

Quantum spacetime,
from geometry to symmetries to observers

Fedele Lizzi

Università di Napoli Federico II
INFN Napoli

A review and some perspectives

Puzzles 2026



DIPARTIMENTO
DI **FISICA**
ETTORE PANCINI

UNIVERSITÀ DEGLI STUDI DI NAPOLI FEDERICO II





I dedicate this talk to Rafael Sorkin, who left us two days ago. Rafael was a great scientist, a wonderful human being and a personal friend. I wish to remember him with a picture taken at a social occasion, Rafael is the second from the left, next to his wife Fatma.

Why Geometry?

Geometry is tied closely to physics. Galilei wrote in the *Saggiatore* (1623).

Philosophy is written in this great book which is continually open before our eyes I mean the universe but it cannot be understood unless one first learns to understand the language and to recognize the characters in which it is written. It is written in the language of mathematics, and its characters are triangles, circles, and other geometrical figures, without which it is humanly impossible to understand a single word of it; without these, one is wandering in vain through a dark labyrinth.

La filosofia scritta in questo grandissimo libro che continuamente ci sta aperto innanzi a gli occhi (io dico l'universo), ma non si pu intendere se prima non s'impara a intender la lingua, e conoscer i caratteri, ne quali scritto. Egli scritto in lingua matematica, e i caratteri son triangoli, cerchi, ed altre figure geometriche [. . .] senza questi un aggirarsi vanamente per un oscuro laberinto.

Several physical theories do not just use geometry as a tool, they **are** geometry

Classical (analytic) mechanics is nothing but the symplectic geometry of phase space.

General relativity has taught us that gravity is the Riemannian geometry of curved spacetime.

But the geometric view of mechanics had a crisis with the advent of the **quantum world** with the uncertainty principle

All at once the notion of a point in phase space becomes untenable, and the geometry made of points need to be generalised to describe the quantum phase space

The algebra of commuting functions on phase space becomes the algebra of quantum **noncommuting** operators.

Why Quantum Spacetime?

The dynamical variable of general relativity is spacetime itself, encoded, in the metric $g^{\mu\nu}$, or in derived quantities: Riemann tensor, Christoffel symbols.

It is natural then to think of g as a field and quantize it using the usual procedures of quantum field theory. This does work the same way as it does for the other physical interactions. The resulting theory is not UV complete renormalizable. It works as effective theory, but in the ultraviolet an infinite number of counterterms are needed.

There are important conceptual differences between $g^{\mu\nu}(x)$ and a regular field. The latter is defined on background spacetime, with its topology, metric and all structures. Not to speak of diffeomorphism invariance!

The asymptotic freedom programme is based on the theory running towards an UV fixed point. While this does not imply automatically a phase transition, it suggests that the geometry of spacetime is substantially altered. An interesting discussion point.

The idea is then to quantize spacetime itself. There are more motivations to introduce some sort of “pointless” spaces. Without any ambition to be complete I may mention, in chronological order:

- 1936 Bronstein: concentrating enough energy to localize the gravitational field itself disturbs the geometry, creating what later will be called a black hole.
- 1947 Snyder, and later Yang, consider noncommuting coordinates, in the hope of solving the problems of infinities in quantum field theory.
- 1964 Mead uses a “gravitational” version of the Heisenberg microscope to argue that there should be a uncertainty in the measure of positions of the order of the Planck length squared.
- 1994 Doplicher, Fredenhagen and Roberts: to avoid gravitational collapse needs spacetime uncertainty relations and noncommuting coordinates.
- 1999 Seiberg and Witten find that in some limit of string theory, coordinates do not commute.

In general all of these paper do not cite the previous ones, except SW who cite Snyder, but not Yang.

From Commutative to Noncommutative Geometry

Hence we need of a generalization of geometry, along the lines of what happens for quantum mechanics. This was first advocated by Von Neumann, who called it **pointless geometry**.

The Noncommutative Geometry terminology stems from the programme impulsed by Connes to unify commutative and noncommutative geometry.

