STABILITY, HORNDESKI & BEYOND

Beyond Horndeski, Degeneracy and Disformal Transformations

Lecture 5 — from second-order equations to degenerate higher-order scalar–tensor theories

instabilities $\to$ characteristics $\to$ k-essence/Galileons $\to$ Horndeski $\to$ degenerate theories

Beyond-Horndeski interactions

Lecture 4 set $F_4 = F_5 = 0$ to reach Horndeski. Now we keep exactly those terms.

The retained interactions

Quartic and quintic antisymmetric terms

$$L_4^{\rm bH} = F_4(X,\phi)\,\varepsilon^{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma}{}_{\sigma}\,\phi_\mu\phi_\alpha\phi_{\nu\beta}\phi_{\rho\gamma}$$

$$L_5^{\rm bH} = F_5(X,\phi)\,\varepsilon^{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma\delta}\,\phi_\mu\phi_\alpha\phi_{\nu\beta}\phi_{\rho\gamma}\phi_{\sigma\delta}$$

with the shorthand

$$\phi_\mu \equiv \nabla_\mu\phi, \qquad \phi_{\mu\nu} \equiv \nabla_\mu\nabla_\nu\phi, \qquad X \equiv \phi_\mu\phi^\mu$$

What is special about them

- The scalar equation can carry third derivatives of the metric, and the metric equation third derivatives of the scalar.
- These higher time derivatives are not independent: the theory still propagates two tensor modes and one scalar mode.
- Health is no longer read off from the order of the equations — it follows from a degeneracy of the kinetic structure.

Higher-order equations, same degrees of freedom

A simultaneous quartic + quintic extension is consistent only on a degeneracy surface.

The mechanism

- Naively the extra time derivatives look like a new Ostrogradsky mode.
- A constraint removes the would-be extra canonical pair, so no ghost appears.
- F_4 and F_5 cannot both be arbitrary — they are tied by a degeneracy relation.

Quartic + quintic degeneracy

relating the apparently independent functions

$$X\, G_{5,X}\, F_4 = 3\, F_5 \left(G_4 - 2X\, G_{4,X}\right)$$

The conceptual step

Horndeski removes the dangerous mode by forcing second-order equations. Beyond Horndeski permits higher-order equations but arranges the kinetic structure so a constraint still removes the extra mode.

The scalar–tensor kinetic matrix — setup

Introduce a 3+1 foliation and locate where the velocities live.

Foliation

unit normal n^a, spatial metric, lapse N, shift N^a

$$\gamma_{ab} = g_{ab} + n_a n_b, \qquad t^a = N\, n^a + N^a, \qquad t^a \partial_a = \frac{\partial}{\partial t}$$

decompose the scalar gradient

$$A_a \equiv \nabla_a \phi, \qquad A_* \equiv n^a A_a, \qquad A_a{}^b \equiv \gamma_a{}^b A_b$$

Where the velocities sit

- The extra scalar acceleration is carried by the time derivative of the normal projection A^*.
- The metric velocities are contained in the extrinsic curvature $K_a{}^b$.
- Degeneracy will be a statement about the joint kinetic matrix of these two velocity sets.

The kinetic Lagrangian and its Hessian

Collect the velocity-dependent part; the Hessian is the object that must be degenerate.

Velocity content

velocity part of the second derivative

$$\nabla_{(a}A_{b)} = \frac{1}{N} n_a n_b \dot{A}_* + \Lambda_{ab}{}^{cd} K_{cd} + (\text{no independent velocities})$$

extrinsic curvature

$$K_{ab} = \frac{1}{2N}\left(\dot{\gamma}_{ab} - D_a N_b - D_b N_a\right)$$

The scalar–tensor kinetic matrix and its degeneracy condition

Kinetic sector

schematic quadratic form in the velocities

$$L_{\text{kin}} = \tfrac{1}{2}\mathcal{A}\,\dot{A}_*^2 + \mathcal{B}^{ab}\dot{A}_* K_{ab} + \tfrac{1}{2}\mathcal{K}^{ab,cd} K_{ab} K_{cd}$$

- The scalar acceleration and the metric velocities mix through the cross term $\mathcal{B}^{ab}$.
- Collecting the coefficients gives one Hessian for both velocity sets.

$$\boxed{\mathcal{K} = \begin{pmatrix} \mathcal{A} & \mathcal{B}^{cd} \\ \mathcal{B}^{ab} & \mathcal{K}^{ab,cd} \end{pmatrix}, \qquad \mathcal{K}\begin{pmatrix} v_* \\ v_{cd} \end{pmatrix} = 0}$$

Degenerate theory: the kinetic matrix $\mathcal{K}$ admits a null vector (v, v_cd).*

Momenta, inversion, and the primary constraint

Degeneracy is precisely the failure to solve all velocities for the momenta.

