

Lecture 5: Λ CDM, Type-II Minimally Modified Gravity, and Maple Workshop

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Lecture 5

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Why Minimally Modified Gravity?

- Many extensions of GR add scalar, vector, or massive spin-2 polarizations.
- Extra gravitational modes are common, but they are not logically required.
- **Minimally modified gravity** changes the dynamics while retaining the two local gravitational modes of GR.
- The central tool is a modified nonlinear constraint structure.

Two Modes Do Not Force GR

- A degree-of-freedom count fixes the dimension of physical phase space, not the Hamiltonian on it.
- Altering constraints can modify the background and perturbation equations without activating a scalar graviton.
- The nonlinear Hamiltonian count must be established before specializing to FLRW.
- VCDM is a particularly simple realization of this idea.

What VCDM Is—and Is Not

- VCDM is **not** a vector-tensor or generalized-Proca theory.
- Its name reflects the promotion of the constant Λ in Λ CDM to a function $V(\phi)$.
- The scalar ϕ is auxiliary and supplies no independent scalar initial data.
- The gravitational sector retains two tensor modes while $V(\phi)$ controls cosmological evolution.

Construction Logic

- Start from the ADM Hamiltonian of GR and perform a canonical transformation.
- Introduce a cosmological constant in the transformed frame and add the constraint needed to prevent a scalar graviton.
- Transform back and couple matter minimally in the original frame.
- The inverse transformation turns the constant into $V(\phi)$ and selects a preferred foliation.

Minimally Modified Gravity: Type-I vs Type-II

- **Minimally Modified Gravity (MMG):** Theories with the same two local gravitational degrees of freedom as GR, but with different dynamics.
- **Type-I theories:** Possess an Einstein frame. The gravitational Hamiltonian can be transformed to that of GR; modifications reside in matter couplings.
- **Type-II theories:** Possess no such frame.
- VCDM is a remarkably simple Type-II theory.

Construction by a Canonical Detour (Steps 1-3)

Algorithm to construct VCDM:

- 1 Start from the ADM Hamiltonian of GR.
- 2 Perform a canonical transformation to variables $(\mathfrak{N}, \mathfrak{N}^i, \Gamma_{ij})$.
- 3 Add a cosmological constant in that transformed frame.

Construction by a Canonical Detour (Steps 4-6)

- 4 Add a gauge-fixing constraint so that no scalar graviton is activated.
- 5 Transform back to the original ADM variables.
- 6 Couple ordinary matter minimally in the original frame.

Note: The cosmological constant introduced in step 3 becomes a potential $V(\phi)$ of an auxiliary field after the inverse transformation.

The Λ CDM Hamiltonian

After convenient field redefinitions, the Hamiltonian is:

$$H_{\Lambda\text{CDM}} = \int d^3x \left[N \mathcal{H}_0^{\text{GR}} + N^i \mathcal{H}_i^{\text{GR}} + \sqrt{\gamma} \lambda_C \left(M_{\text{Pl}}^2 \phi - \frac{\pi^{ij} \gamma_{ij}}{\sqrt{\gamma}} \right) \right. \\ \left. + \lambda_\phi \pi_\phi + \sqrt{\gamma} M_{\text{Pl}}^2 \lambda_{\text{gf}}^i \partial_i \phi + N \sqrt{\gamma} M_{\text{Pl}}^2 V(\phi) \right] + H_m.$$

- ϕ is tied to the trace of the gravitational momentum.
- λ_ϕ imposes the primary constraint $\pi_\phi \approx 0$.
- Spatial homogeneity of ϕ is enforced on each preferred slice ($\partial_i \phi = 0$).

Key Idea of VCDM

Key Idea

The symbol V in VCDM emphasizes that a function $V(\phi)$ replaces the constant vacuum term of Λ CDM, while the gravitational sector still has exactly two local tensor modes. The auxiliary field changes the constraint surface, not the dimension of the physical phase space.