Without going into details the starting point is a series of theorems due mainly to Gelfand, Naimark and Moser which show that all topological informations of (Hausdorff) topological spaces can be inferred from the algebra of continuous functions, and that the reverse is also true, all associative commutative algebra of a certain kind (C^* -algebras) enable the reconstruction of a topological space.

All these C^* -algebras can be represented as operators on a Hilbert space.

As physicists we could say that the informations are encoded in the **fields**.

In Connes programme the whole geometry is reconstructed as properties of the algebra acting on a Hilbert space, and a generalization of the Dirac operator which gives the metric and differential structures. Integrals become traces and in general the usual geometric structures are described as properties of operators.

This algebraic structure is very solid and it survives the jump of having the algebra being noncommutative. We are led to a **Noncommutative geometry**. Since not relevant to this meeting, I skip the applications to the standard model.

This programme has been **very** successful already: Quantum Mechanics. In fact many of the techniques used in noncommutative geometry were originally introduced in this context.

Noncommutativity is an hallmark of quantizations, and it appears prominently, and many of the approaches to quantum gravity use, one way or another, a description based on operator, this is true of LQG, spin foams and GFT.

Operators “are infinite matrices”, this gives a solid background to matrix models of quantum gravity, BFSS and especially IKKT.

One way to obtain a noncommutative geometry is to consider the algebra generated by a $\star$ deformation of the usual product. The paradigmatic, and somewhat naive, example is the Grönewold-Moyal product θ , which can be written as

$$(f \star g)(x) = f(x) \exp \left[\frac{i}{2} \theta^{\mu\nu} \left(\overleftarrow{\partial}_\mu \overrightarrow{\partial}_\nu \right) \right] g(x),$$

so that

$$x^\mu \star x^\nu - x^\nu \star x^\mu = [x^\mu, x^\nu]_\star = i\theta^{\mu\nu}$$

The antisymmetric θ must be constant, otherwise the product is not associative. Non constant commutation relation require different products.

This is a spacetime version of the x and p commutation rule of quantum mechanics. For many it has become synonymous of noncommutative geometry.

Field theory with the Grönewold-Moyal product

The popularity of the $\star$ product is due to the fact Seiberg and Witten found that spacetime coordinates, seen as fields on the worldsheet, in a particular limit, have this kind of noncommutativity.

Field theories with this $\star$ product have been intensively studied, including some variations of the product. Let me mention some relevant aspects.

QED, i.e. the $U(1)$ gauge theory is noncommutative since $[A^\mu, A^\nu]_\star \neq 0$.

Ultraviolet/Infrared mixing. Some ultraviolet divergences (in nonplanar diagrams) are mitigated for high momenta in one direction p_μ . But they are substituted by infrared divergences in the dual direction $\theta^{\mu\nu} p_\nu$.

This can be seen as a manifestation of Heisenberg's principle, short distances localization (ultraviolet momentum) in one direction, imply delocalization (infrared) in the other.

Some work has been done towards actual physical predictions, looking at the lagrangian of the standard model and substituting the usual product with a $\star$ product. For example searching for decays forbidden in the usual standard model, but allowed for the presence of extra terms in the action.

Such noncommutativity might show up in quadrupole correlation in cosmic microwave background, or other big bang signatures.

Likewise it is also possible to write gravitational actions, like Einstein-Hilbert, with the deformed product, and study its effects.

These predictions depend on the scale of θ , which is expected to be Planckian, but also crucially by the fact that $\theta^{\mu\nu}$ selects two privileged background directions. Lorentz symmetry is broken, and I will return on this, but first let me ask a question:

In which sense this has to do with quantum gravity?

We are not doing a field theory of the gravitational spacetime. Gravity may be introduced as a background, but it is stretching the point to say that this is quantum gravity.

Here I do not want to spoil what I will say during the panel discussion we will have tomorrow, but there is an element of novelty in what I have presented so far:

I am renouncing to locality in a “controlled” way. The same way we gave up locality in phase space. We are substantially modifying the arena in which we play.

More, we are showing that we can play in different arenas!

Quantum Spacetime requires Quantum Symmetries

Hopf algebras/Quantum groups developed in parallel with NCG. Here just a few features.

