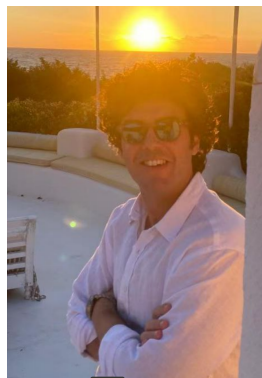


Consistency constraints on effective field theories coupled to gravity

Anna Tokareva
Hangzhou Institute for Advanced study

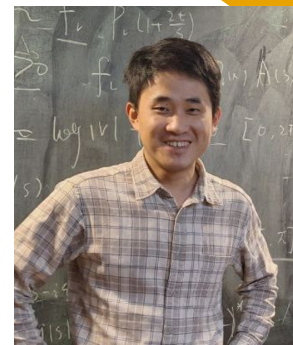
My collaborators



Giorgio di Russo,
postdoc at HIAS



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now postdoc in Pisa U



Jiaxin Cheng, 3rd year
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Co-author of 'WGC in
the sky' paper

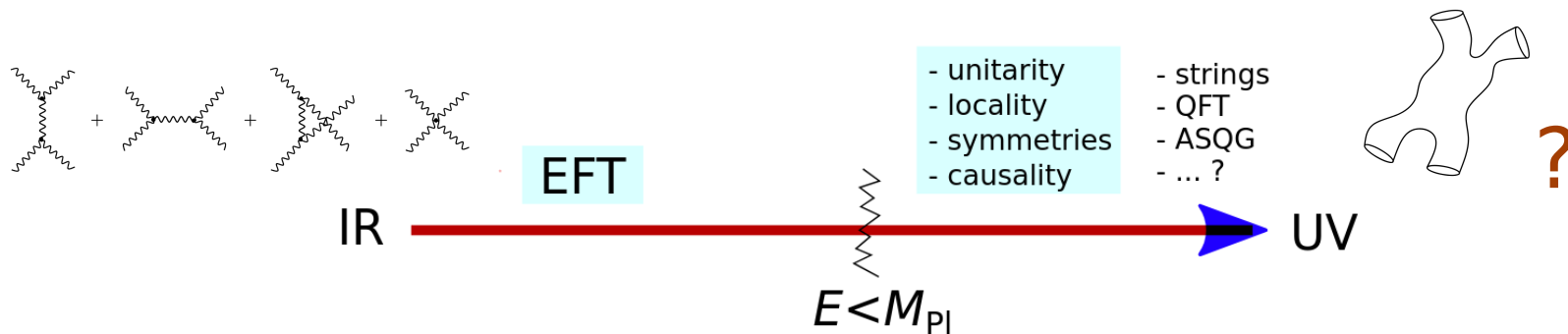


Yongjun Xu, final year
PhD student at HIAS



Laurentiu Rodina,
PI in BIMSA (Yau center)

EFT and quantum gravity



$$S = M_{\text{Pl}}^{D-2} \int d^D x \sqrt{-g} \left\{ \frac{R}{2} + \frac{c_{\text{GB}}}{\Lambda^2} \mathcal{G} + \frac{c_3}{\Lambda^4} R^3 + \frac{c_1}{\Lambda^6} \mathcal{C}^2 + \frac{c_2}{\Lambda^6} \tilde{\mathcal{C}}^2 + \dots \right\}$$

$$\mathcal{G} := R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma},$$

$$R^3 := R_{\alpha\beta\rho\sigma}R^{\rho\sigma}{}_{\mu\nu}R^{\mu\nu\alpha\beta}, \quad \mathcal{C} := R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}, \quad \tilde{\mathcal{C}} := \frac{1}{2}R^{\alpha\beta\mu\nu}\epsilon_{\mu\nu\rho\sigma}R^{\rho\sigma}{}_{\alpha\beta}$$

Lectures in Quantum Gravity, arXiv:2412.08690

I.Basile, L.Buoninfante, F.di Filippo, B.Knorr, A.Platania, A.Tokareva

UV-IR connections

Partial wave expansion: $\mathcal{A}(s, \theta) = 32\pi \sum_{l=0}^{\infty} \left(l + \frac{1}{2}\right) f_l(s) P_l(\cos \theta)$

QFT axioms in a language of the scattering amplitudes:

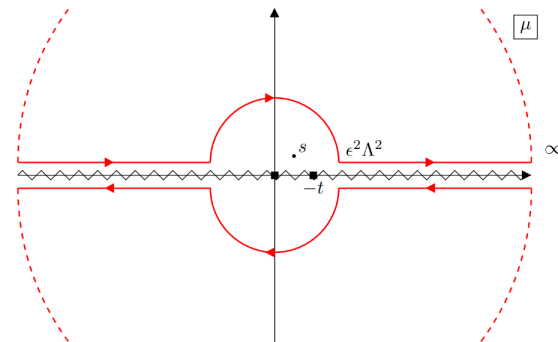
- Lorentz invariance $\Rightarrow \mathcal{A} = \mathcal{A}(s, t, u)$,
- Unitarity $\Rightarrow 2\text{Im} f_l(s) > |f_l(s)|^2 \sqrt{\frac{s-4m^2}{s}} > 0$, $|f_l(s)| < 1$,
- Causality $\Rightarrow \mathcal{A}(s, t)$ is an analytic function outside the real axes,
- Locality $\Rightarrow \mathcal{A}(s, t) < s^N t^N$ for finite N .

Amplitude in the scalar EFT:

$$\mathcal{A}(s, t) = g_2(s^2 + t^2 + u^2) + g_3stu + g_4(s^2 + t^2 + u^2)^2 + g_5(s^2 + t^2 + u^2)stu + \dots$$

$$C(n, m) \equiv \frac{n!}{2\pi i} \int_{\text{arcs}} d\mu \frac{\partial_t^m \mathcal{A}(\mu, 0)}{\mu^{n+1}} = \frac{\partial_t^m}{\partial t^n} \frac{\partial^n}{\partial s^n} \mathcal{A}(s, t)$$

$$I_{s,u}(n, m) \equiv \int_{\epsilon^2 \Lambda^2}^{\infty} \frac{d\mu}{\pi} \frac{\partial_t^m \text{Disc}_s \mathcal{A}_{s,u}(\mu, 0)}{\mu^{n+1}} > 0$$



Not all EFTs are allowed!

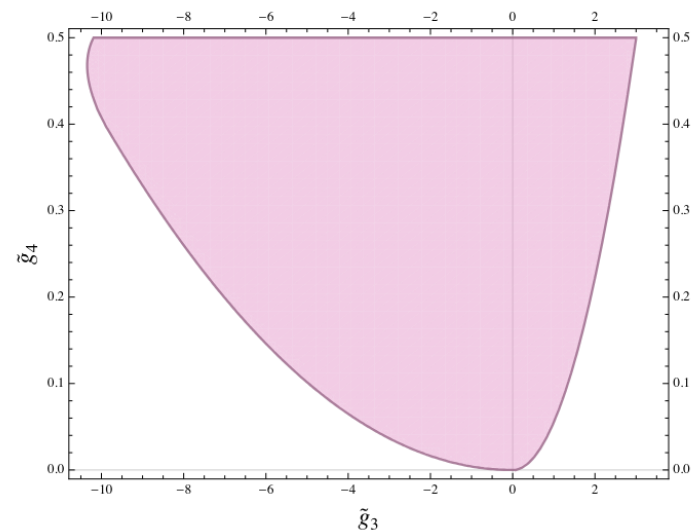
$$L = -\frac{1}{2} (\partial_\mu \varphi)^2 + \boxed{\frac{g_2}{2} (\partial_\mu \varphi)^4} + \frac{g_3}{3} (\partial_\mu \varphi)^2 (\partial_\rho \partial_\sigma \varphi)^2 + \\ + 4g_4 (\partial_\rho \partial_\sigma \varphi \partial_\rho \partial_\sigma \varphi)^2 + \dots$$



$$A(s, t) = g_2 (s^2 + t^2 + u^2) + g_3 stu + g_4 (s^2 + t^2 + u^2)^2 + \dots$$

Positivity bound: $g_2 > 0$ (Arkani-Hamed et al, 2008)

Galileon symmetry: $\varphi \rightarrow \varphi + a_\mu x^\mu \Rightarrow g_2 = 0$



(Caron-Huot, Van Duong 2021)

Large hierarchy between EFT Wilson coefficients is typically forbidden

Not all theories are allowed!

A few recent results:

- compact bounds on ratios of EFT coefficients with null constraints (Simon Caron-Huot, Vincent Van Duong 2011.02957, Andrew J. Tolley, Zi-Yue Wang, Shuang-Yong Zhou 2011.02400)
- Analytical expressions for the shapes of the allowed parameter space of simple EFTs from moment problem (Li-Yuan Chiang, Yu-tin Huang, Wei Li, Laurentiu Rodina, He-Chen Weng 2105.02862)
- species bound: the number of light states is constrained! (Simon Caron-Huot, Yue-Zhou Li 2408.06440)
- gravity cannot have a UV completion with the finite number of states (S. Caron-Huot, Yue-Zhou Li 2408.06440 & Kelian Häring, Alexander Zhiboedov 2410.21499)

- ...