Lagrangian Form

Legendre transforming gives the VCDM Lagrangian:

$$\mathcal{L}_{\text{VCDM}} = N\sqrt{\gamma} \left[\frac{M_{\text{Pl}}^2}{2} \left({}^{(3)}R + K_{ij}K^{ij} - K^2 - 2V(\phi) \right) \right. \\ \left. - \frac{M_{\text{Pl}}^2}{N} \lambda_{\text{gf}}^i \partial_i \phi - \frac{3M_{\text{Pl}}^2}{4} \lambda^2 - M_{\text{Pl}}^2 \lambda (K + \phi) \right] + \mathcal{L}_m.$$

- For $V_\phi = 0$, it reduces to GR with a cosmological constant.
- For $V_\phi \neq 0$, preferred slicing remains (genuinely Type-II).

Nonlinear Constraint Count

- Canonical pairs (γ_{ij}, π^{ij}) and $(\phi, \pi_\phi) \rightarrow 14$ phase-space variables.
- **First-class:** 3 extended momentum constraints (spatial diffeomorphisms).
- **Second-class:** 4 independent constraints (Hamiltonian-type, π_ϕ , trace-momentum relation, and scalar part of $\partial_i \phi = 0$).

Degrees of Freedom

$$N_{\text{dof}}^{\text{VCDM}} = \frac{1}{2}(14 - 2 \times 3 - 4) = 2.$$

Minisuperspace and FLRW Background

Take an FLRW ansatz:

$$N = N(t), \quad N^i = 0, \quad \gamma_{ij} = a^2(t)\delta_{ij}, \quad H = \frac{\dot{a}}{Na}.$$

The minisuperspace Lagrangian per comoving volume:

$$L_{\text{mini}} = Na^3 M_{\text{Pl}}^2 \left[-3H^2 - V(\phi) - \frac{3}{4}\lambda^2 - \lambda(3H + \dot{\phi}) \right] + L_m.$$

- The gauge-fixing term vanishes on the homogeneous ansatz.

Multiplier and Auxiliary Equations

Varying multipliers and fields gives:

- **Varying λ :** $\lambda = -2H - \frac{2}{3}\phi$
- **Lapse equation:** $\phi^2 = 3V + \frac{3\rho}{M_{\text{Pl}}^2}$
- **Varying ϕ :** $\phi = \frac{3}{2}V_\phi - 3H$

Raychaudhuri and Continuity Equations

Evolution of ϕ and Hubble parameter:

$$\frac{\dot{\phi}}{N} = \frac{3}{2} \frac{\rho + P}{M_{\text{Pl}}^2}, \quad \frac{\dot{H}}{N} = \frac{(\rho + P)(3V_{\phi\phi} - 2)}{4M_{\text{Pl}}^2}.$$

Minimal matter coupling yields:

$$\frac{\dot{\rho}}{N} + 3H(\rho + P) = 0.$$

- For $V_{\phi\phi} = 0$ this reduces to the GR result $\dot{H}/N = -(\rho + P)/(2M_{\text{Pl}}^2)$.

The Friedmann Equation in VCDM

Combining the lapse and ϕ equations yields the familiar Friedmann form:

$$3M_{\text{Pl}}^2 H^2 = \rho + \rho_\phi,$$

where the effective dark energy density is:

$$\rho_\phi = M_{\text{Pl}}^2 (V - \phi V_\phi) + \frac{3}{4} M_{\text{Pl}}^2 V_\phi^2.$$

The effective pressure is:

$$P_\phi = -\rho_\phi - \frac{3}{2}(\rho + P) V_{\phi\phi}.$$

Background Reconstruction

Proposition

For any prescribed expanding FLRW history $H(\mathcal{N})$ (with $\mathcal{N} = \ln(a/a_0)$) and ordinary matter history ($\rho + P > 0$), one can construct a local Λ CDM potential $V(\phi)$ realizing that history.

$$\phi(\mathcal{N}) = \phi_0 + \int_{\mathcal{N}_0}^{\mathcal{N}} \frac{3}{2} \frac{\rho(\nu) + P(\nu)}{M_{\text{Pl}}^2 H(\nu)} d\nu$$

- Monotonicity permits inversion to find $\mathcal{N}(\phi)$.

