Hamiltonian analysis of quadratic duality-symmetric action in the Sen formalism

partly based on 2506.07219 "Duality-symmetric D3-brane action with twisted self-dual 2-form doublet"

Pichet Vanichchapongjaroen

The Institute for Fundamental Study "The Tah Poe Academia Institute", Naresuan University, Phitsanulok 65000, Thailand

pichetv@nu.ac.th

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Disclaimer

To simplify the slides for students, I have

- omitted heavy citations, and
- skipped many topics.

For all the references and other part of the discussions, please see in the arXiv paper 2506.07219.

Outline

Motivation

Duality symmetry of Maxwell's equations

Maxwell's equations in free space

$$\begin{aligned} \vec{\nabla}\cdot\vec{E} &= 0, & \vec{\nabla}\times\vec{B}-\partial_t\vec{E} &= \vec{0}, \\ \vec{\nabla}\cdot\vec{B} &= 0, & \vec{\nabla}\times\vec{E}+\partial_t\vec{B} &= \vec{0}, \end{aligned} \tag{1}$$

is invariant under duality transformation $\vec{E}\mapsto\vec{B}$, $\vec{B}\mapsto-\vec{E}$, or more generally, under $SO(2)$ duality rotation

$$\begin{pmatrix}\vec{E} & \vec{B}\end{pmatrix}\mapsto\begin{pmatrix}\vec{E} & \vec{B}\end{pmatrix}\begin{pmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix}. \tag{2}$$

Maxwell's action

In terms of $(\vec{E}, \vec{B})$, the Maxwell's action reads

$$S[A_0, \vec{A}] = \frac{1}{2} \int d^4x \, (\vec{E}^2 - \vec{B}^2), \tag{3}$$

where

$$\vec{E} = \vec{\nabla} A_0 - \partial_t \vec{A}, \qquad \vec{B} = \vec{\nabla} \times \vec{A}. \tag{4}$$

The fields $\vec{E}$ and $\vec{B}$ satisfy Bianchi identity: two Maxwell's equations

$$\vec{\nabla} \cdot \vec{B} = 0, \quad \vec{\nabla} \times \vec{E} + \partial_t \vec{B} = \vec{0} \quad \text{(Bianchi)}. \tag{5}$$

Varying the action wrt $\vec{A}, A_0$ gives the other two Maxwell's equations.

$$\vec{\nabla} \cdot \vec{E} = 0, \quad \vec{\nabla} \times \vec{B} - \partial_t \vec{E} = \vec{0} \quad \text{(EOM)}, \tag{6}$$

Maxwell's action under duality rotation

Under the $SO(2)$ duality rotation, the action transforms as

$$S[A_0, \vec{A}] \mapsto \cos 2\theta S[A_0, \vec{A}] - \sin 2\theta \int d^4x \vec{E} \cdot \vec{B}. \qquad (7)$$

Symmetry only when $\cos 2\theta = 1$, i.e. trivial $\vec{E} \mapsto \pm\vec{E}, \vec{B} \mapsto \pm\vec{B}$.

Natural question: Can duality symmetry be made manifest at the action level?

Quick answer: Yes. But need to introduce extra pair of electric and magnetic fields.

We will first discuss EOM level (with two set of fields), then discuss how to construct action.

Introducing an extra pair of electric and magnetic field

- First consider EOM level.
- Define another pair of "electric" $\vec{D}$ and "magnetic" $\vec{H}$ fields by

$$\vec{D} = \frac{\delta S}{\delta \vec{E}} = \vec{E}, \qquad \vec{H} = -\frac{\delta S}{\delta \vec{B}} = \vec{B}. \tag{8}$$

