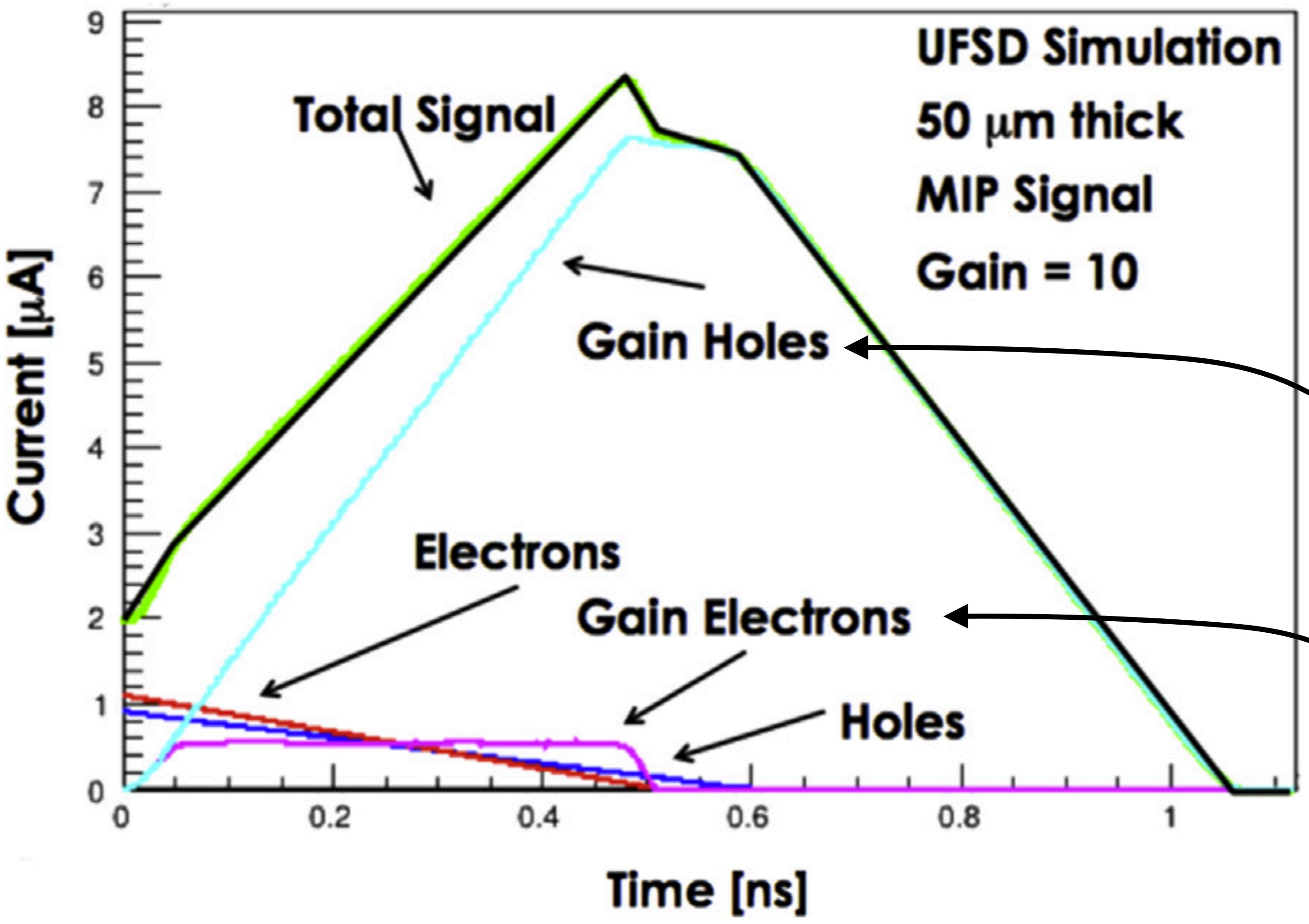
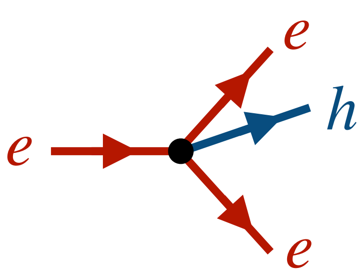