The issue is **covariance**. Let the noncommutative spacetime be defined by

$$R^{\mu\nu}(x) \equiv [x^\mu, x^\nu] - C^{\mu\nu}(x) = 0$$

We are acting on composite systems, or on products. Take the Leibnitz rule defined by a vector field X . Its action on a product is captured by this **primitive coproduct**.

$$\Delta_0 X = X \otimes 1 + 1 \otimes X$$

which is just the ordinary Leibniz rule

$$X \triangleright (ab) = (X \triangleright a)b + a(X \triangleright b).$$

For noncommutative coordinate relations, in general $X \triangleright R^{\mu\nu}(x) \neq 0$

κ -Minkowski and deformed translation

Consider

$$[x^0, x^i] = \frac{i}{\kappa} x^i, \quad [x^i, x^j] = 0$$

The relativistic symmetry is the κ -Poincaré Hopf algebra. It actually predates the noncommutative space.

In the Majid–Ruegg bicrossproduct basis spatial translations already have a nonprimitive coproduct,

$$\Delta P_i = P_i \otimes \mathbb{1} + e^{-P_0/\kappa} \otimes P_i$$

With this coproduct the defining κ -Minkowski relation is covariant:

$$P_i \triangleright \left([x^0, x^j] - \frac{i}{\kappa} x^j \right) = 0$$

This at the level of Lie algebra. At the level of the group the deformation can be seen, in the spirit of what I said before, as making the functions on the group, which is manifold, noncommutative.

Usually a translation is a unitary operator $\exp i a^\mu P_\mu$. The a 's are coordinate functions on the group. And they commute.

For a quantum group they no longer are commuting quantities, for the case at hand they have the same commutation relations of the coordinates:

$$[a^0, a^i] = \frac{i}{\kappa} a^i, \quad [a^i, a^j] = 0$$

Let me now show how quantum groups may solve a puzzle in NCG. I chose this one because I worked on it with Manfredonia, Mercati and Poulain, and because it will lead me to the final theme of my talk.

The commutation relation $[x^0, x^i] = \frac{i}{\kappa} x^i$ leads naturally to an uncertainty:

$$\Delta x^0 \Delta x^i \geq \frac{1}{2\kappa} |\langle x^i \rangle|$$

This means that it is possible to sharply localise in space and time only states located at the origin. A sharply spatially localized event far from the origin must be increasingly delocalized in time.

What is so special about the origin?

The choice of the origin has to do with the choice of an observer/reference frame.* A different observer will have a different origin. A different observer will have a different origin.

*For the purpose of this talk the two are synonymous.

The Quantum Information Frontier

In order to understand quantum spacetime, we may need to quantize the observers as well.



Transformations among observers are now **quantum**.

The joint state describing the event and the transformation relating the two observers lives in a tensor-product Hilbert space.

Transformations among operators must be done with quantized transformations, with the noncommuting a 's.

A cannot sharply localize B. Nevertheless, she knows that B can sharply localize events close to his own origin, although A cannot assign those same events equally sharp coordinates..

As a metafore, imagine we are having a video call, and you know generically I am in Napoli. You cannot localise a pen on my table more than knowing it is in Napoli. But you have no problems in knowing that I can localise it precisely.

This is coherent with the principle of relative locality introduced by Amelino-Camelia, Kowalski-Glickman, Freidel and Smolin.

Quantum Transformations in ordinary Spacetime

Ordinarily a transformation is labelled by a point g of a group, and acts through a unitary operator $U(g)$.

$$\rho \longmapsto U(g)\rho U^\dagger(g).$$

But a quantum reference frame cannot in general have a sharply defined relative position or velocity. The transformation itself must therefore be described by a state on the group, $\rho_R(g)^*$.

$$\mathcal{T}_p(\rho) = \int_G dg \rho_R(g) U(g)\rho U^\dagger(g)$$

A sharp transformation corresponds to $\rho_R(g) = \delta(g - g_0)$. For a genuine distribution the transformation is **smeared**: a pure state may become mixed.