Finding the constraints

$$\frac{1}{2\pi i} \oint_{arc} \frac{A(\mu, t)}{(\mu - s)^{n+1}} = \int_r^\infty \frac{d\mu}{\pi} \frac{\text{Disc}_s A(\mu, t)}{(\mu - s)^{n+1}} + (-1)^n \int_{r-t}^\infty \frac{d\mu}{\pi} \frac{\text{Disc}_u A(\mu, t)}{(\mu - u)^{n+1}} + \frac{1}{2\pi i} \oint_\infty \frac{A(\mu, t)}{(\mu - s)^{n+1}}$$

To compute in EFT:

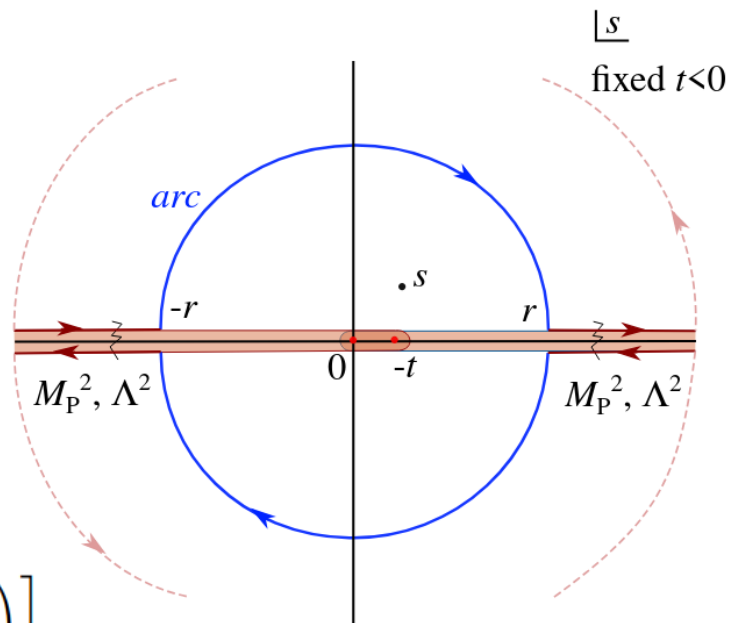
$$M_n(t) \equiv \frac{1}{2\pi i} \oint_{arc} \frac{A(\mu, t)}{\mu^{n+1}} + (-1)^n \int_{\epsilon^2}^{r-t} \frac{d\mu}{\pi} \frac{\text{Disc}_u A(\mu, t)}{(\mu - u)^{n+1}}.$$

We relate them to sums and integrals of positive-definite functions

$$\text{Disc}_s A(\mu, t) = 16\pi \sum_{j=\text{even}}^\infty (2j+1) \text{Im} f_j(\mu) P_j \left(1 + \frac{2t}{\mu} \right)$$

$$[X(\mu, j)] \equiv \sum_{j \text{ even}} 16(2j+1) \int_r^\infty d\mu \text{Im} f_j(\mu) X(\mu, j).$$

$$M_2(t) = \left[\left(\frac{1}{\mu^3} + \frac{1}{(\mu+t)^3} \right) P_j \left(1 + \frac{2t}{\mu} \right) \right]$$



Finding the constraints

$$A_{\text{EFT}}(s, t) = g_2(s^2 + t^2 + u^2) + g_3stu + g_4(s^2 + t^2 + u^2)^2 \\ + g_5(stu)(s^2 + t^2 + u^2) + g_6(s^2 + t^2 + u^2)^3 + \dots$$

$$g_2 = \left[\frac{1}{\mu^3} \right], \quad g_3 = \left[\frac{3}{\mu^4} \right] - \left[\frac{2L^2}{\mu^4} \right] \quad g_4 = \left[\frac{1}{2\mu^5} \right]$$

g_4 term is unique, so we have a relation implied by the crossing symmetry:

$$6M_4(t) |_{t=0} = \partial_t^2 M_2(t) |_{t=0} \quad \Rightarrow \quad \left[\frac{L^2(L^2 - 8)}{\mu^5} \right] = 0 \quad \text{- null constraint}$$

It is crucial for obtaining two-sided bounds on g_3 !

Finding the constraints

$$a_{k,q} = \left[\frac{L^{2q}}{\mu^{k+1}} \right] \text{ - moments of positive-definite functions}$$

Their values are lying in a compact space (also called a-geometry in 2012.15849)
The shape is carved by the following set of inequalities:

- ‘relaxing’ inequalities $a_{2,0} - a_{3,0} \geq 0, \quad a_{3,0} - a_{4,0} \geq 0, \quad a_{3,1} - a_{4,1} \geq 0, \dots$

- Grams inequalities $f_1(\mu, L^2) = 1/\mu^{3/2}, f_2(\mu, L^2) = 1/\mu^{5/2}, f_3(\mu, L^2) = L^2/\mu^{5/2}$

$$\begin{vmatrix} \int f_1^2 & \int f_1 f_2 & \int f_1 f_3 \\ \int f_2 f_1 & \int f_2^2 & \int f_2 f_3 \\ \int f_3 f_1 & \int f_3 f_2 & \int f_3^2 \end{vmatrix} \geq 0 \Rightarrow \begin{vmatrix} a_{2,0} & a_{3,0} & a_{3,1} \\ a_{3,0} & a_{4,0} & a_{4,1} \\ a_{3,1} & a_{4,1} & a_{4,2} \end{vmatrix} \geq 0, \quad \begin{vmatrix} a_{2,0} & a_{3,0} \\ a_{3,0} & a_{4,0} \end{vmatrix} \geq 0, \quad \begin{vmatrix} a_{2,0} & a_{3,1} \\ a_{3,1} & a_{4,2} \end{vmatrix} \geq 0, \quad \begin{vmatrix} a_{4,0} & a_{4,1} \\ a_{4,1} & a_{4,2} \end{vmatrix} \geq 0.$$

- polytope boundaries $(L^2 - 6)(L^2 - 20) \geq 0 \quad L^2 \in \{0, 6, 20, 42, \dots\}$

$$\left[\frac{(L^2 - 6)(L^2 - 20)}{\mu^5} \right] \geq 0, \quad a_{4,2} - 26a_{4,1} + 120a_{4,0} \geq 0.$$

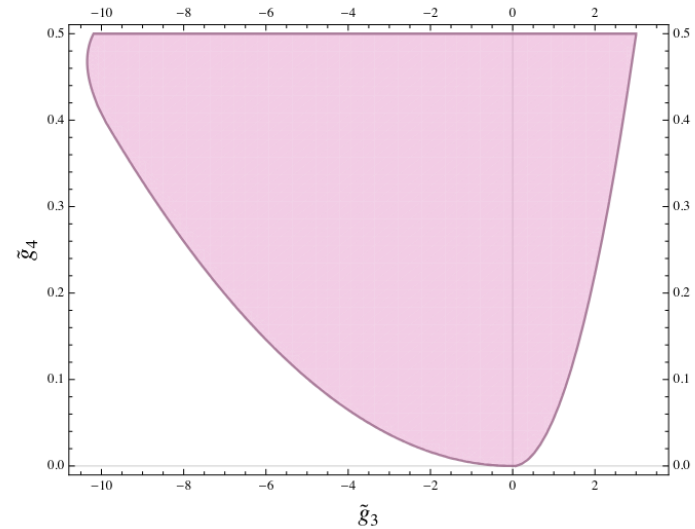
Finding the constraints

Solution to the moments problem:

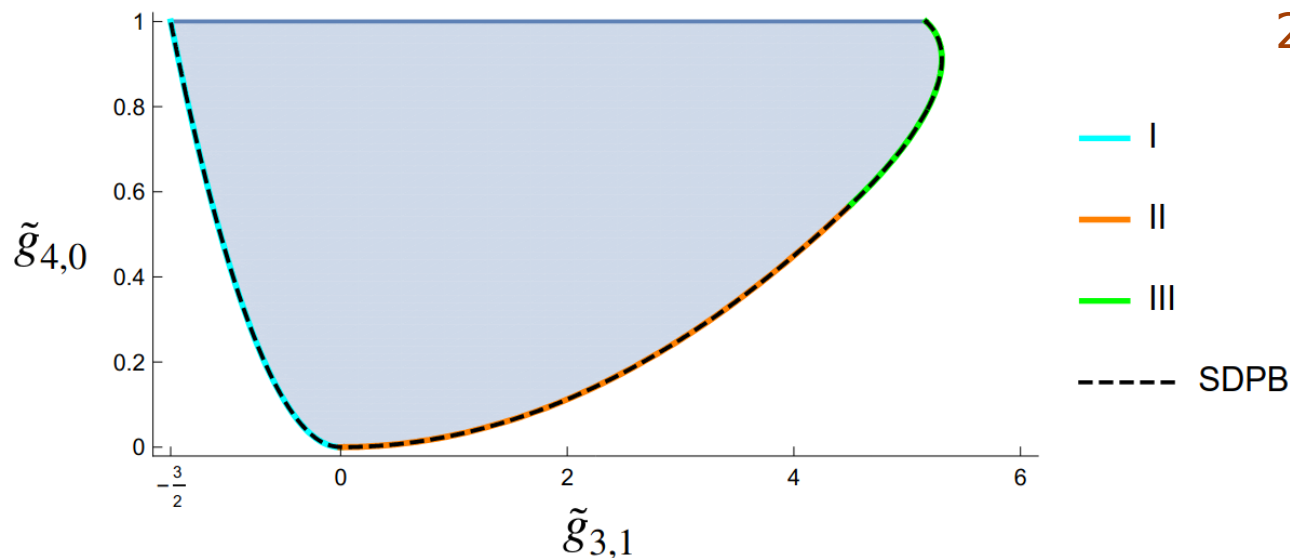
$$0 < \tilde{a}_{3,1} < \sqrt{\frac{160}{3}} \tilde{a}_{4,0}, \quad 0 < \tilde{a}_{4,0} \leq \frac{5}{6},$$
$$\tilde{a}_{3,1} < \sqrt{\frac{80}{9}(\tilde{a}_{4,0} - \tilde{a}_{4,0}^2)} + \frac{20}{3} \tilde{a}_{4,0}, \quad \frac{5}{6} < \tilde{a}_{4,0} \leq 1.$$

Most optimal bound requires $n=5$ null constraint and all listed inequalities for moments $[1/m^6]$

- Next step – relation to g_n
- requires projecting out a -s which are not related to couplings