  Let us call these relations as **constitutive relations**.
- Let us define C_0 and $\vec{C}$ via $\vec{H} = \vec{\nabla} C_0 - \partial_t \vec{C}, \quad \vec{D} = -\vec{\nabla} \times \vec{C}$.
- So $\vec{D}$ and $\vec{H}$ automatically satisfy Bianchi identity

$$\vec{\nabla} \cdot \vec{D} = 0, \quad \vec{\nabla} \times \vec{H} - \partial_t \vec{D} = \vec{0}, \tag{9}$$

  which is equivalent to EOM for $\vec{E}$ and $\vec{B}$.
- Recall Bianchi identity for $\vec{E}$ and $\vec{B}$ is

$$\vec{\nabla} \cdot \vec{B} = 0, \quad \vec{\nabla} \times \vec{E} + \partial_t \vec{B} = \vec{0}. \tag{10}$$

Duality symmetry

- Equations (8)-(10) (constitutive relations and Bianchi identities) are invariant under duality rotation

$$\begin{pmatrix}\vec{E} & \vec{H}\end{pmatrix} \mapsto \begin{pmatrix}\vec{E} & \vec{H}\end{pmatrix}\begin{pmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix}, \tag{11}$$

$$\begin{pmatrix}\vec{B} & \vec{D}\end{pmatrix} \mapsto \begin{pmatrix}\vec{B} & \vec{D}\end{pmatrix}\begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}. \tag{12}$$

Non-linear generalisation

- Generalisation to non-linear theory. Generalise S. Eq.(8) is modified. However, eq.(9)-(10) remain the same.
- Convenient to define

$$F_{i0} = E_i, \qquad F_{ij} = \epsilon_{ijk} B_k, \tag{13}$$

$$G_{i0} = H_i, \qquad G_{ij} = -\epsilon_{ijk} D_k, \tag{14}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \qquad G_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu. \tag{15}$$

- Note: for linear theory, $F_{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} G_{\rho\sigma}$.
- Actually, not all S admits duality symmetry. For this, Gaillard-Zumino (GZ) condition [Gaillard-Zumino '81]

$$F_{\mu\nu} F_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma} + G_{\mu\nu} G_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma} = 0 \tag{16}$$

must be satisfied.

Other generalisations

- Couple to curved metric.
- Couple to axion C_0 and dilaton ϕ. Duality symmetry group enhanced from $SO(2)$ to $SL(2, \mathbb{R})$.
- Couple to a pair of 2-form sources. In the context of D3-brane, this pair is (B_2, C_2) in SUGRA background.

In this talk, we are only discussing linear theory coupled to curved metric(s).

Looking for duality symmetric action

Strategy 1

Construct action of $A_\mu, \hat{A}_\mu$ with constitutive relations as EOM.

This is a standard strategy, made possible e.g. by the PST formalism. Possible to construct and extend. But we are not discussing this strategy here.

Looking for duality symmetric action (alt)

Strategy 2

Construct an action with $F_{\mu\nu}, G_{\mu\nu}$ such that they satsify constitutive relations. At EOM level,
$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, G_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu$.

- Made possible in the Sen formalism with key features:
  - * Actually, even $F_{\mu\nu}$ and $G_{\mu\nu}$ are NOT independent fields (i.e. composite)
  - Contains ghost d.o.f.
  - Two metrics. One is standard. The other couples to ghosts.
- We will follow this strategy.

Goal

- Paper goal: Construct the following actions in the Sen formalism
  - Duality-symmetric action with non-linear twisted self-duality condition (equivalently, constitutive relation).
  - Duality-symmetric D3-brane action coupled to background type IIB supergravity.
- Goal for this talk:
  - Describe the linear system in curved spacetime.
  - Describe how to apply constraint analysis on this system.

Linear twisted self-duality in the Sen formalism

The Sen formalism for twisted self-duality (II)

The physical system (on-shell consideration)