Composition of transformations corresponds to convolution of the distributions. Generic states therefore form a semigroup, not a group.

*Do not confuse the state ρ of the particle with the state ρ_R of the transformation.

A surprising consequence: a thermal reference frame

Consider two Galilean reference frames with uncertain relative velocity v . A particle with sharp momentum p in one frame is described in the other by

$$\rho' = \int dv \rho_R(v) |p + mv\rangle \langle p + mv|$$

Thus uncertainty in the motion of the reference frame produces mixing in the observed system.

If the reference frame of mass M is itself thermal at temperature T_R , $p_R(v)$ is Gaussian. A particle at rest becomes a thermal momentum distribution with

$$T' = \frac{m}{M} T_R.$$

In the classical-frame limit $M \rightarrow \infty$ (or $T_R \rightarrow 0$), $T' \rightarrow 0$ and the usual sharp transformation is recovered.

Quantum observers/reference frame have of appeared in the literature also from the information theory point of view, for example in the work of Brukner, Giacomini, Castro-Ruiz, Hohn. . .

There the emphasis is in another fundamental quantum aspect: **entanglement**.

Not only a state can be in an entangled state, but also the reference frame can be entangled. It becomes necessary therefore to define transformations between frames which are fully quantum objects.

Localisability becomes therefore a frame dependent concept, and this leads to a quantum spacetime.

This approach is therefore in some sense dual to the one we proposed here.

Conclusions (?)

The first and foremost puzzle is obviously having the “true” structure of spacetime at the Planck scale. I am advocating that it will be described by some theory of operators. It may not be usual operators on a Hilbert space. For example we may give up associativity.

I do not claim that the models which start “easily” from noncommuting variables are an ultimate answer, or that we know which those are. Nevertheless this activity is teaching us useful lessons.

Symmetries may not be in the way we think. Relation among observers may be different, the low energy limit may emerge in unexpected ways. Quantum information may play a central role.

Noncommutative geometry is the starting, not the conclusion.

References: Early Quantum Spacetime

- M. Bronstein, *Quantum theory of weak gravitational fields*, originally published in 1936; English translation and republication in Gen. Rel. Grav. **44** (2012) 267–283. [\[DOI\]](#).
- H.S. Snyder, *Quantized Space-Time*, Phys. Rev. **71** (1947) 38, [\[DOI\]](#); C.N. Yang, *On Quantized Space-Time*, Phys. Rev. **72** (1947) 874, [\[DOI\]](#).
- C.A. Mead, *Observable Consequences of Fundamental-Length Hypotheses*, Phys. Rev. **143** (1966) 990, [\[DOI\]](#).
- S. Doplicher, K. Fredenhagen, J.E. Roberts, *The quantum structure of space-time at the Planck scale and quantum fields*, Commun. Math. Phys. **172** (1995) 187, [hep-th/0303037](#).
- N. Seiberg, E. Witten, *String Theory and Noncommutative Geometry*, JHEP **09** (1999) 032, [hep-th/9908142](#).

References: NCG and Moyal Field Theory

- The standard reference for the spectral/algebraic approach is [A. Connes](#), *Noncommutative Geometry*, Academic Press (1994). An electronic version is available from Connes' webpage: [\[PDF\]](#).
- Two useful reviews of noncommutative field theory are [M.R. Douglas](#), [N.A. Nekrasov](#), *Noncommutative Field Theory*, Rev. Mod. Phys. **73** (2001) 977, [hep-th/0106048](#), and [R.J. Szabo](#), *Quantum Field Theory on Noncommutative Spaces*, Phys. Rep. **378** (2003) 207, [hep-th/0109162](#).
- For UV/IR mixing: [S. Minwalla](#), [M. Van Raamsdonk](#), [N. Seiberg](#), *Noncommutative Perturbative Dynamics*, JHEP **02** (2000) 020, [hep-th/9912072](#).
- On the connection between the commutation relations and UV/IR mixing: [S. Galluccio](#), [F. Lizzi](#), [P. Vitale](#), *Translation Invariance, Commutation Relations and Ultraviolet/Infrared Mixing*, JHEP **09** (2009) 054, [arXiv:0907.3640](#).
- For the matrix-model point of view: [T. Banks](#), [W. Fischler](#), [S.H. Shenker](#), [L. Susskind](#), *M Theory As A Matrix Model: A Conjecture*, [hep-th/9610043](#), and [N. Ishibashi](#), [H. Kawai](#), [Y. Kitazawa](#), [A. Tsuchiya](#), *A Large-N Reduced Model as Superstring*, [hep-th/9612115](#).