Conjugate momenta

to the scalar acceleration

$$p_* = \frac{\partial L_{\rm kin}}{\partial \dot{A}_*} = \mathcal{A}\,\dot{A}_* + \mathcal{B}^{ij} K_{ij}$$

to the spatial metric

$$n\,\pi^{ij}_{\rm kin} = \frac{\partial L_{\rm kin}}{\partial K_{ij}} = \mathcal{B}^{ij}\dot{A}_* + \mathcal{K}^{ij,kl} K_{kl}$$

- If $\mathcal{K}$ is nondegenerate these invert: the scalar acceleration is an independent velocity.

The constraint

- For a null vector (v*, v_ij), contracting the momenta makes the velocities cancel:

$$\boxed{v_*\, p_* + v_{ij}\,\pi^{ij}_{\rm kin} = \mathcal{C}\big(\gamma_{ij}, A_*, A_i{}^b, \dots\big)}$$

- This primary constraint's preservation generates a secondary constraint, removing the extra pair.

What replaces “second order”

The criterion is not the highest derivative in one equation — it is whether the kinetic matrix is nondegenerate or degenerate.

Why the beyond-Horndeski structure is degenerate

Antisymmetrization correlates the velocity coefficients rather than leaving them free.

Correlated coefficients

- The double-ε contractions antisymmetrize the second derivatives of φ.
- After the 3+1 split, the coefficients of the velocity structures are not independent:

$$A_1 = -A_2, \qquad A_3 = -A_4, \qquad A_5 = 0$$

- Together with $F = G_4$, this sits the theory on a degenerate surface of coefficient space.

Missing from the Legendre map

a combination that determines no independent velocity

$$v_* \frac{\partial L_{\text{kin}}}{\partial \dot{A}_*} + v_{ij} \frac{\partial L_{\text{kin}}}{\partial K_{ij}} \quad \text{does not determine an independent velocity}$$

- This absent combination is exactly the origin of the primary constraint.

Why we stop here

The explicit null vector and the full algebraic degeneracy conditions lead into the DHOST classification, which is beyond the scope of this course.

Disformal transformations

An invertible field redefinition of the metric — it cannot create a new degree of freedom.

The map and its inverse

general disformal metric

$$\tilde{g}_{\mu\nu} = C(X,\phi)\, g_{\mu\nu} + D(X,\phi)\, \phi_\mu \phi_\nu$$

algebraic inverse metric

$$\tilde{g}^{\mu\nu} = \frac{1}{C}\left(g^{\mu\nu} - \frac{D}{C+DX}\phi^\mu\phi^\nu\right)$$

the transformed kinetic variable

$$\tilde{X} \equiv \tilde{g}^{\mu\nu}\phi_\mu\phi_\nu = \frac{X}{C+DX}$$

Invertibility

metric invertibility

$$C \neq 0, \qquad C + DX \neq 0$$

if C or D depends on X, also invert $X \mapsto \tilde{X}$

$$\frac{d\tilde{X}}{dX} = \frac{C - XC_X - X^2 D_X}{(C+DX)^2}$$

hence one additionally requires

$$\boxed{C - XC_X - X^2 D_X \neq 0}$$

What disformal maps do and don't change

They rearrange the form and order of the equations, not the physical content.

$$\text{Horndeski} \subset \text{beyond Horndeski / GLPV} \subset \text{wider degenerate scalar–tensor space}$$

Restricted vs general

- Transformations depending only on φ preserve the Horndeski form.
- Once X-dependence is allowed, invertible maps move into beyond Horndeski and wider degenerate families.
- The detailed DHOST classes are defined by the full degeneracy conditions, not this nesting alone.

A second view of degeneracy

If an invertible disformal redefinition maps a healthy scalar–tensor theory into one whose equations look higher order, the number of physical degrees of freedom is unchanged. The transformed theory must therefore possess the corresponding constraint structure — degeneracy is guaranteed indirectly, without re-deriving it from scratch.

Matter coupling and the physical frame

The equivalence must be stated carefully once matter is present.

