Horndeski Theory

The most general scalar–tensor theory with second-order field equations.

$$S = \int d^4x \sqrt{-g} \sum_{i=2}^{5} \mathcal{L}_i$$

The goal

Couple the higher-derivative scalar interactions suggested by Galileons to a dynamical metric, while keeping BOTH the scalar and the metric field equations no higher than second order.

$$G_2(X,\phi) \qquad G_3(X,\phi) \qquad G_4(X,\phi) \qquad G_5(X,\phi)$$

Four free functions of X and ϕ.

The roadmap

- **Why naive covariantization fails** — curvature terms appear.
- **Curvature counterterms** restore 2nd-order equations.
- **The full action** L_2–L_5 and its hierarchy.
- **Antisymmetric construction** — a second route.

The Horndeski problem

Ask for a general local action whose field equations stay strictly second order.

Start from a very general local action

$$S = \int d^4x\, F\big[\, g, \partial g, \partial^2 g, \ldots;\ \phi, \partial\phi, \partial^2\phi, \ldots \big]$$

Arbitrary nonlinear interactions of a metric $g_{\mu\nu}$ and one scalar φ, in four dimensions.

Demand field equations of the restricted form

$$\mathcal{E}\big[\, g, \partial g, \partial^2 g;\ \ \phi, \partial\phi, \partial^2\phi \big] = 0$$

No derivatives higher than second order — in both the scalar and the metric equations.

THE QUESTION

How can the higher-derivative scalar interactions suggested by Galileons be coupled consistently to a dynamical metric, while keeping both the scalar and the metric equations second order?

Why second order — the Ostrogradsky link

The demand of Lecture 4 is a direct response to the instability of Lecture 1.

The danger

A nondegenerate dependence on higher time derivatives carries an extra degree of freedom whose Hamiltonian is linear in a momentum — a ghost.

$$\det\left(\frac{\partial^2 L}{\partial \ddot{q}_a\,\partial \ddot{q}_b}\right) \neq 0 \implies \text{Ostrogradsky ghost}$$

Nondegenerate higher-derivative kinetic term.

The strategy

Requiring second-order equations removes the premise of the theorem: there is no nondegenerate higher-derivative sector to host the ghost.

$$\text{2nd-order field equations} \implies \text{premise of the theorem evaded}$$

A strong (sufficient) way to guarantee a healthy spectrum.

Lecture 1: higher-order equations $\not\Rightarrow$ Ostrogradsky ghost WITHOUT the nondegeneracy assumption — we return to this subtlety at the end.

Why naive covariantization fails

A flat-space Galileon with second-order equations need not stay second order once curved.

Flat quartic Galileon

$$\mathcal{L}_4^{\text{flat}} = F(X)\left[(\Box\phi)^2 - \phi_{\mu\nu}\phi^{\mu\nu}\right], \qquad \phi_{\mu\nu} = \partial_\mu\partial_\nu\phi$$

$$F(X) \propto X \quad \text{(ordinary quartic Galileon)}$$

Naive covariantization $\partial \to \nabla$

$$\mathcal{L}_4^{\text{naive}} = F(X)\left[(\Box\phi)^2 - \nabla_\mu\nabla_\nu\phi\,\nabla^\mu\nabla^\nu\phi\right]$$

The integrations by parts now require commuting covariant derivatives.

The commutator generates a curvature term

Integrating by parts forces the Riemann tensor to appear — with a third derivative.

Commuting covariant derivatives on the vector ∇ρφ

$$[\nabla_\mu, \nabla_\nu]\nabla_\rho\phi = R_{\mu\nu\rho}{}^{\sigma}\nabla_\sigma\phi$$

A representative combination generated in the scalar variation

$$\nabla_\mu\nabla_\nu\phi^{\mu\nu} - \Box^2\phi = \nabla_\mu(R^{\mu\nu}\phi_\nu) = R^{\mu\nu}\phi_{\mu\nu} + \frac{1}{2}\,\phi^\mu\nabla_\mu R$$

$$\nabla_\mu R^{\mu\nu} = \tfrac{1}{2}\nabla^\nu R \qquad \text{(contracted Bianchi)}$$

The last term carries a third derivative of the metric. Conversely, the metric variation of the naive quartic generates third derivatives of the scalar.

$$\text{2nd-order flat Galileon} \;\not\Rightarrow\; \text{2nd-order covariant theory}$$

Curvature counterterms

Add a curvature coupling whose coefficient is tuned to the scalar interaction.