References: Quantum Groups and κ -Minkowski

- The basic bicrossproduct construction and the introduction of κ -Minkowski: S. Majid, H. Ruegg, *Bicrossproduct structure of κ -Poincaré group and non-commutative geometry*, Phys. Lett. B **334** (1994) 348, [hep-th/9405107](#).
- For an introduction to noncommutative spaces, quantum groups, twists and deformed symmetries: P. Aschieri, M. Dimitrijević, P. Kulish, F. Lizzi, J. Wess, *Noncommutative Spacetimes: Symmetries in Noncommutative Geometry and Field Theory*, Lecture Notes in Physics **774**, Springer (2009), [\[DOI\]](#).
- For localization and quantum reference frames in κ -Minkowski: F. Lizzi, M. Manfredonia, F. Mercati, T. Poulain, *Localization and Reference Frames in κ -Minkowski Spacetime*, Phys. Rev. D **99** (2019) 085003, [arXiv:1811.08409](#).
- For the principle of relative locality: G. Amelino-Camelia, L. Freidel, J. Kowalski-Glikman, L. Smolin, *The Principle of Relative Locality*, Phys. Rev. D **84** (2011) 084010, [arXiv:1101.0931](#).

References: Quantum Observers and Quantum Reference Frames

- A basic reference for modern quantum reference frames is [F. Giacomini](#), [E. Castro-Ruiz](#), [C. Brukner](#), *Quantum mechanics and the covariance of physical laws in quantum reference frames*, Nature Commun. **10** (2019) 494, [arXiv:1712.07207](#).
- For the extension to spacetime and relativistic clocks: [F. Giacomini](#), *Spacetime Quantum Reference Frames and superpositions of proper times*, Quantum **5** (2021) 508, [arXiv:2101.11628](#).
- For observer dependence of subsystems, correlations and thermodynamics: [P.A. Höhn](#), [I. Kotecha](#), [F.M. Mele](#), *Quantum Frame Relativity of Subsystems, Correlations and Thermodynamics*, [arXiv:2308.09131](#).
- For the mixed transformations and the thermal example discussed here: [G. Fiore](#), [F. Lizzi](#), *Mixed states for reference frames transformations*, [arXiv:2507.05758](#).

References: The Quantum Information Frontier

- For the idea that entanglement may be directly related to the emergence of spacetime: [M. Van Raamsdonk](#), *Building up spacetime with quantum entanglement*, Gen. Rel. Grav. **42** (2010) 2323, [arXiv:1005.3035](#).
- For the quantum-error-correction interpretation of holography: [F. Pastawski](#), [B. Yoshida](#), [D. Harlow](#), [J. Preskill](#), *Holographic quantum error-correcting codes: Toy models for the bulk/boundary correspondence*, JHEP **06** (2015) 149, [arXiv:1503.06237](#).
- For the interface between quantum information and loop quantum gravity: [E. Bianchi](#), [E.R. Livine](#), *Loop Quantum Gravity and Quantum Information*, [arXiv:2302.05922](#).

These are obviously only entry points into a rapidly developing literature.

The kist was compiled by ChatGPT. Here is our last exchange:

Fedele: Good! If anyone complains that I left out a reference to their work, I'll send them to Sam Altman.

ChatGPT: Perfect. You can say that the selection was made by an artificial intelligence, so moral responsibility is distributed nonlocally.

Sam Altman would probably pass the blame back to me, though, so a safer formulation might be:

"These are only entry points into the literature, not an attempt at completeness."

That way nobody can really complain. And if they still do, that already counts as an interesting experimental test of **observer dependence**.