Frames

matter minimally coupled to the tilde metric

$$\tilde{S}[\tilde{g},\phi] \;+\; S_m[\tilde{g},\psi_m]$$

after the field redefinition

$$S[g,\phi] \;+\; S_m\big[C\,g_{\mu\nu} + D\,\phi_\mu\phi_\nu,\;\psi_m\big]$$

The Jordan frame

- Gravitational actions are related by a field redefinition, but matter is now nonminimally coupled to g.
- The metric to which matter is minimally coupled defines the physical Jordan frame.

instabilities → characteristics → k-essence/Galileons → Horndeski → degenerate theories

Final perspective

Two lessons close the core course.

The structural lesson

- The absence of an Ostrogradsky mode is a statement about constraints and the number of independent initial data.
- It is not simply a statement about the differential order of the field equations.

The viability lesson

- Health of the theory does not by itself guarantee phenomenological viability.
- One must still understand nonlinear screening, characteristic propagation, and compact-object solutions.

$\text{instabilities} \rightarrow \text{characteristics} \rightarrow k\text{-essence/Galileons} \rightarrow \text{Horndeski} \rightarrow \text{degenerate theories}$

PART 5 · SUPPLEMENTARY

Applications

Black holes: shift symmetry & no-hair

A shift-symmetric current, evaluated on a static spherical background.

Symmetry and ansatz

shift symmetry gives a conserved current

$$\phi \;\rightarrow\; \phi + c$$

$$\nabla_\mu J^\mu = 0$$

static, spherically symmetric background

$$ds^2 = -h(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2, \qquad \phi = \phi(r)$$

Radial current

only the radial component survives

$$r^2\sqrt{\frac{h}{f}}\,J^r = C_J$$

its invariant norm

$$J^2 = J_\mu J^\mu = \frac{(J^r)^2}{f}$$

regularity at the horizon (f → 0)

$$C_J = 0, \qquad J^r = 0 \;\; \text{outside the horizon}$$

The no-hair conclusion and its assumptions

For a broad analytic class the scalar is forced to be constant.

Forcing the trivial scalar

broad analytic shift-symmetric class

$$J^r = f\,\phi'\,\mathcal{F}(\phi', f, h, f', h', \dots)$$

- Asymptotic flatness and a canonical weak-field branch make F tend to a finite nonzero constant.

hence

$$\boxed{J^r = 0 \implies \phi' = 0}$$

Five assumptions

1. static, spherically symmetric metric and scalar;
2. asymptotic flatness, constant scalar at infinity;
3. finite current norm at horizon and infinity;
4. a canonical weak-field kinetic branch;
5. analyticity of the Horndeski functions near X = 0.

How to read it

A no-hair theorem is a statement about assumptions — a hairy solution must evade at least one.

Loophole I — time-dependent scalar

A stationary metric can coexist with a linearly time-dependent scalar.

Ansatz and action

shift symmetry ⇒ only ∂ϕ appears

$$\phi = q\,t + \psi(r)$$

a useful Horndeski example

$$S = \frac{1}{16\pi G}\int d^4x\,\sqrt{-g}\,\left[R - \eta(\nabla\phi)^2 + \beta\,G_{\mu\nu}\nabla^\mu\phi\nabla^\nu\phi - 2\Lambda\right]$$

in the G-function convention

$$G_2 = -\eta X - 2\Lambda, \qquad G_4 = 1 - \frac{\beta}{2}X$$

Reduction of the quartic term

$$L_4 = G_4 R - 2G_{4,X}\left[(\Box\phi)^2 - \phi_{\mu\nu}\phi^{\mu\nu}\right]$$

$$= R - \frac{\beta}{2}X\,R + \beta\left[(\Box\phi)^2 - \phi_{\mu\nu}\phi^{\mu\nu}\right]$$

using, up to a boundary term

$$(\Box\phi)^2 - \phi_{\mu\nu}\phi^{\mu\nu} \;\doteq\; R_{\mu\nu}\phi^\mu\phi^\nu$$

one obtains

$$L_4 \;\doteq\; R + \beta\,G_{\mu\nu}\phi^\mu\phi^\nu$$

- The loophole occurs within the Horndeski class itself.

Loophole II — scalar–Gauss–Bonnet hair

Curvature sources the scalar; the coupling evades the analyticity assumption.

Coupling and scalar equation

$$L = R - (\nabla\phi)^2 + 2\alpha\,\phi\,\mathcal{G}$$

the Gauss–Bonnet invariant

$$\mathcal{G} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2$$

topological in 4D, but the scalar equation is nontrivial

$$\Box\phi + \alpha\,\mathcal{G} = 0$$

Why it evades no-hair

- Curvature sources φ, so the current argument no longer forces φ′ = 0.

in Horndeski language the coupling is nonanalytic

$$G_5 \sim \ln X$$

- This violates the analyticity assumption near X = 0.