Final result



$$\text{Region I: } \tilde{g}_{31} = -\frac{3}{2}\sqrt{\tilde{g}_{40}},$$

$$0 \leq \tilde{g}_{40} \leq 1$$

$$\text{Region II: } \tilde{g}_{31} = \frac{1}{2}\sqrt{\frac{427}{3}}\tilde{g}_{40},$$

$$0 \leq \tilde{g}_{40} \leq \frac{243}{427}$$

$$\text{Region III: } \tilde{g}_{31} = \frac{30}{7}\tilde{g}_{40} + \frac{37}{42}\sqrt{\tilde{g}_{40}(21 - 20\tilde{g}_{40})},$$

$$\frac{243}{427} \leq \tilde{g}_{40} \leq 1$$

Crossing-symmetric dispersion relations

$$x = -(st + tu + us), \quad y = -stu, \quad a \equiv \frac{y}{x}, \quad \mathcal{M}^{\text{tree}}(x, a) = 8\pi G \frac{x}{a} + \sum_{m,n} c_{mn} x^m a^n \quad (n \leq m)$$

$$\mathcal{M}(x, a) = \alpha_0(a) + x \int_0^\infty \frac{ds'}{\pi} \frac{\text{Im } \mathcal{M}(s', \tau(s', a))}{(s')^3} \frac{(2s' - 3a)(s')^2}{x(a - s') - (s')^3}, \quad \tau(s', a) = -\frac{s'}{2} \left(1 - \sqrt{\frac{s' + 3a}{s' - a}} \right)$$

Sinha, Zahed (2012.04877)

Advantage: tree-level amplitude lead to truncated left hand side,
all null constraints are included

$$k = 2 : \quad -\frac{8\pi G}{a} + 2g_2 - a g_3 = \left\langle \frac{2s' - 3a}{(s')^3} P_J^{(D)} \left(\sqrt{\frac{s' + 3a}{s' - a}} \right) \right\rangle,$$

$$k = 4 : \quad 4g_4 - 2a g_5 + a^2 g_6 = \left\langle \frac{(s' - a)(2s' - 3a)}{(s')^6} P_J^{(D)} \left(\sqrt{\frac{s' + 3a}{s' - a}} \right) \right\rangle,$$

Crossing-symmetric dispersion relations

Chebyshev sampling in a : $a_i = -\frac{1}{6} + \frac{1}{6} \cos\left(\frac{(2i-1)\pi}{2N_a}\right), \quad i = 1, \dots, N_a,$

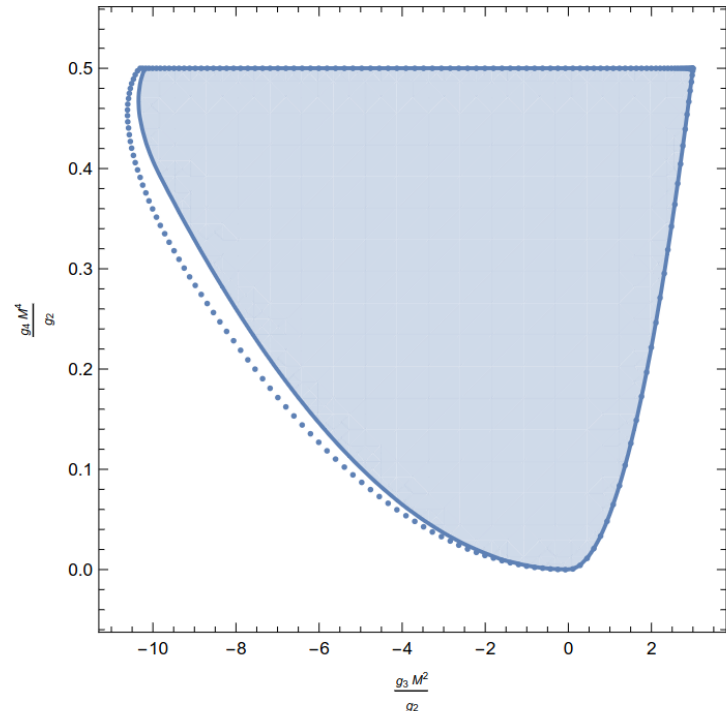
Discretised s-integral

$$\sum_{J,z} n_{J,d} \frac{1}{\pi} \frac{1}{N_\mu} \left(\frac{z}{N_\mu}\right)^{1-\frac{d}{2}} \rho(z, J) B(2, a, N_\mu/z, J, 4) = 2g_2 - g_3 a.$$

Bounds are given by linear program

$$0 \leq \rho\left(\frac{i}{N_\mu}, J\right) \leq 2. \quad - \text{ optimization variables}$$

Direct access to the spectrum of UV theory!



The whole calculation (Mathematica)

```
(* Partial-wave normalization *)
n[l_, d_] := ((4 Pi)^(d/2) (d + 2 l - 3)
  Gamma[d + l - 3])/(Pi Gamma[(d - 2)/2]
  Gamma[l + 1]);

(* Gegenbauer polynomial *)
P[l_, d_, x_] := Hypergeometric2F1[
  -l, l + d - 3, (d - 2)/2, (1 - x)/2];

(* k=2 dispersive kernel *)
B[a_, mu_, l_, d_] := (2 mu - 3 a)/mu^3
  P[l, d, Sqrt[mu + 3 a]/Sqrt[mu - a]];

(* Discretization parameters *)
NN = 20; nn = 300; jj = 40;

(* Chebyshev sampling in a *)
a[i_] := N[-1/6 + (1/6) Cos[
  ((2 i - 1) Pi)/(2 NN)], 30];

(* Variables *)
sc-camel-var-r = Flatten@Table[rho[z, l],
  {z, 1, nn}, {l, 0, jj, 2}];
vars = Join[{g2, g3, k2}, varR];

(* k=2 sum rule *)
eq2[ai_, d_] :=
  Sum[rho[z, l] B[ai, nn/z, l, d] n[l, d]
    (1/Pi) (1/nn) (z/nn)^(d/2 - 3),
    {l, 0, jj, 2}, {z, 1, nn}]
  + k2/ai - 2 g2 + g3 ai;

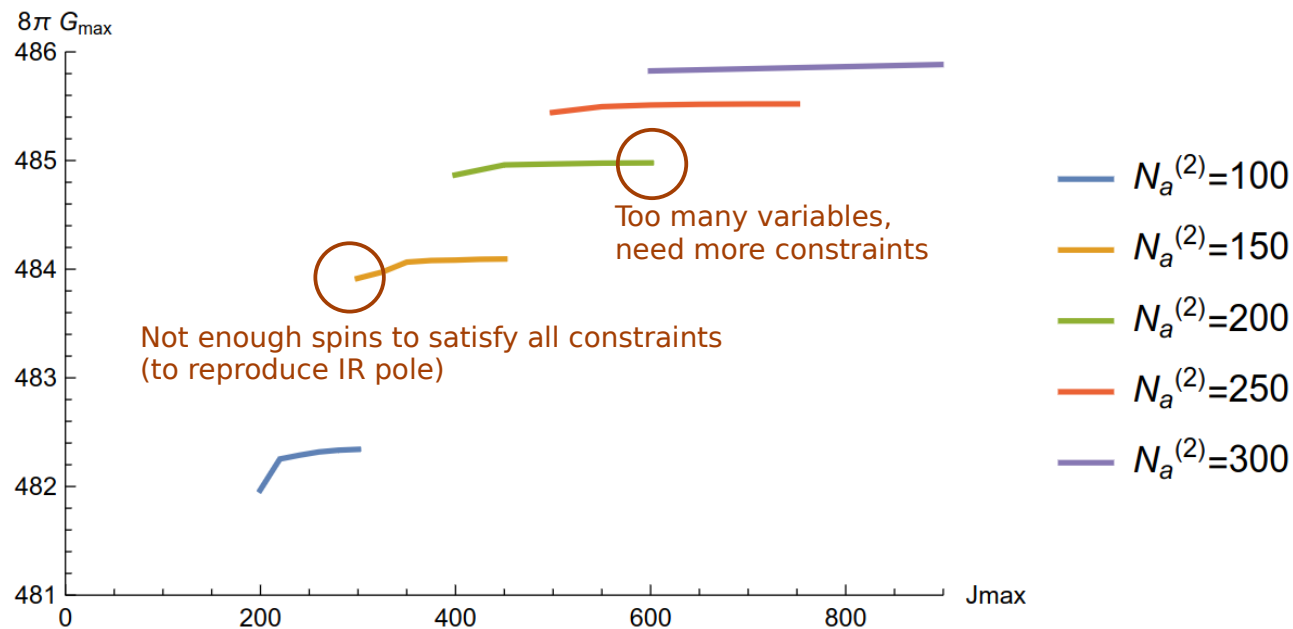
(* Sum rules, positivity, scale *)
cons[d_] := Join[
  (eq2[#, d] == 0) & /@ Table[a[i], {i, 1, NN}],
  Thread[varR >= 0], {k2 == 1}];

(* Solve the LP *)
g2 /. LinearOptimization[g2, cons[6],
  vars, Method -> "CLP",
  Tolerance -> 10^-12]
```