Introduce $G_4(X, \phi)$ tuned to the derivative interaction

$$F(X,\phi) = -2\,G_{4,X}(X,\phi)$$

The completed quartic Lagrangian

$$\mathcal{L}_4^H = G_4(X,\phi)\,R - 2G_{4,X}(X,\phi)\,\left[(\Box\phi)^2 - \phi_{\mu\nu}\phi^{\mu\nu}\right]$$

Shorthand for the quartic scalar structure

$$Q \equiv (\Box\phi)^2 - \phi_{\mu\nu}\phi^{\mu\nu}$$

Next: show the two variations cancel the dangerous ∇R term term-by-term.

Explicit cancellation — the scalar variation

Vary the derivative part of L_4^H and isolate the term that produces a derivative of curvature.

Variation of the derivative part (keep only highest-metric-derivative terms)

$$\delta S_{\mathrm{der}} \supset -4\int d^4x\,\sqrt{-g}\,G_{4,X}\,[\Box\phi\,\Box\delta\phi - \phi^{\mu\nu}\nabla_\mu\nabla_\nu\delta\phi]$$

Integrate by parts ($\doteq$ up to total derivative)

$$\doteq -4\int d^4x\,\sqrt{-g}\,G_{4,X}\,\left[\Box^2\phi - \nabla_\mu\nabla_\nu\phi^{\mu\nu}\right]\delta\phi + \cdots$$

Using Eq. (6), the dangerous piece is

$$\delta S_{\mathrm{der}} \supset +2\int d^4x\,\sqrt{-g}\,G_{4,X}\,\phi^\mu\nabla_\mu R\,\delta\phi + \cdots$$

Explicit cancellation — the curvature variation

Varying the X-dependence of G_4R produces exactly the opposite term.

Recall, in this convention

$$\delta X = 2\,\phi^\mu \nabla_\mu \delta\phi$$

Vary the X inside the curvature term

$$\delta S_R \supset \int d^4x\,\sqrt{-g}\;2G_{4,X} R\,\phi^\mu \nabla_\mu \delta\phi$$

Integrate by parts and keep the highest-metric-derivative contribution

$$\doteq -2\int d^4x\,\sqrt{-g}\,\nabla_\mu(G_{4,X} R\,\phi^\mu)\,\delta\phi \;\supset\; -2\int d^4x\,\sqrt{-g}\,G_{4,X}\,\phi^\mu \nabla_\mu R\,\delta\phi + \cdots$$

CANCELLATION

$$\underbrace{+2\,G_{4,X}\,\phi^\mu \nabla_\mu R}_{\text{from }\delta S_{\text{der}}} \;+\; \underbrace{-2\,G_{4,X}\,\phi^\mu \nabla_\mu R}_{\text{from }\delta S_R} \;=\; 0$$

The full Horndeski action

Four arbitrary functions of X and ϕ; the lower two sectors reproduce earlier lectures.

The action

$$S = \int d^4x \sqrt{-g} \sum_{i=2}^{5} \mathcal{L}_i$$

Four free functions

$$G_2(X,\phi) \quad G_3(X,\phi) \quad G_4(X,\phi) \quad G_5(X,\phi)$$

L_2 — k-essence sector

$$\mathcal{L}_2 = G_2(X,\phi)$$

L_3 — generalized cubic Galileon

$$\mathcal{L}_3 = G_3(X,\phi)\,\Box\phi$$

The full Horndeski action — quartic and quintic

Relative coefficients are fixed so higher derivatives cancel in both field equations.

L_4 — curvature + tuned quadratic

$$\mathcal{L}_4 = G_4\, R - 2G_{4,X}\left[(\Box\phi)^2 - \phi_{\mu\nu}\phi^{\mu\nu}\right]$$

L_5 — Einstein coupling + tuned cubic

$$\mathcal{L}_5 = G_5\, G_{\mu\nu}\phi^{\mu\nu} + \tfrac{1}{3}G_{5,X}\left[(\Box\phi)^3 - 3\Box\phi\,\phi_{\mu\nu}\phi^{\mu\nu} + 2\phi_{\mu\nu}\phi^{\nu\rho}\phi_{\rho}{}^{\mu}\right]$$

CONVENTION

Signs and numerical coefficients in L_3–L_5 vary between references.