**Electrons** arriving at **gain layer** are **multiplied**



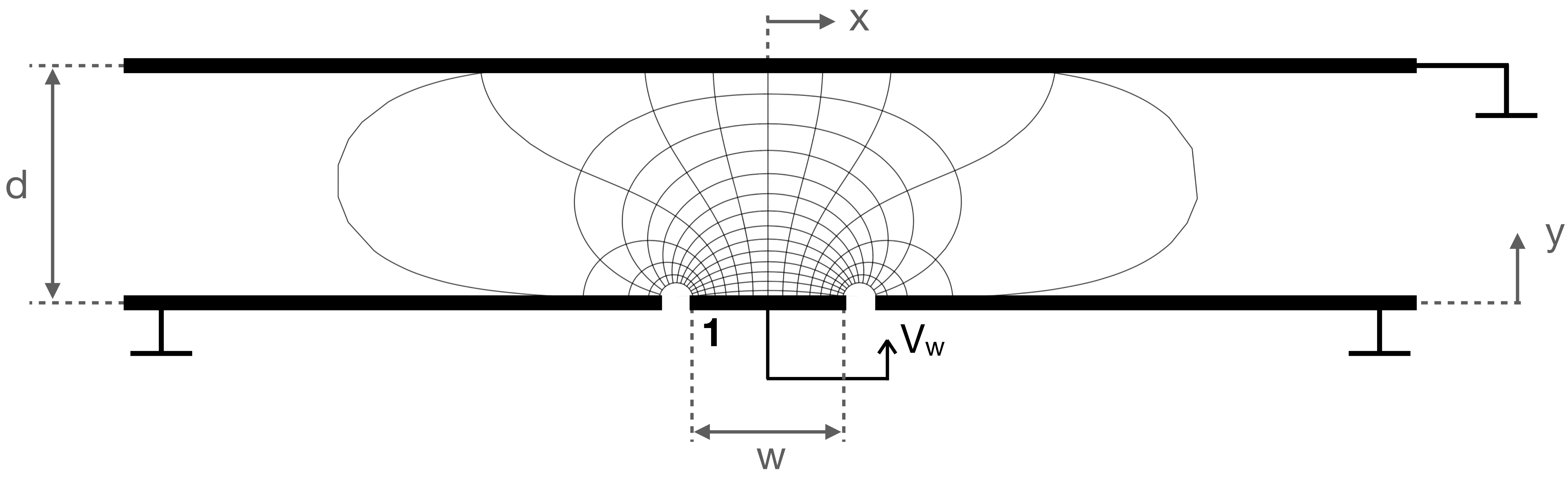
**Gain holes** produced at y=d drift through the full detector and **dominate the total signal**

**Gain electrons** produced at y=d are not important for total signal

# The real world is not a parallel plate capacitor!

**Realistic detector geometries have more complicated weighting fields …**

**→ revisit the strip geometry from earlier**



**Weighting field for electrode 1:**

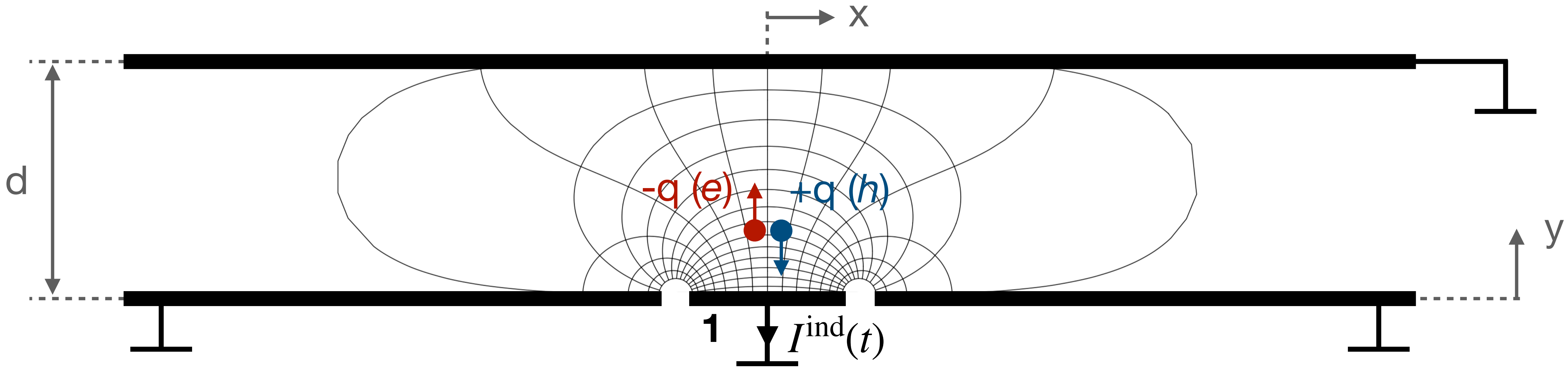
$$E_1^y = \frac{V_w}{2d}\left[\frac{\sinh\left(\pi\frac{x+w/2}{d}\right)}{\cosh\left(\frac{x+w/2}{d}\right)-\cos\left(\frac{\pi y}{d}\right)} - \frac{\sinh\left(\pi\frac{x-w/2}{d}\right)}{\cosh\left(\frac{x-w/2}{d}\right)-\cos\left(\frac{\pi y}{d}\right)}\right]$$