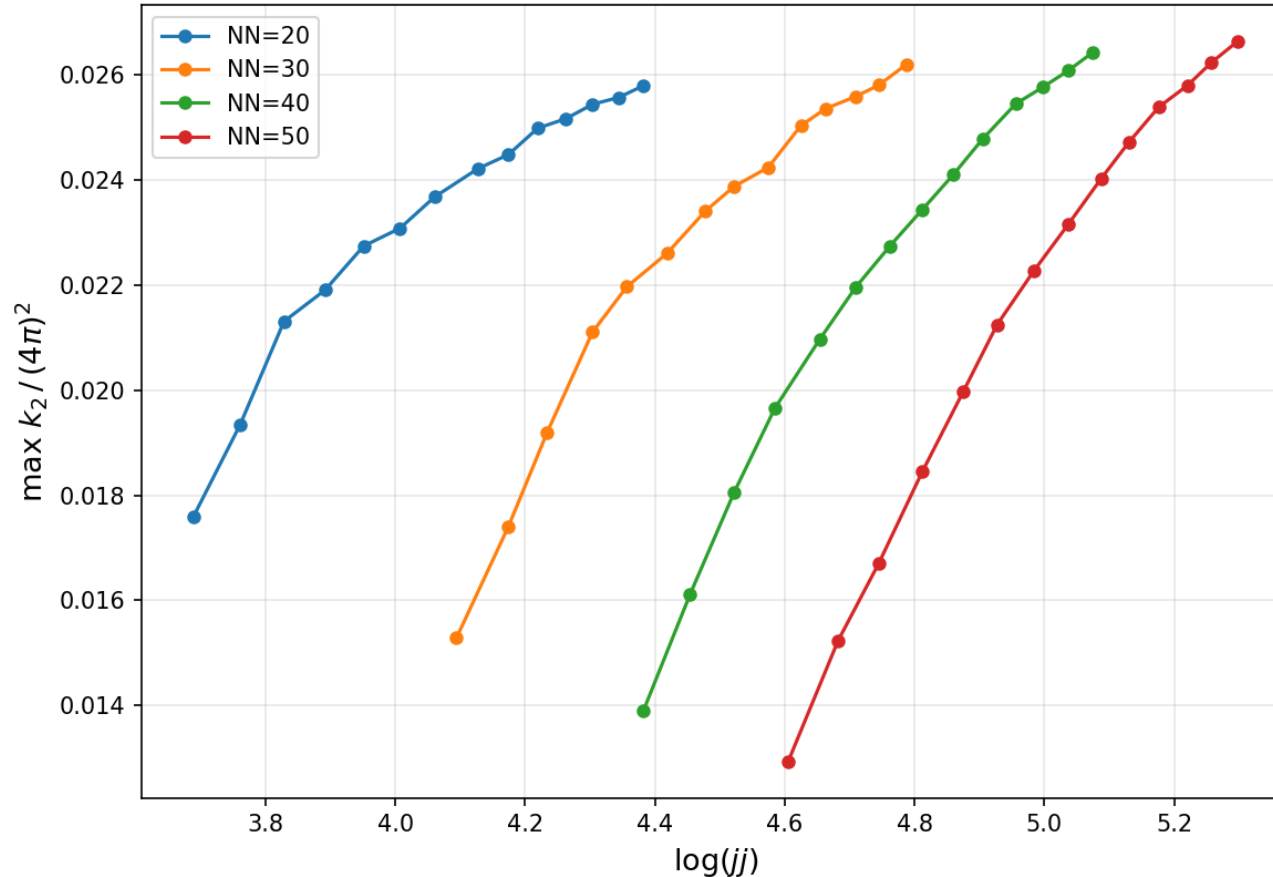
Idea belongs to
Yongjun Xu

Numerical stability



$$2N_a^{(2)} \lesssim J_{\max} \lesssim 3N_a^{(2)}$$

Numerical stability in 4d



Some bounds depend on spin cutoff...

- a reflection of the problem that S-matrix is not a proper observable because of soft emission

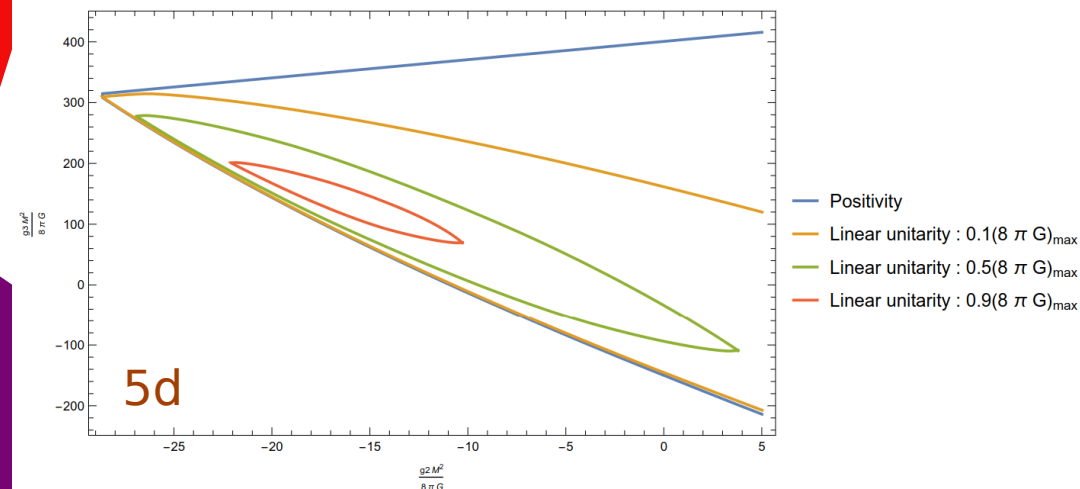
Results in higher dimensions

2604.15235

Guangzhuo Peng, Laurentiu Rodina, Anna Tokareva, Yongjun Xu

Sampled dispersion relations,
linear program with $0 < \rho < 2$
(existing bounds are reproduced)

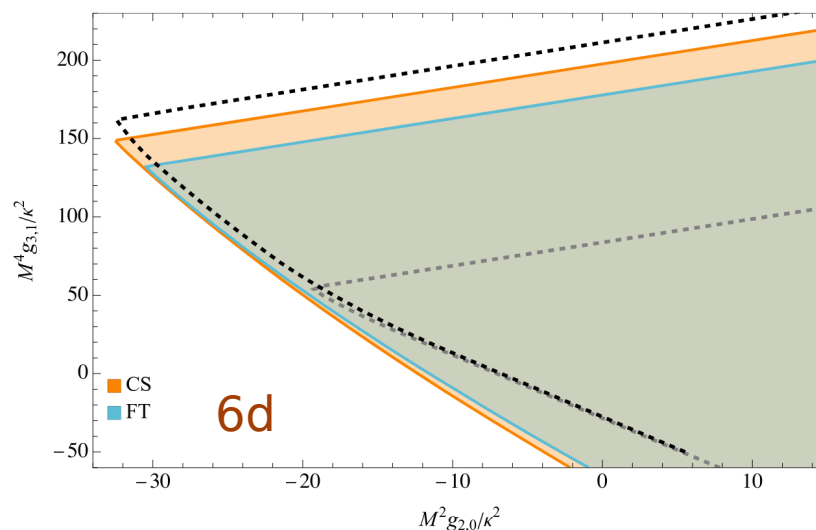
Non-projective constraints



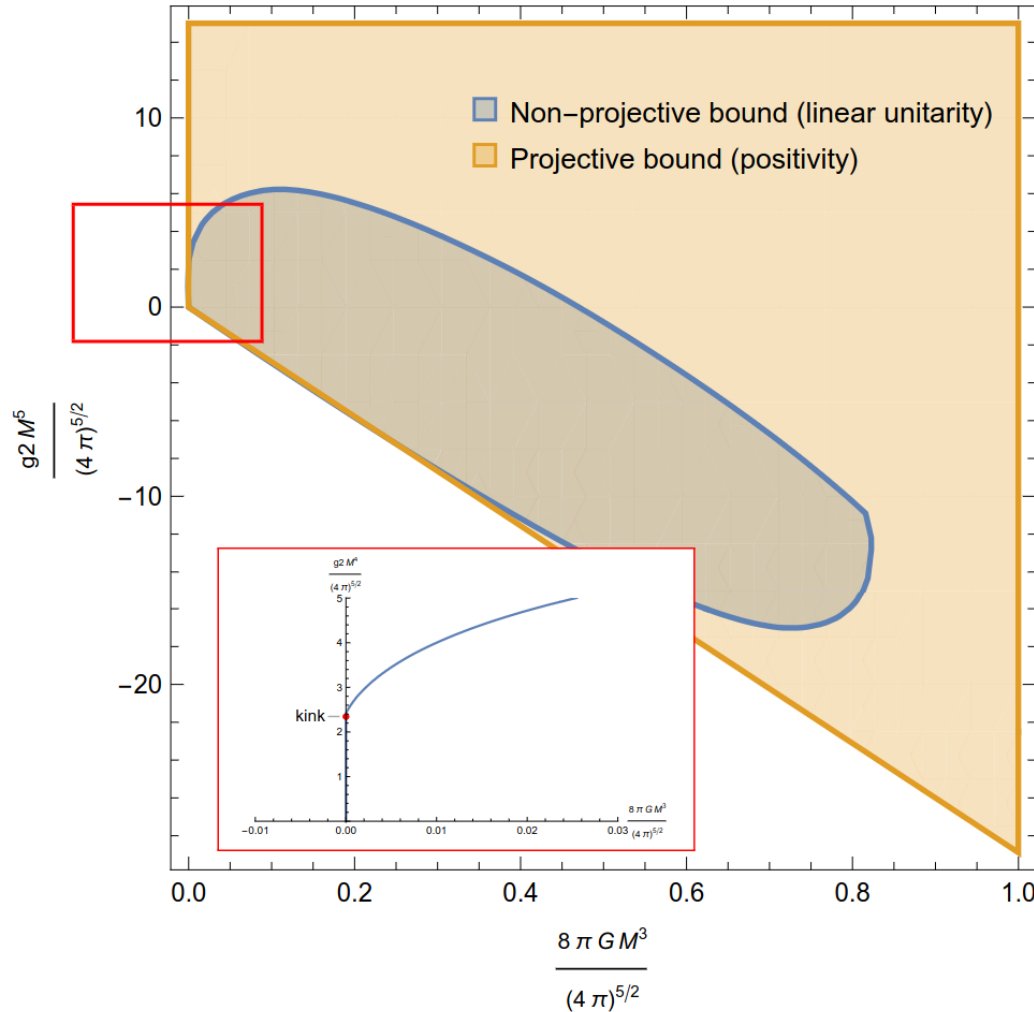
2501.18465

Carl Beadle, Giulia Isabella, Davide Perrone,
Sara Ricossa, Francesco Riva, Francesco Serra
2501.17949
Cyuan-Han Chang, Julio Parra-Martinez