- There are two sectors, uncoupled from each other.
  - Physical sector: $\vec{A}_\mu = (A^I_\mu) = (A_\mu, \hat{A}_\mu)$ is coupled to physical metric $g_{\mu\nu}$.
  - Unphysical (ghost) sector: $(\bar{A}^I_\mu) = (\bar{A}_\mu, \hat{\bar{A}}_\mu)$ is coupled to unphysical metric $\bar{g}_{\mu\nu}$.
  - NB: In the extensions (see e.g. the paper 2506.07219), other external fields can be included as long as they remain in one sector.
- For linear theory, physical field strength doublet $\vec{f} = (f, \hat{f})$ satisfies twisted self-duality (TSD) condition

$$\vec{f} = *\vec{f}, \qquad * \text{ is Hodge star} \bullet \text{ along with } \epsilon^{IJ} \tag{17}$$

which is equivalent to $f_{\mu\nu} = \frac{1}{2}\frac{1}{\sqrt{-g}}\epsilon^{\mu\nu\rho\sigma}\hat{f}_{\rho\sigma}$.

The Sen formalism for twisted self-duality (III)

The action level

- Independent fields: 1-form doublet $\vec{P}$ and 2-form doublet $\vec{Q}$ such that $\vec{Q} = \bar{*}\vec{Q}$ where $\bar{*}$ (is (unphysical) Hodge star $\bar{\bullet}$ along with ϵ^{IJ}).
- Composite field $\vec{R}(\vec{Q}, g, \bar{g})$ satisfies $\vec{R} = -\bar{*}\vec{R}$.
- Composite field $\vec{H}(\vec{Q}, g, \bar{g}) = \vec{Q} - \vec{R}$ satisfies $\vec{H} = *\vec{H}$.

The quadratic action

The action is (NB: Don't worry about details.)

$$\begin{aligned} S_{\text{Sen}} &= -\int \Omega\left(\frac{1}{4} d\vec{P} \wedge \bar{*} d\vec{P} - \vec{Q} \wedge d\vec{P} + \frac{1}{2}\vec{Q} \wedge \vec{R}\right) \\ &= \int \left(\frac{1}{4} dP^I \wedge \bar{\bullet} dP^I + \epsilon_{IJ} Q^I \wedge dP^J - \frac{1}{2}\Omega_{IJ} Q^I \wedge R^J\right). \end{aligned} \tag{18}$$

Independent fields: 1-form doublet $\vec{P} = (P, \hat{P})$ and 2-form doublet $\vec{Q} = (Q, \hat{Q})$ subject to $\vec{Q} = \bar{*}\vec{Q}$, which can be solved to eliminate $\hat{Q}$. Action becomes (NB2: Don't worry about details.)

$$\begin{aligned} S_{\text{Sen}} = \frac{1}{4}\int \left(dP \wedge \bar{\bullet} dP + d\hat{P} \wedge \bar{\bullet} d\hat{P}\right) + \int Q \wedge (d\hat{P} + \bar{\bullet} dP) \\ - \int Q \wedge \bar{\bullet}(\bar{\bullet} + \bullet)^{-1}(-\bar{\bullet} + \bullet) Q. \end{aligned} \tag{19}$$

The $SO(2)$ duality

The action is invariant under $SO(2)$ transformation

$$\vec{P}' = \vec{P}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \qquad \vec{Q}' = \vec{Q}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{20}$$

The equations of motion

- EOM of $\vec{P}$ and $\vec{Q}$:

$$d\left(\frac{1}{2}\bar{*}d\vec{P}+\vec{Q}\right)=0, \qquad \vec{R}=\frac{1}{2}d\vec{P}-\frac{1}{2}\bar{*}d\vec{P}. \tag{21}$$

- The first equality of eq.(21) implies (if spacetime has no hole)

$$\frac{1}{2}\bar{*}d\vec{P}+\vec{Q}=\frac{1}{2}d\vec{\xi}, \tag{22}$$

where $\vec{\xi}$ is a 2-tuple 1-form which satisfies

$$d(\vec{\xi}+\vec{P})=\bar{*}d(\vec{\xi}+\vec{P}). \tag{23}$$

- So $\vec{\xi}+\vec{P}$ is unphysical field.
- EOM also implies that $\vec{\xi}-\vec{P}$ is physical field satisfying

$$d(\vec{\xi}-\vec{P})=*d(\vec{\xi}-\vec{P}). \tag{24}$$

Uncoupling at EOM level

Essentially, by eliminating $\vec{Q}$, the remaining fields are

- Physical field $\vec{\xi} - \vec{P}$ satisfies

$$d(\vec{\xi} - \vec{P}) = *d(\vec{\xi} - \vec{P}). \tag{25}$$

- Unphysical field $\vec{\xi} + \vec{P}$ satisfies

$$d(\vec{\xi} + \vec{P}) = \bar{*}d(\vec{\xi} + \vec{P}). \tag{26}$$

Actually, being physical or unphysical field is not clear at this point. It actually requires Hamiltonian analysis.