Reconstructing VCDM Potential

Once $\mathcal{N}(\phi)$ is known, the potential is obtained analytically:

$$V(\phi) = \frac{1}{3}\phi^2 - \frac{\rho(\mathcal{N}(\phi))}{M_{\text{Pl}}^2}.$$

- **Pitfall:** A reconstruction is not yet a stable model!
- One must test invertibility, regularity of $V(\phi)$, scalar perturbations, and the range of validity of the preferred foliation.

Worked Example: Λ CDM

If the target history obeys GR with a cosmological constant:

$$3M_{\text{Pl}}^2 H^2 = \rho + M_{\text{Pl}}^2 \Lambda \implies \phi = C - 3H$$

Substitution yields a linear potential:

$$V(\phi) = \frac{2C}{3}\phi - \frac{C^2}{3} + \Lambda.$$

- Although the potential is linear, the effective energy density is $\rho_\phi = M_{\text{Pl}}^2 \Lambda$.

Tensor Perturbations in VCDM

The tensor action is exactly that of GR:

$$S_T^{(2)} = \frac{M_{\text{Pl}}^2}{8} \sum_{s=+, \times} \int dt d^3x N a^3 \left[\frac{\dot{h}_s^2}{N^2} - \frac{(\partial_i h_s)^2}{a^2} \right].$$

Thus, VCDM tensors are massless and luminal, unlike MTMG tensors.

Properties of Gravitational Waves in VCDM

In cosmic time ($N = 1$), the propagation equation is standard:

$$\ddot{\gamma}_{ij} + 3H\dot{\gamma}_{ij} + \frac{k^2}{a^2}\gamma_{ij} = 0.$$

- ① **No Graviton Mass:** The graviton remains exactly massless.
- ② **Speed of Light:** The propagation speed is strictly $c_T^2 = 1$, instantly satisfying GW170817 constraints.
- ③ **Standard Friction:** Gravitational waves are damped solely by the cosmic expansion.

Scalar Perturbations: Sound Speed

- In the scalar sector, the only propagating modes are those supplied by matter.
- For one barotropic perfect fluid, the high- k sound speed and no-ghost coefficient are the GR ones:

$$c_s^2 = \left(\frac{\partial P}{\partial \rho} \right)_s, \quad Q_s = \frac{a^2 \rho^2}{2k^2(\rho + P)} > 0.$$

The displayed no-ghost inequality assumes $\rho + P > 0$; gradient stability additionally requires $c_s^2 > 0$.

Dust Action in Λ CDM

For dust, integrating all auxiliary perturbations yields:

$$S_{\delta}^{(2)} \supset \frac{1}{2} \int d^4x N a^3 \frac{\left(\frac{3}{2} \rho_m + \frac{k^2 M_{\text{Pl}}^2}{a^2} \right) a^2 \rho_m \dot{\delta}_m^2}{\left(\frac{3}{2} \rho_m + \frac{k^2 M_{\text{Pl}}^2}{a^2} - \frac{9}{4} \rho_m V_{\phi\phi} \right) k^2 N^2}.$$