Smeared dispersion relations,
positivity only



Bound on Planck mass/(EFT cutoff)



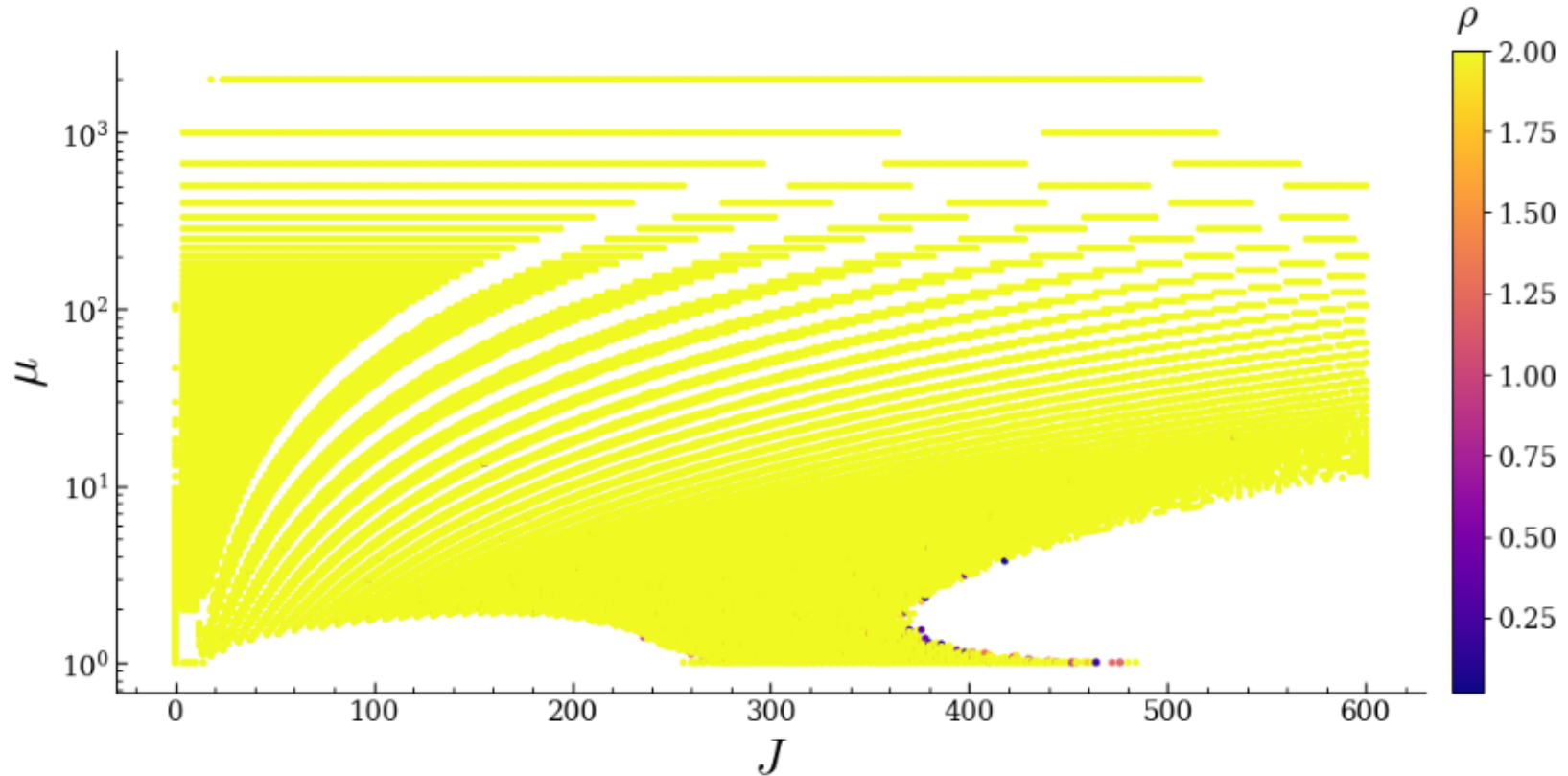
5d

Assumption:
tree-level EFT amplitude
is correct up to cutoff scale

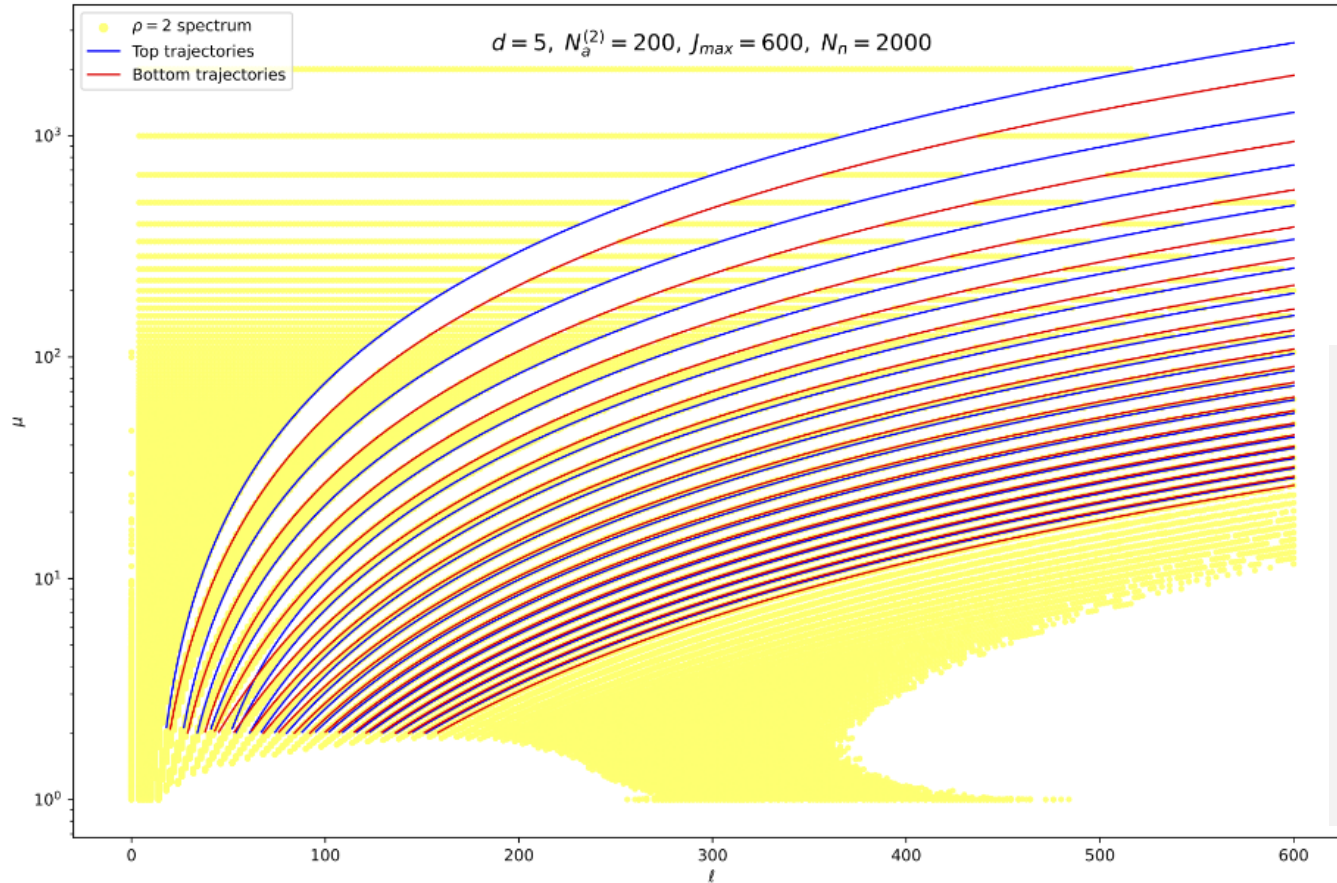
If IR loops are important
the bound on g_2 is changing!

Extremal spectrum

Spectrum maximizing $8\pi G$ ($D=5$)