**Heubrandtner et al.**, “Static Electric Fields in an Infinite Plane Condensor with One or Three Homogeneous Layers” [link]

# The real world is not a parallel plate capacitor!

**Realistic detector geometries have more complicated weighting fields …**

→ **revisit the strip geometry from earlier**



For **w** ≫ **d**, the situation reverts to the parallel-plate geometry

For **w** ~ **d**, the weighting field increases towards the strip
→ signal dominated by moving charges close to the strip

**… similar for weighting field for *pixel* geometry**

**Heubrandtner et al.**, "Static Electric Fields in an Infinite Plane Condensor with One or Three Homogeneous Layers" [link]

# Summary

**Electrical signals in detectors are <u>exclusively</u> due to induction by moving charged particles**

**Signals in silicon detectors may be computed with the Ramo-Shockley theorem** *(itself a special case of a more general fact)*

$$I^{\mathrm{ind}}(t) = -\frac{q}{V_w}\mathbf{E}_n(\mathbf{x}_q(t); V_w) \cdot \dot{\mathbf{x}}_q(t)$$

Weighting field of electrode

Particle trajectory

Depending on the **geometry** and the **charge trajectory**, either electrons or holes are responsible for the **dominant part of the signal**

**Any questions?** Feel free to contact me at *philipp.windischhofer@cern.ch*

# References

**W. Shockley**, *Currents to Conductors Induced by a Moving Point Charge*,
Journal of Applied Physics. 9 (10): 635 (1938)

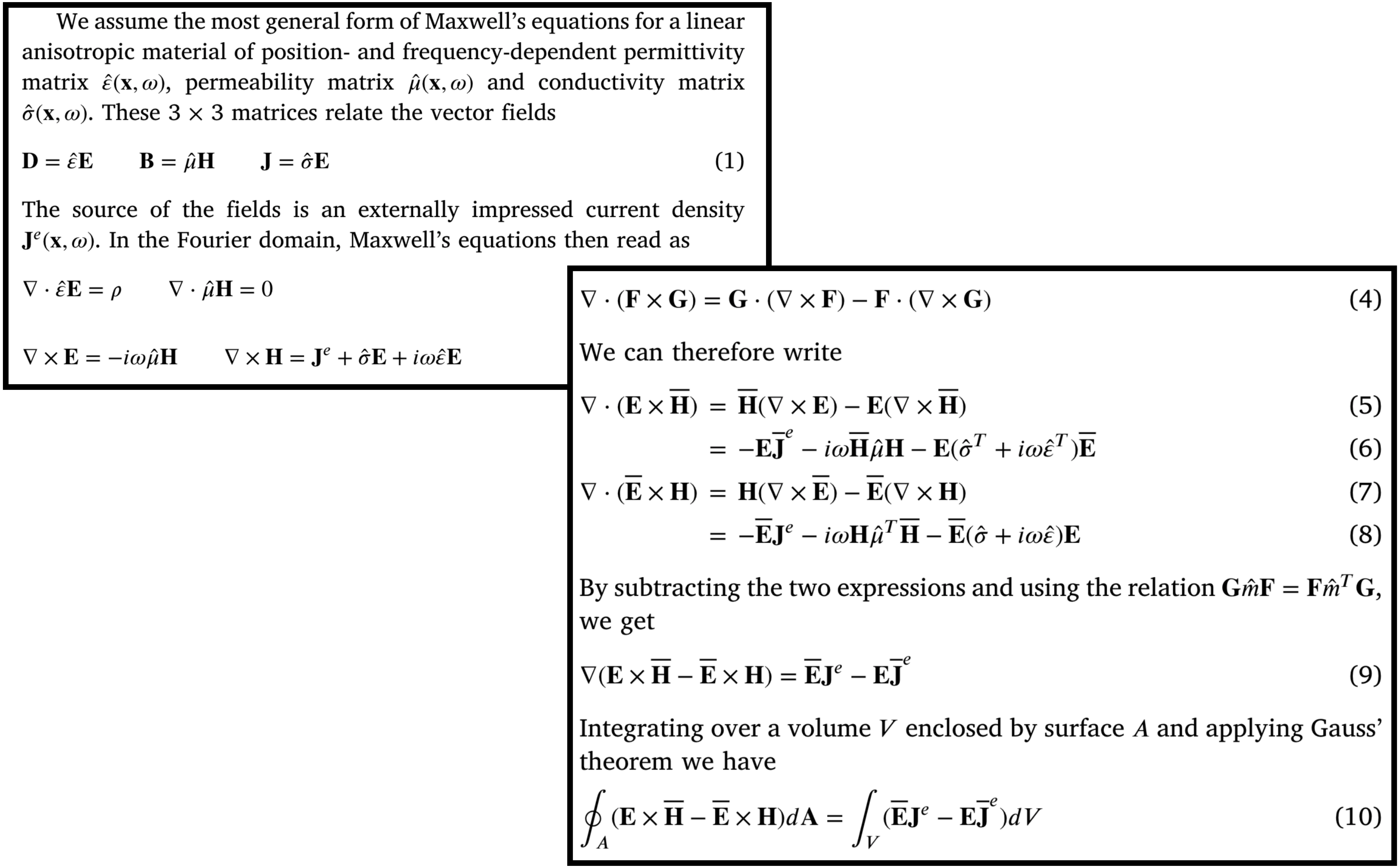
**S. Ramo**, *Currents induced in electron motion*, PROC. IRE 27, 584 (1939)