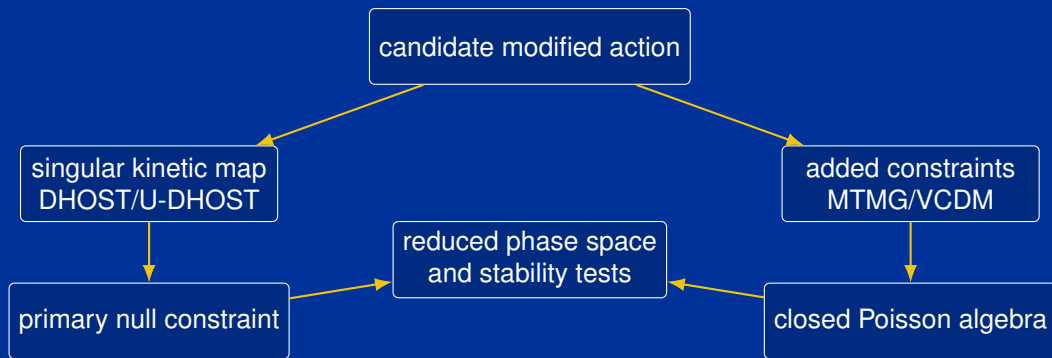
- The potential curvature $V_{\phi\phi}$ changes the kinetic normalization, friction, and effective gravitational coupling.
- At high k , the modification is suppressed and GR is recovered locally!

Four Theories, Two Mechanisms

Theory	Defining Mechanism	Local Gravity Modes
DHOST	Covariant kinetic degeneracy	$2T + 1S$
U-DHOST	Degeneracy on scalar foliation	$2T + 1S$ (plus elliptic)
MTMG	Two added second-class constraints	$2T$ massive
VCDM	Canonical construction + gauge fixing	$2T$ massless

- MTMG and VCDM have no gravitational scalar at all.
- Any scalar cosmological mode comes entirely from matter.

A Constraint-Flow View



Goal of the Workshop

- The final lecture turns conceptual distinctions into a working calculation protocol.
- **Goal:** Not asking a computer algebra system to “find the physics”.
- **Real Goal:** Make the computer verify a sequence of physics decisions whose meaning is already understood.

A Universal Background Workflow (Setup)

For any ADM action, adopt the ansatz:

$$N = N(t), \quad N^i = 0, \quad h_{ij} = a^2(t)\delta_{ij}.$$

Keep the lapse arbitrary and define $H = \dot{a}/(Na)$. Keep the homogeneous fluid off shell with the Schutz–Sorkin action:

$$L_m = -Na^3 \left[\rho(n) + \frac{n}{N} \dot{\ell} \right].$$

Its matter equations give

$$\frac{d}{dt}(a^3 n) = 0, \quad \dot{\ell} = -N\rho_{,n}, \quad P = n\rho_{,n} - \rho.$$

On these equations, the metric variation becomes

$$\delta S_m = \int dt Na^3 \left(-\rho \frac{\delta N}{N} + 3P \frac{\delta a}{a} \right).$$

The action and the compact response formula are two levels of the same description.

A Universal Background Workflow (Steps)

- 1 Substitute the homogeneous ansatz into the action, but **do not set** $N = 1$.
- 2 Remove only genuine total derivatives; record discarded boundary terms.
- 3 Vary N , a , all homogeneous fields, and every multiplier.
- 4 Solve multiplier constraints first. Substitute them into the remaining equations.
- 5 Verify the Noether identity.
- 6 Only now set $N = 1$ or change to e-fold time.

Three Background Checks

For a complete background calculation, supplement the algebraic residuals in scripts with these checks:

- 1 **The GR or lower-theory limit.**
- 2 **The matter continuity identity:** one background equation must follow from the others.
- 3 **Dimensional homogeneity** of every Friedmann term.

Universal Scalar-Perturbation Workflow

Scalar mode metric decomposition:

$$ds^2 = -N^2(1 + 2\epsilon\alpha)dt^2 + 2N\epsilon\partial_i\chi dt dx^i + a^2 [(1 + 2\epsilon\zeta)\delta_{ij} + 2\epsilon\partial_i\partial_j E] dx^i dx^j$$

Keep bookkeeping parameter ϵ :

- $S^{(0)}$: Background equations.
- $S^{(1)}$: Must vanish! (Strong test of signs and integrations by parts).
- $S^{(2)}$: Contains constraints and physical kinetic/gradient matrices.