Extremal spectrum



Quadratically-shaped bands -
not linear Regge trajectories

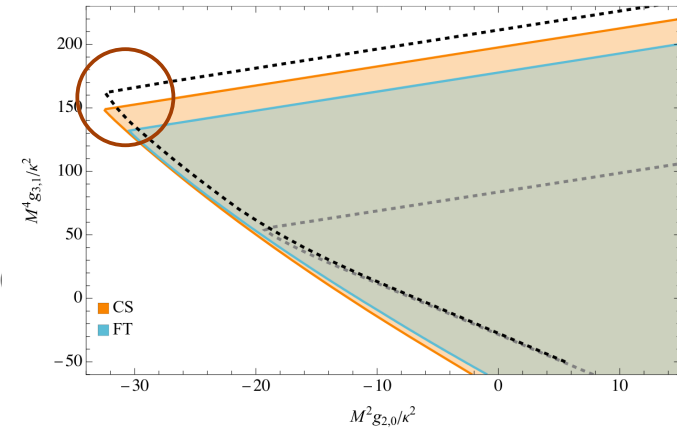
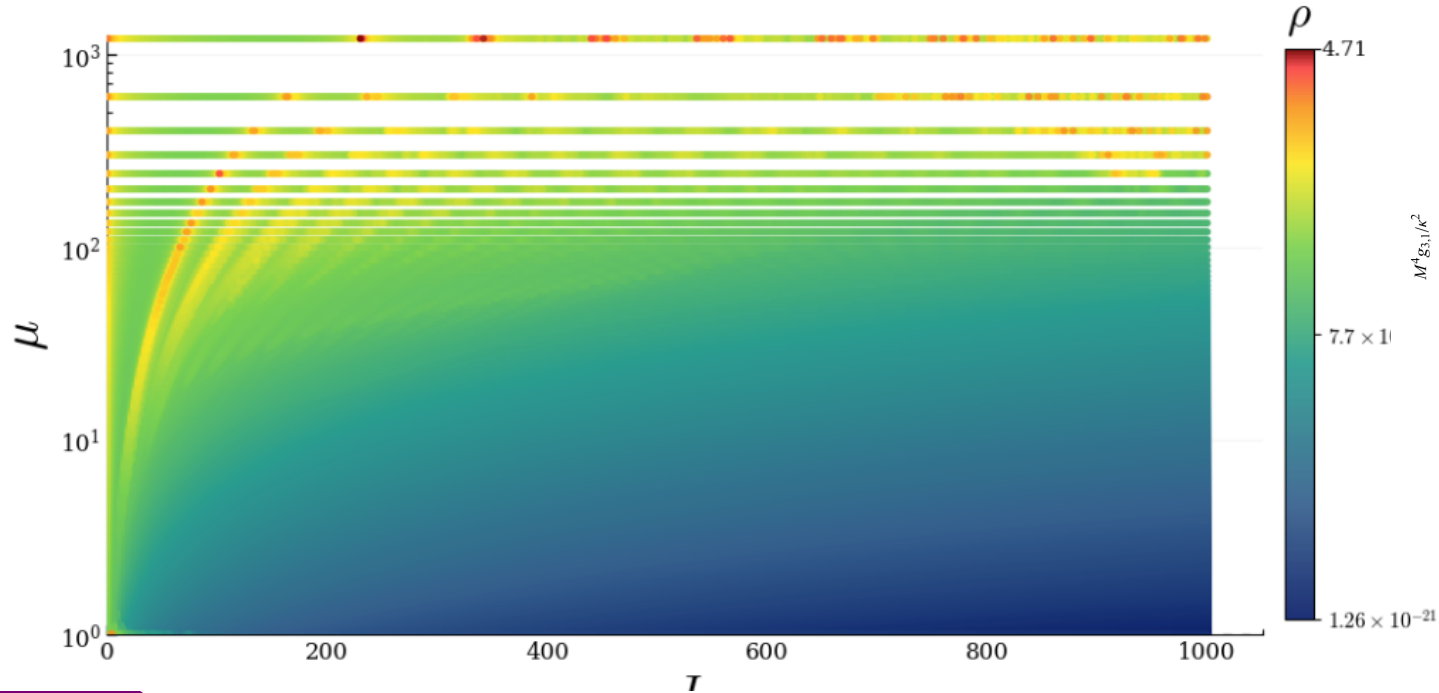
$$\mu_n \simeq b_{0,n} + b_{1,n}J + b_{2,n}J^2$$

$$b_{2,n} = \frac{0.0364}{(1.243 + n)^2} \quad (\text{top})$$

$$b_{2,n} = \frac{0.0318}{(1.481 + n)^2} \quad (\text{bottom})$$

Extremal spectrum

Spectrum minimizing $g_2/8\pi G$ ($D=6$)



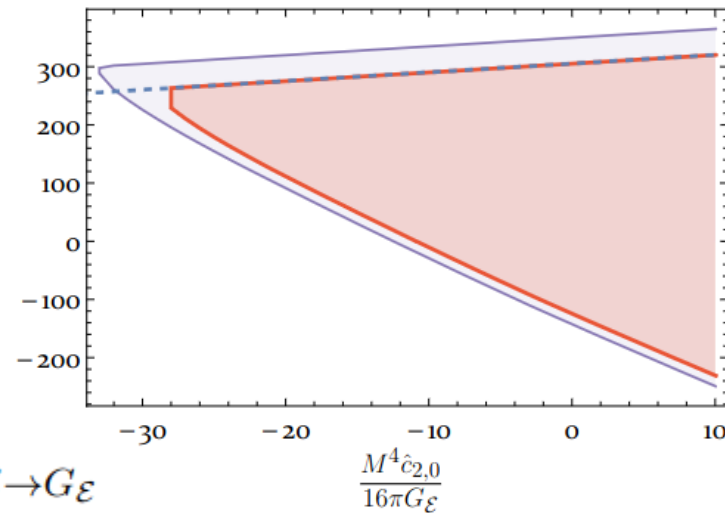
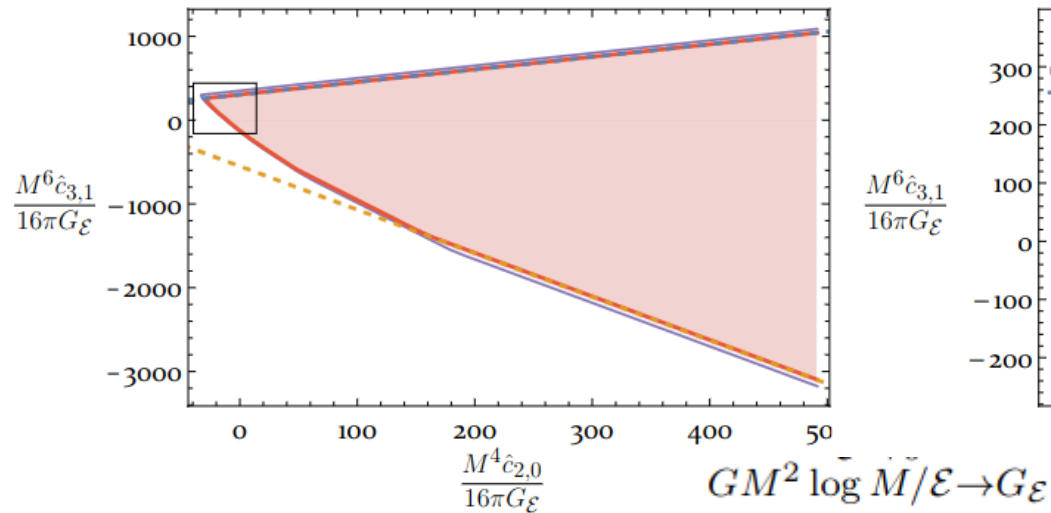
Approach to 4d gravity

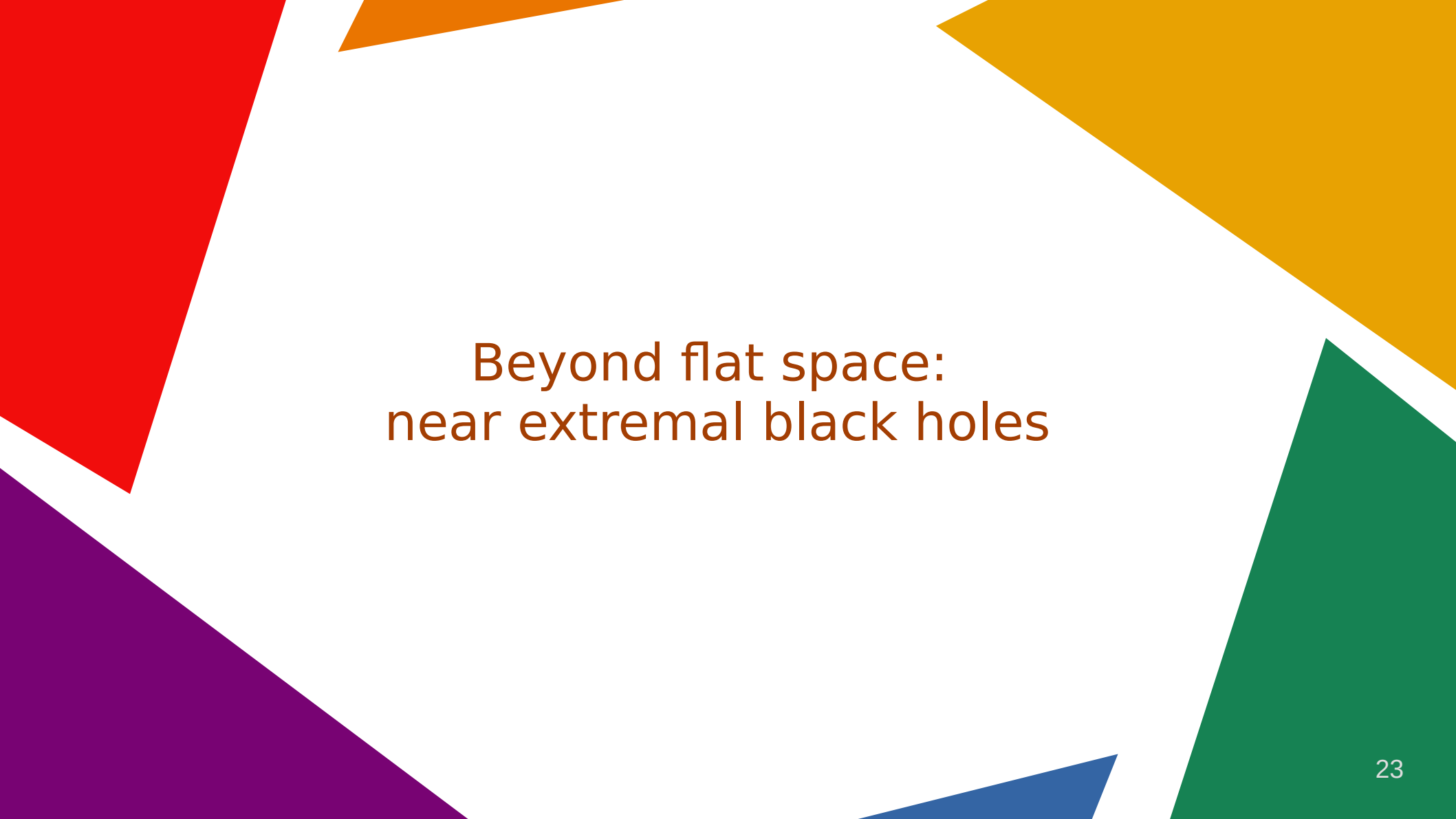
$$\mathcal{M}^{\text{tree,GR}} = 8\pi G \left(\frac{tu}{s} + \frac{us}{t} + \frac{st}{u} \right) + \hat{c}_{2,0} \left(\frac{s^2 + t^2 + u^2}{2} \right) + \hat{c}_{3,1} stu + \hat{c}_{4,0} \left(\frac{s^2 + t^2 + u^2}{2} \right)^2 + \dots$$

$$\mathcal{W}_{\text{GR}}(s, t) = \exp \left[-G \frac{(\mathcal{E}/\mu)^{2\epsilon}}{\pi\epsilon} \left(s \log(-s/\mu^2) + t \log(-t/\mu^2) + u \log(-u/\mu^2) \right) \right]$$

$$\mathcal{M}_{\mathcal{E}}^{\epsilon} \equiv \frac{\mathcal{M}^{\epsilon}}{\mathcal{W}}$$

stripped
amplitude



The background of the slide is white, featuring several large, overlapping triangles in various colors: red, orange, yellow, green, purple, and blue. These triangles are positioned around the edges of the slide, creating a dynamic, geometric frame.