**W. Riegler, P. Windischhofer**, *Signals induced on electrodes by moving charges: a general theorem for Maxwell's equations based on Lorentz-reciprocity,*
Nucl. Instrum. Meth. A 980, 164471 (2020)

**W. Riegler**, *Signals in particle detectors*, CERN Academic Training Lectures,
https://indico.cern.ch/event/843083/

# Backup

# Maxwell’s equations and Lorentz reciprocity



We assume the most general form of Maxwell’s equations for a linear anisotropic material of position- and frequency-dependent permittivity matrix $\hat{\varepsilon}(\mathbf{x},\omega)$, permeability matrix $\hat{\mu}(\mathbf{x},\omega)$ and conductivity matrix $\hat{\sigma}(\mathbf{x},\omega)$. These $3 \times 3$ matrices relate the vector fields

$$\mathbf{D} = \hat{\varepsilon}\mathbf{E} \qquad \mathbf{B} = \hat{\mu}\mathbf{H} \qquad \mathbf{J} = \hat{\sigma}\mathbf{E} \tag{1}$$

The source of the fields is an externally impressed current density $\mathbf{J}^e(\mathbf{x},\omega)$. In the Fourier domain, Maxwell’s equations then read as

$$\nabla \cdot \hat{\varepsilon}\mathbf{E} = \rho \qquad \nabla \cdot \hat{\mu}\mathbf{H} = 0$$

$$\nabla \times \mathbf{E} = -i\omega\hat{\mu}\mathbf{H} \qquad \nabla \times \mathbf{H} = \mathbf{J}^e + \hat{\sigma}\mathbf{E} + i\omega\hat{\varepsilon}\mathbf{E}$$

$$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G}) \tag{4}$$

We can therefore write

$$\begin{aligned}
\nabla \cdot (\mathbf{E} \times \overline{\mathbf{H}}) &= \overline{\mathbf{H}}(\nabla \times \mathbf{E}) - \mathbf{E}(\nabla \times \overline{\mathbf{H}}) & (5)\\
&= -\mathbf{E}\overline{\mathbf{J}}^e - i\omega\overline{\mathbf{H}}\hat{\mu}\mathbf{H} - \mathbf{E}(\hat{\sigma}^T + i\omega\hat{\varepsilon}^T)\overline{\mathbf{E}} & (6)\\
\nabla \cdot (\overline{\mathbf{E}} \times \mathbf{H}) &= \mathbf{H}(\nabla \times \overline{\mathbf{E}}) - \overline{\mathbf{E}}(\nabla \times \mathbf{H}) & (7)\\
&= -\overline{\mathbf{E}}\mathbf{J}^e - i\omega\mathbf{H}\hat{\mu}^T\overline{\mathbf{H}} - \overline{\mathbf{E}}(\hat{\sigma} + i\omega\hat{\varepsilon})\mathbf{E} & (8)
\end{aligned}$$

By subtracting the two expressions and using the relation $\mathbf{G}\hat{m}\mathbf{F} = \mathbf{F}\hat{m}^T\mathbf{G}$, we get

$$\nabla(\mathbf{E} \times \overline{\mathbf{H}} - \overline{\mathbf{E}} \times \mathbf{H}) = \overline{\mathbf{E}}\mathbf{J}^e - \mathbf{E}\overline{\mathbf{J}}^e \tag{9}$$

Integrating over a volume $V$ enclosed by surface $A$ and applying Gauss’ theorem we have

$$\oint_A (\mathbf{E} \times \overline{\mathbf{H}} - \overline{\mathbf{E}} \times \mathbf{H})d\mathbf{A} = \int_V (\overline{\mathbf{E}}\mathbf{J}^e - \mathbf{E}\overline{\mathbf{J}}^e)dV \tag{10}$$

# Drift velocities in silicon