Hamiltonian analysis

The Lagrangian

- The Lagrangian is

$$\begin{aligned}\mathcal{L} =& \frac{1}{2}\sqrt{-\bar{g}}\partial_\mu P^I_\nu \partial_\rho P^I_\sigma \bar{g}^{\mu[\rho}\bar{g}^{\sigma]\nu} + \sqrt{-\bar{g}}Q^I_{\mu\nu}\partial_\rho P^I_\sigma \bar{g}^{\mu\rho}\bar{g}^{\nu\sigma} \\ & - \frac{1}{4}\sqrt{-\bar{g}}Q^I_{\mu\nu}R^I_{\rho\sigma}\bar{g}^{\mu\rho}\bar{g}^{\nu\sigma}.\end{aligned} \tag{27}$$

- The phase space variables are P^I_μ, Q^I_{ij} and their conjugate momenta. The spacetime metrics are external fields. One may use the twisted self-duality conditions (i.e. $\vec{Q} = \bar{*}\vec{Q}$, $\vec{R} = -\bar{*}\vec{R}$ and $\vec{H} = *\vec{H}$) to realise that $Q^I_{\mu\nu}, R^I_{\mu\nu}, H^I_{\mu\nu}$ depend on Q^I_{ij} and external fields.

Conjugate momenta and primary constraints

- Conjugate momenta for P^I_μ are

$$\pi^0_I \equiv \frac{\partial \mathcal{L}}{\partial \dot{P}^I_0} = 0, \tag{28}$$

$$\pi^i_I \equiv \frac{\partial \mathcal{L}}{\partial \dot{P}^I_i} = \sqrt{-\bar{g}}\partial_\rho P^I_\sigma \bar{g}^{0[\rho}\bar{g}^{\sigma]i} + \sqrt{-\bar{g}}Q^I_{\mu\nu}\bar{g}^{\mu 0}\bar{g}^{i\nu}. \tag{29}$$

- Conjugate momenta for Q^I_{ij} are

$$\Pi^{ij}_I \equiv \frac{\partial \mathcal{L}}{\partial \dot{Q}^I_{ij}} = 0. \tag{30}$$

- Therefore, the primary constraints are $\pi^0_I \approx 0$, and $\Pi^{ij}_I \approx 0$.

The total Hamiltonian

- The ADM decomposition is applied on the metrics $g, \bar{g}$.
- The total Hamiltonian is

$$\begin{aligned}\mathcal{H} =& \pi^i_I \partial_0 P^I_i - \mathcal{L} + \zeta^I \pi^0_I + \theta^I_{ij} \Pi^{ij}_I \\ =& -\frac{\bar{N}}{\sqrt{\bar{\gamma}}} \bar{\gamma}_{ij} \pi^i_I \pi^j_I + 2\pi^b_I \bar{N}^a \partial_{[a} P^I_{b]} - 2\frac{\bar{N}}{\sqrt{\bar{\gamma}}} \bar{\gamma}_{ij} \pi^{+i}_I \epsilon^{jab} \epsilon^{IJ} Q^J_{ab} \\ & - \frac{1}{2} \bar{N} \sqrt{\bar{\gamma}} \partial_a P^I_b \bar{\gamma}^{a[m} \bar{\gamma}^{n]b} \partial_m P^I_n \\ & - \frac{1}{2} \bar{N} \sqrt{\bar{\gamma}} Q^I_{ab} \bar{\gamma}^{a[m} \bar{\gamma}^{n]b} H^I_{mn} + \pi^i_I \partial_i P^I_0 + \zeta^I \pi^0_I + \theta^I_{ij} \Pi^{ij}_I,\end{aligned} \tag{31}$$

where

$$\pi^{\pm i}_I \equiv \pi^i_I \pm \frac{1}{2} \epsilon_{IJ} \epsilon^{iab} \partial_a P^J_b. \tag{32}$$