Beyond flat space:
near extremal black holes

Einstein-Maxwell EFT

$$S_4 = \int d^4x \sqrt{-g} \left[\frac{R}{2\kappa} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + g_4 (F_{\mu\nu} F^{\mu\nu})^2 + h_4 R^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} + f_4 (F_{\mu\nu} \tilde{F}^{\mu\nu})^2 \right]$$

Reissner-Nordström black hole

$$ds^2 = -G(r)dt^2 + \frac{dr^2}{G(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

$$G(r) = 1 - \frac{2M}{r} + \frac{\kappa Q^2}{2r^2} + \boxed{\sum_{n=f_4, g_4, h_4} n \mathcal{S}_n(r)},$$

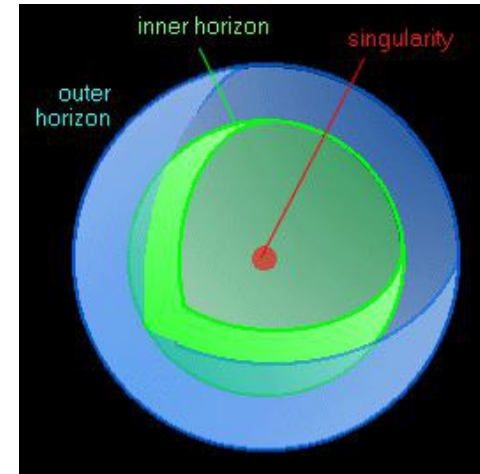
$$F_{\mu\nu} = \left(\frac{Q}{r^2} + \sum_{n=f_4, g_4, h_4} n \delta A_n(r) \right) dt \wedge dr.$$

$$\mathcal{S}_{g_4}(r) = -\frac{8Q^4}{5r^6}.$$

$$\mathcal{S}_{h_4}(r) = -\frac{64Q^4}{5r^6} + \frac{28MQ^2}{r^5} - \frac{16Q^2}{r^4}.$$

$$\mathcal{S}_{f_4}(r) = 0.$$

EFT corrections



Extremality condition: $M = Q - \frac{2g_4 + h_4}{5Q}$

Weak gravity conjecture

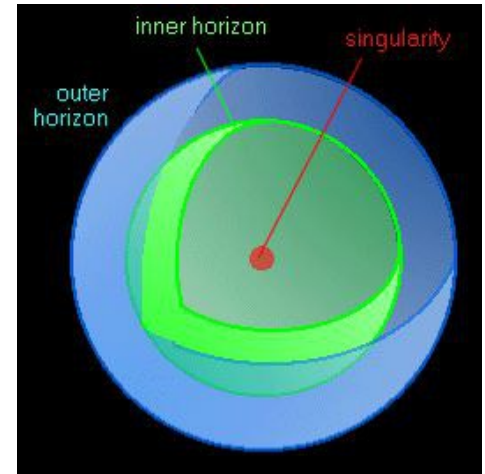
Arkani-Hamed, Motl, Nicolis, Vafa,
hep-th/0601001

All black holes must be able to evaporate

There must be states with $Q=M$ which are objects without naked singularities

Extremality condition: $M = Q - \frac{2g_4 + h_4}{5Q}$

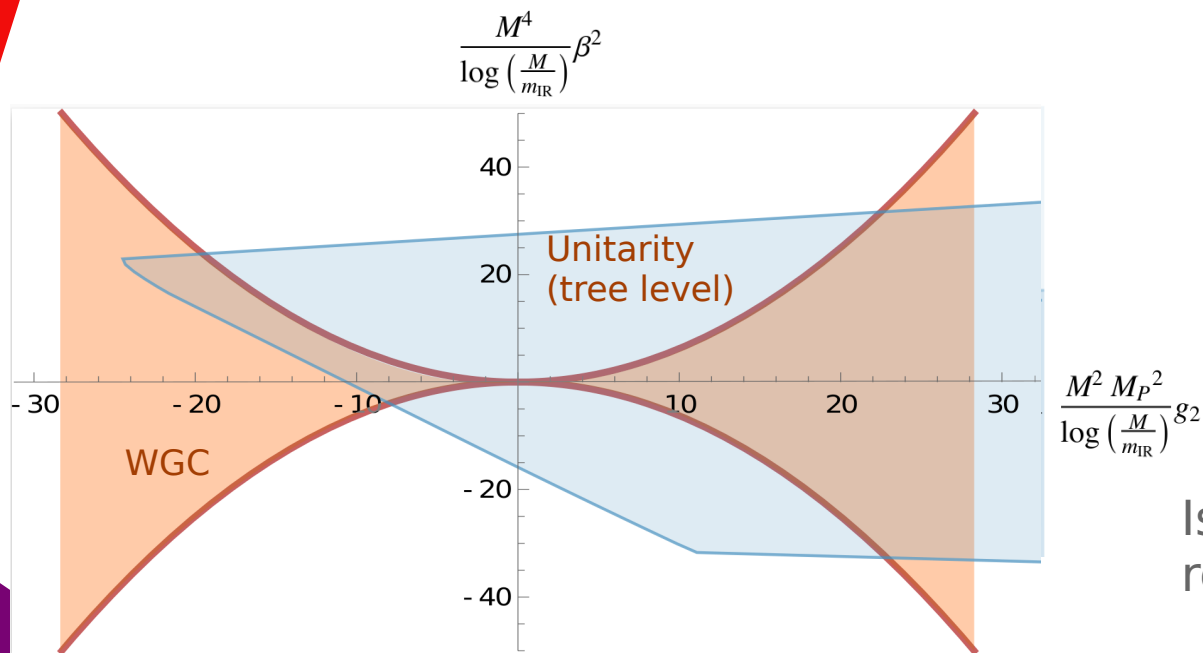
$$2g_4 + h_4 \geq 0$$



Can it be also obtained from dispersion relations?

Bounds on Einstein-Maxwell EFT

J. Henriksson, B. McPeak, F. Russo,
A. Vichi, 2203.08164



Constraints are
very different from WGC,

In general, they are much
weaker!

Is black hole evaporation
related to causality at all?

Quasinormal modes for scalar field

AT, G. di Russo, 2510.06813

Our result:

$$\omega_{ZDM} = -\frac{i(\ell + n + 1)}{Q^2} \sqrt{M^2 - Q^2} - \frac{4ig_4(1 + \ell + n)}{5Q^2 \sqrt{M^2 - Q^2}} - \frac{2ih_4(\ell + n + 1)}{5Q^2 \sqrt{M^2 - Q^2}}$$

QNM causality: (S. Melville, 2401.05524) $\delta(-\text{Im } \omega) > 0$

All corrections must make black holes more stable than in pure GR



$$\delta(-\text{Im } \omega) \approx h_4 + 2g_4 > 0$$

Exact match
with WGC!

WGC and photophilic reheating

J. Cheng, AT, 2512.10890

$$S_{int} = \int d^4x \sqrt{-g} (\mathcal{L}_{\phi\phi} + \mathcal{L}_{\phi\gamma\gamma} + \mathcal{L}_{\gamma\gamma h}),$$

$$\mathcal{L}_{\phi\phi} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2,$$

$$\mathcal{L}_{\phi\gamma\gamma} = \frac{\alpha}{\Lambda_1} \phi F_{\mu\nu} F^{\mu\nu},$$

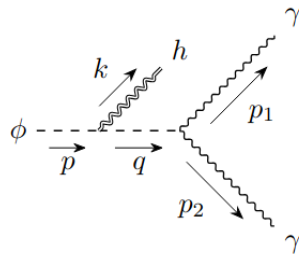
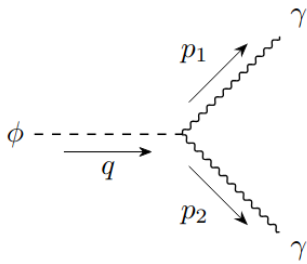
$$\mathcal{L}_{\gamma\gamma h} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{\beta}{\Lambda_2^2} R_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}.$$

What is a cutoff scale?