The Correct Elimination Order (Quadratic Level)

Separate fields into three groups:

- 1 **Gauge variables:** Fix them only after checking that the gauge is complete and admissible.
- 2 **Auxiliary variables:** Vary first and solve their constraint equations.
- 3 **Dynamical variables:** Compute the reduced kinetic matrix *only after* all auxiliary fields have been eliminated.

Note: "Solve" may mean inverting an elliptic operator with stated boundary conditions, as in U-DHOST.

Primary Route in the Detailed Files: Fundiff

```
mini_action := Physics[Intc](Lmini,t):  
eom_q := simplify(subs(tt1=t,  
    convert(Physics[Fundiff](  
        mini_action,q(tt1)),D)))
```

- Used by the detailed `_cdx.mpl` files and throughout `udhost_opt.mpl`.
- Handles arbitrary derivative order; `equatio` is only a wrapper around the same call.

Why the Short Files Still Define EL2

$$E_q = \frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} + \frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{q}}.$$

- The short companion scripts implement this formula directly as EL2, using placeholders for q , $\dot{q}$, $\ddot{q}$.
- It is transparent and does not depend on the Physics package.
- There is no competing convention: `Fundiff` is the main route; EL2 is a compact teaching tool and an independent cross-check.

Maple Procedures: Integration by Parts

Checking IBP replacements explicitly:

```
IBP_D2_D1 := proc(L::algebraic, q::algebraic, rt::name)
    local t, C;
    t := rt;
    C := coeff(coeff(expand(L), diff(q,t,t)), diff(q,t));
    expand(L - C*diff(q,t,t)*diff(q,t)
           - 1/2*diff(C,t)*diff(q,t)^2);
end proc;
```

- The difference must have an identically vanishing Euler-Lagrange derivative.

Pitfall: Expression Swell

Expression Swell versus Physics

Commands such as `expand` and `simplify` can turn a one-line factorized constraint into thousands of terms.

Solution: Preserve factors that encode branches, Hessian rank, or stability. Expand only to extract a coefficient, and immediately factor the result again.

A Research-Grade Verification Ladder (1-4)

- 1 **Kinematics:** inverse metric times metric is the identity; determinant agrees with ADM determinant.
- 2 **Background:** $S^{(1)}$ vanishes on the background equations.
- 3 **Gauge:** pure-gauge perturbations give a boundary term.
- 4 **Constraints:** number and differential type of auxiliary equations agree with Hamiltonian count.

A Research-Grade Verification Ladder (5-8)

- 5 **Reduction:** substituting auxiliary solutions back into equations agrees with varying the reduced action.
- 6 **UV Limit:** high- k kinetic and gradient matrices reproduce known limits.
- 7 **Theory Limit:** DHOST $\rightarrow$ Horndeski/GR, U-DHOST $\xi \rightarrow 0$, VCDM $V_{\phi\phi} \rightarrow 0$, MTMG $m \rightarrow 0$.
- 8 **Independent Route:** reproduce one key equation by a second method (e.g. ADM vs covariant expansion).

Supplied Maple Scripts

- `dhost_bg_only_cdx.mpl`: covariant FLRW invariants, class- Λ homogeneous comparison, and background variations.
- `udhost_opt.mpl`: shift-symmetric quadratic scalar perturbations and auxiliary reduction; it overwrites `actiogen1.txt`.
- `mtmg_bg_only_cdx.mpl`: vacuum homogeneous square root, N , a , λ variations, and factored multiplier constraint.
- `vcdm_bg_only_cdx.mpl`: homogeneous VCDM with a Schutz-type fluid and continuity residual.

Summary and Conclusion

- **VCDM** is a type-II minimally modified gravity theory with two local gravitational tensor modes and no scalar graviton.
- In the tensor sector, VCDM behaves exactly as GR (massless, $c_T = 1$), naturally evading GW constraints.
- **Maple Workflow:** Formalizing constraints, elimination orders, and stability checks is crucial. A rigorous verification ladder prevents computer algebra errors.