Secondary constraints

- Requiring $\dot{\pi}_I^0 \approx 0$, and $\dot{\Pi}_I^{ij} \approx 0$ gives secondary constraints

$$\partial_i \pi_I^i \approx 0, \qquad H_{jk}^I - \epsilon_{ijk} \Omega^{IJ} \pi_J^{+i} \approx 0. \tag{33}$$

- Requiring the time derivative of the secondary constraints to vanish on the constraint surface does not produce further constraints.

Constraints, gauge fixing, reducing phase space

- First-class constraints: $\pi_I^0 \approx 0, \partial_i \pi_I^i \approx 0$.
- Second-class constraints: $\Pi_I^{ij} \approx 0, H_{jk}^I - \epsilon_{ijk}\Omega^{IJ}\pi_J^{+i} \approx 0$.
- Impose gauge-fixing conditions $P_0^I \approx 0, \partial_i P_i^I \approx 0$ to eliminate first-class constraints.
- Note: $\partial_i P_i^I \approx 0$ is NOT covariant and does not contain any metric.
- Reclassify: $\pi_I^0 \approx 0, \partial_i \pi_I^i \approx 0$, $\Pi_I^{ij} \approx 0, H_{jk}^I - \epsilon_{ijk}\Omega^{IJ}\pi_J^{+i} \approx 0$, $P_0^I \approx 0, \partial_i P_i^I \approx 0$ are second-class.
- Use second-class constraints to eliminate $\pi_I^0, P_0^I, \Pi_I^{ij}, Q_{ij}^I$ and the longitudinal components of π_I^i and P_i^I (for each I).

The reduced phase space

- In reduced phase space, Hamiltonian is separated as

$$\mathcal{H} = \mathcal{H}_- + \mathcal{H}_+, \tag{34}$$

where

$$\mathcal{H}_- = -\frac{1}{2}\frac{\bar{N}}{\sqrt{\bar{\gamma}}}\bar{\gamma}_{ij}\pi_I^{-i}\pi_I^{-j} + \frac{1}{2}\epsilon_{IJ}\pi^{-Ii}\bar{N}^a\epsilon_{aij}\pi^{-Jj}, \tag{35}$$

$$\mathcal{H}_+ = \frac{1}{2}\frac{N}{\sqrt{\gamma}}\gamma_{ij}\pi_I^{+i}\pi_I^{+j} - \frac{1}{2}\epsilon_{IJ}\pi^{+Ii}N^a\epsilon_{aij}\pi^{+Jj} \tag{36}$$

where $\pi_I^i = \epsilon^{ijk}\partial_j\psi_{Ik}/2$.

- The sign of $\mathcal{H}_-$ is incorrect. So $\psi_{Ii} - \Omega_{IJ}P_i^J$ is unphysical. Note: it is coupled to $\bar{g}_{\mu\nu}$.
- The sign of $\mathcal{H}_+$ is correct. So $\psi_{Ii} + \Omega_{IJ}P_i^J$ is physical. Note: it is coupled to $g_{\mu\nu}$.

The moral of the story

The moral of the story

- Constructed action should
  - satisfy all the required symmetries, and
  - produced expected equations of motion.
- Lesson from the Sen formalism. Gauge fields are not independent fields. Field strengths are also not independent fields.
- Ghosts are not always bad as long as they are completely decoupled.
- Ghosts (due to incorrect sign of kinetic energy) are not discovered at EOM level but at Hamiltonian level.
- Decoupling of ghosts might be realised after fixing the gauge to eliminate first-class constraints and then go to reduced phase space (by solving all the resulting second-class constraints).

Thank you