The hierarchy and its meaning

Each sector connects Horndeski back to the previous lectures.

$$\mathcal{L}_2 = G_2(X,\phi) \;\supset\; \text{k-essence} \quad (G_2 = -K)$$

Contains the k-essence sector of Lectures 2–3.

$$\mathcal{L}_3 = G_3(X,\phi)\,\Box\phi \;\supset\; \text{cubic Galileon}$$

Generalizes the cubic Galileon of Lecture 3.

$$\mathcal{L}_4 :\; \text{nonminimal } G_4 R \;+\; \text{tuned } [(\Box\phi)^2 - \phi_{\mu\nu}\phi^{\mu\nu}]$$

Nonminimal curvature coupling with a tuned quadratic in $\phi_{\mu\nu}$.

$$\mathcal{L}_5 :\; \text{Einstein coupling } G_{\mu\nu}\phi^{\mu\nu} \;+\; \text{tuned cubic}$$

Einstein-tensor coupling with a tuned cubic in $\phi_{\mu\nu}$.

Antisymmetric construction — Levi–Civita tensors

A second route to the same tuning: products of two Levi–Civita tensors.

Definition (mostly-plus)

$$\varepsilon_{\alpha\beta\gamma\delta} = \begin{cases} +\sqrt{-g}, & (\alpha\beta\gamma\delta) \text{ even perm. of } 0123 \\ -\sqrt{-g}, & (\alpha\beta\gamma\delta) \text{ odd perm. of } 0123 \\ 0, & \text{two indices coincide} \end{cases}$$

Normalization

$$\varepsilon_{0123} = +\sqrt{-g}\,, \qquad \varepsilon^{0123} = -\frac{1}{\sqrt{-g}}$$

Products of two Levi–Civita tensors organize the scalar interactions in an explicitly antisymmetric form — automatically producing determinant-like contractions.

The antisymmetric scalar densities

Contract two epsilons with gradients and Hessians of ϕ.

$$\ell_{(2,0)} = -\frac{1}{3!}\,\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\alpha\nu\rho\sigma}\,\phi_\mu\phi^\alpha = X$$

Two gradients → the kinetic term.

$$\ell_{(3,0)} = -\frac{1}{2!}\,\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\alpha\beta\rho\sigma}\,\phi_\mu\phi^\alpha\phi_\nu{}^\beta = X\Box\phi - \phi^\mu\phi_{\mu\nu}\phi^\nu$$

Three fields → the cubic Galileon structure.

$$\ell_{(4,0)} = -\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\alpha\beta\gamma\sigma}\,\phi_\mu\phi^\alpha\phi_\nu{}^\beta\phi_\rho{}^\gamma$$

Four fields → quartic structure.

$$\ell_{(5,0)} = -\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\alpha\beta\gamma\delta}\,\phi_\mu\phi^\alpha\phi_\nu{}^\beta\phi_\rho{}^\gamma\phi_\sigma{}^\delta$$

Five fields → quintic structure.

The quartic density expanded

Antisymmetry produces exactly the determinant-like contractions of the Galileon.

Expanding the double-epsilon contraction of $\ell_{(4,0)}$

$$\ell_{(4,0)} = X\left[(\Box\phi)^2 - \phi_{\mu\nu}\phi^{\mu\nu}\right] - 2\,\phi^{\mu}\phi_{\mu\nu}\phi^{\nu}\Box\phi + 2\,\phi^{\mu}\phi_{\mu\nu}\phi^{\nu\rho}\phi_{\rho}$$

Why these combinations are special

The leading $X\cdot[(\Box\varphi)^2 - \varphi_{\mu\nu}\varphi^{\mu\nu}]$ piece is precisely the quartic Galileon structure. The special combinations are therefore not arbitrary — antisymmetry selects exactly the determinant-like contractions of second derivatives that keep the equations of motion second order.

Curvature structures

Replace one pair of second derivatives with a Riemann tensor.