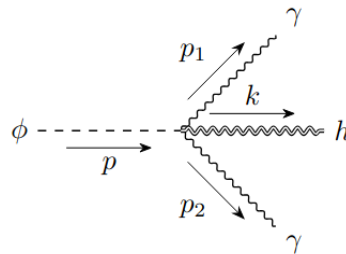
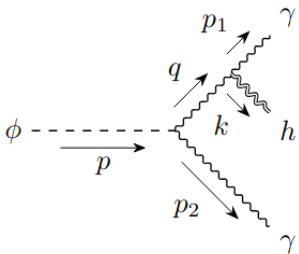
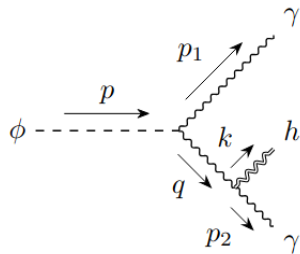
Unitarity bound: $\Lambda_2 \simeq \frac{\Lambda_{UV}^3}{M_P^2}$

WGC bound: $\Lambda_2 \simeq \frac{\Lambda_{UV}^2}{M_P}$

Reheating

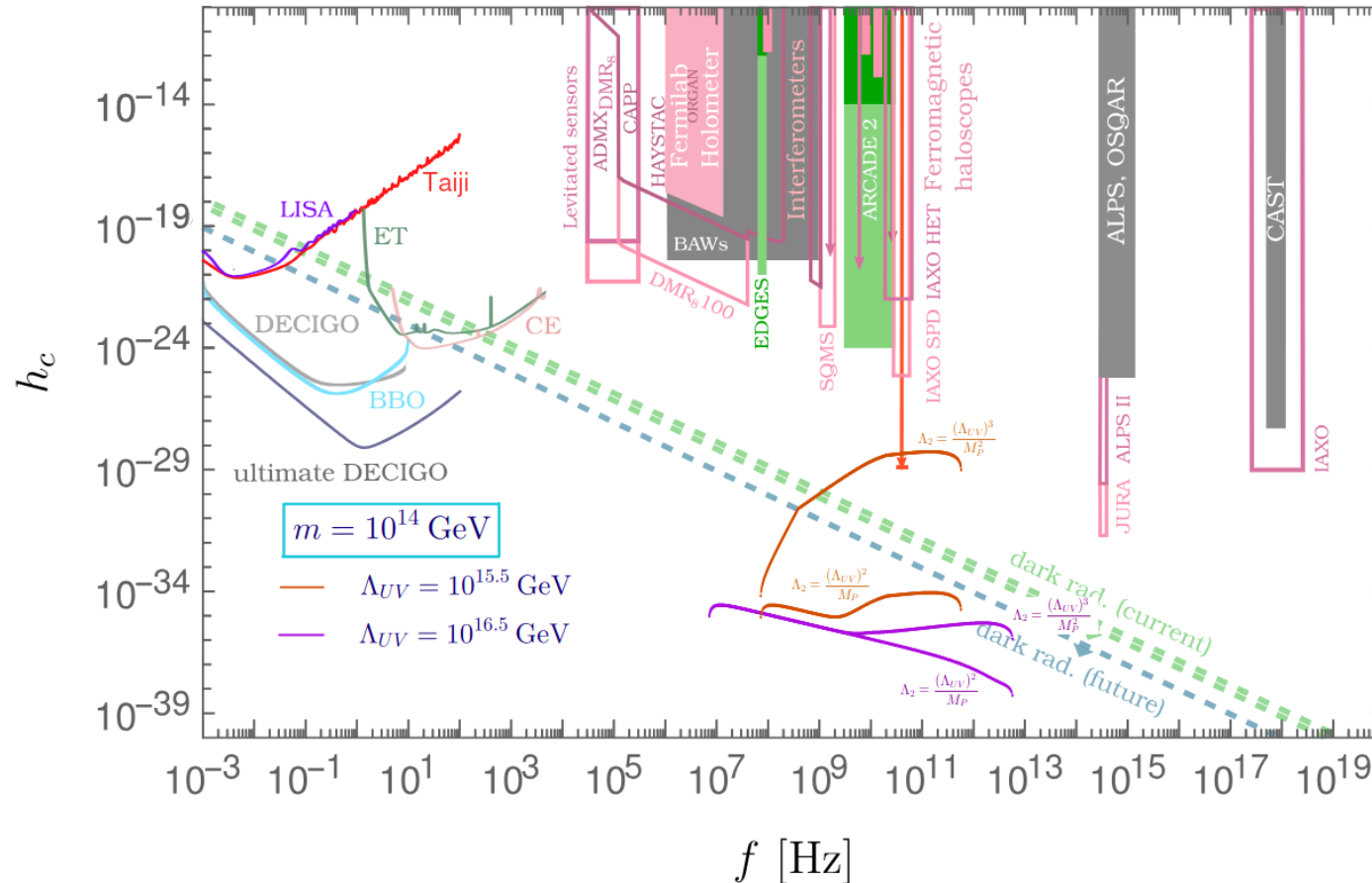


Graviton production



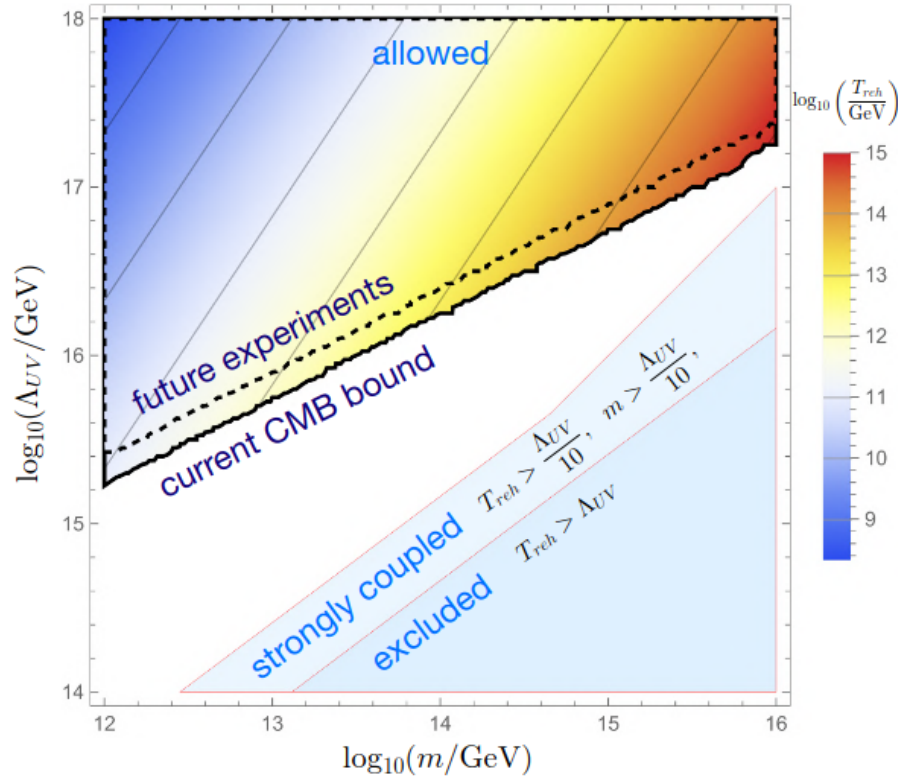
Gravitational waves and CMB constraints

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2512.10890

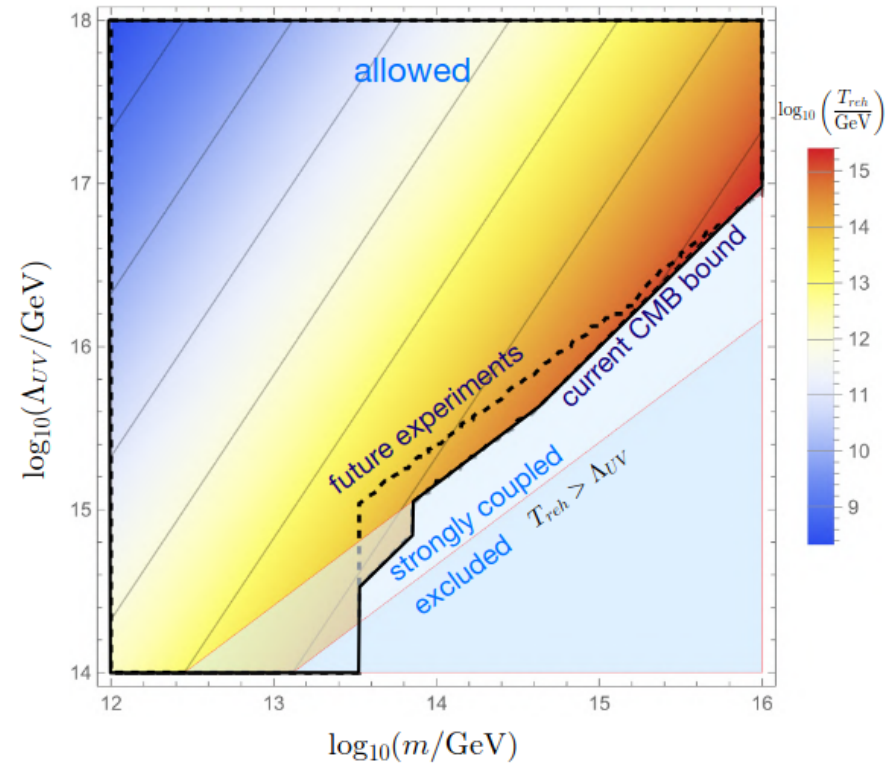


CMB constraints on cutoff scale

Unitarity bound



WGC scaling



Conclusions

- Not everything is allowed in low energy EFT
- The constraints are imposed by the assumptions that the ultimate theory of Nature is unitary, causal, Lorentz-invariant, and local
- After fixing the cutoff scale of EFT we typically obtain bounds which are compact
- The most convenient framework for gravitational EFT constraints – crossing-symmetric dispersion relation
- Linear optimization program allow to implement full (linearized) unitarity and extract the spectra of UV theories completing EFTs living on the boundaries
- In 4d gravity dispersion relations can be formulated for stripped amplitudes providing the bounds on individual couplings which depend on the IR cutoff, however, certain combinations of EFT couplings are stable
- Beyond flat space: black hole Weak Gravity Conjecture coincides with QNM causality statement

Open problems

- EFT constraints with gravity: complete optimization, 4D treatment
- Reproducing some results of numerical bootstrap analytically – application to more complicated EFTs with spinning particles: HEFT, SMEFT, ChPT...
 - * Proofs of conjectures (weak gravity conjecture, breaking of global symmetries by gravity)
 - * EFT constraints beyond flat space: cosmological dS, black holes, ...

Thank you!