Quartic curvature partner — collapses to the Einstein tensor

$$\ell_{(4,1)} = -\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\alpha\beta\gamma\sigma}\,\phi_\mu\phi^\alpha R_{\nu\rho}{}^{\beta\gamma} = -4\,G^{\mu\nu}\phi_\mu\phi_\nu$$

Quintic curvature partner

$$\ell_{(5,1)} = -\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\alpha\beta\gamma\delta}\,\phi_\mu\phi^\alpha\phi_\nu{}^\beta R_{\rho\sigma}{}^{\gamma\delta}$$

These curvature partners are exactly what the covariantization argument demanded: $\ell_{(4,1)}$ reproduces the G_4R coupling, and $\ell_{(5,1)}$ the Einstein-tensor coupling of L_5.

Recovering the Horndeski combinations

Multiply each density by an arbitrary function and integrate by parts.

Attach free functions to the scalar and curvature densities

$$\mathcal{L}_{(n,0)} = f_n(X,\phi)\,\ell_{(n,0)}\,, \qquad \mathcal{L}_{(n,1)} = s_n(X,\phi)\,\ell_{(n,1)}$$

Quadratic

$$\mathcal{L}_{(2,0)} = G_2(X,\phi)\,, \qquad G_2 = X f_2$$

Cubic

$$\mathcal{L}_{(3,0)} = G_3(X,\phi)\,\Box\phi + \text{t.d.}$$

The first two densities reproduce the k-essence and cubic-Galileon sectors directly (up to a total derivative). The nontrivial tuning appears at quartic and quintic order — next slide.

The tuning conditions: $F_4 = F_5 = 0$

Scalar and curvature densities must occur in specific tuned combinations.

Quartic pair

$$\mathcal{L}_{(4,0)} + \mathcal{L}_{(4,1)} = G_4 R - 2G_{4,X}\left[(\Box\phi)^2 - \phi_{\mu\nu}\phi^{\mu\nu}\right] + F_4\, \varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\alpha\beta\gamma\sigma}\phi_\mu\phi^\alpha\phi_\nu{}^\beta\phi_\rho{}^\gamma + \text{t.d.}$$

$$\text{with} \quad F_4 = 0 \iff f_4 + s_{4,X} = 0$$

Quintic pair

$$\mathcal{L}_{(5,0)} + \mathcal{L}_{(5,1)} = G_5 G_{\mu\nu}\phi^{\mu\nu} + \tfrac{1}{3}G_{5,X}\left[(\Box\phi)^3 - 3\Box\phi\,\phi_{\mu\nu}\phi^{\mu\nu} + 2\phi_{\mu\nu}\phi^{\nu\rho}\phi_\rho{}^\mu\right] + F_5\, \varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\alpha\beta\gamma\delta}\phi_\mu\phi^\alpha\phi_\nu{}^\beta\phi_\rho{}^\gamma\phi_\sigma{}^\delta + \text{t.d.}$$

$$\text{with} \quad F_5 = 0 \iff f_5 + \tfrac{4}{3}\, s_{5,X} = 0$$

TWO VIEWS

(1) *curvature counterterms cancel the dangerous higher derivatives;* **(2)** *antisymmetric contractions organize precisely the combinations for which the cancellations occur.*

What Horndeski guarantees

Sufficient for health — but not necessary.

Sufficient condition

$$\text{2nd-order field equations} \implies \text{no nondegenerate Ostrogradsky ghost}$$

But the converse fails — a degenerate kinetic structure can stay healthy

$$\text{higher-order eqs} + \text{degeneracy} \implies \text{still healthy}$$

SUFFICIENT, NOT NECESSARY

Second-order field equations are sufficient for the Horndeski construction, but not necessary for a healthy scalar–tensor theory. This is the conceptual opening to beyond Horndeski.

Connection to the next lecture

From the order of the equations to the degeneracy of the kinetic matrix.

$$\text{Horndeski} \subset \text{beyond Horndeski} \subset \text{DHOST}$$

What changes

The central object is no longer the order of the field equations, but the degeneracy of the kinetic matrix built from the scalar and metric velocities.

Where it leads

Keeping F_4 or F_5 nonzero gives beyond Horndeski, then the wider DHOST family — and naturally, disformal transformations.

NEXT → *Lecture 5 — degeneracy, beyond Horndeski, DHOST, and disformal transformations.*