

Lecture 4: The Minimal Theory of Massive Gravity (MTMG)

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Table of Contents

- 1 Introduction: Ghosts in Massive Gravity
- 2 Minimal Theory of Massive Gravity (MTMG)
- 3 FLRW Background Analysis
- 4 Perturbations & Gravitational Waves
- 5 Maple Practice

The Boulware-Deser Ghost

- In General Relativity, the Hamiltonian is a linear combination of first-class constraints:

$$H_{GR} = \int d^3x \left(N\mathcal{H}_0 + N^i\mathcal{H}_i \right)$$

- Lapse N and shift N^i are non-dynamical Lagrange multipliers.
- 4 constraints eliminate 8 degrees of freedom in phase space (4×2).
- Leaves exactly 2 physical tensor modes from the original 12 phase space variables of h_{ij} .

Generic Massive Gravity

- Giving the graviton a mass breaks diffeomorphism invariance explicitly by adding a potential term $U(g, f)$.
- Generic mass term introduces non-linear dependencies on N and N^i .
- Constraints $\mathcal{H}_0 = 0$ and $\mathcal{H}_i = 0$ are lost.
- All 6 components of spatial metric h_{ij} become dynamical (2 tensor, 2 vector, 2 scalar modes).
- One scalar mode has negative kinetic energy: the **Boulware-Deser (BD) ghost**.

The dRGT Breakthrough

- de Rham, Gabadadze, and Tolley (dRGT) discovered a specific potential $U(g, f)$ keeping Hamiltonian linear in N (after shift redefinition).
- Produces a primary constraint and its secondary partner.
- Their pair projects out the BD ghost.
- Leaves a healthy massive spin-2 field with **5 Degrees of Freedom (DoFs)**.

- With a Minkowski reference metric, standard dRGT has no exactly homogeneous and isotropic solution with a spatially flat FLRW physical metric.
- Open-FLRW and less symmetric self-accelerating branches do exist.
- Their perturbations motivate a careful distinction between a mode removed by constraints and one whose quadratic kinetic term merely vanishes.

Strong Coupling in dRGT

- On known self-accelerating dRGT branches, the quadratic kinetic coefficients of two vector modes and one scalar mode can vanish.
- A vanishing kinetic coefficient is not the statement $c_s^2 = 0$: the quadratic action no longer determines a sound speed for that mode.
- The modes still exist nonlinearly, so the perturbative expansion becomes strongly coupled.

The Core Idea of MTMG

- De Felice and Mukohyama (2015) bypassed these problems.
- Radical approach: **Explicitly break 4D diffeomorphism invariance** in the action.
- Enforce new constraints by hand to eliminate 3 problematic DoFs (2 vector, 1 scalar).
- Leaves only **2 healthy massive tensor modes**.

Removed Mode vs. Vanishing Kinetic Term

- Important distinction:
 - 1 The coefficient of $\dot{q}^2$ vanishes on a solution (strong coupling).
 - 2 A primary-secondary constraint pair removes (q, p_q) from phase space (reduced dimension).
- MTMG changes the generic nonlinear constraint algebra.
- Its absent scalar and vector gravitons are **removed variables**, not zero-kinetic variables.

Physical and Fiducial Vielbeins

- Canonical fields: spatial vielbein e^I_j ; lapse N and shift N^i act as multipliers.
- Fiducial fields: (M, M^i, E^I_j) .
- Triangular ADM form:

$$e^A{}_\mu = \begin{pmatrix} N & 0 \\ N^k e^I_k & e^I_j \end{pmatrix}, \quad E^A{}_\mu = \begin{pmatrix} M & 0 \\ M^k E^I_k & E^I_j \end{pmatrix}$$

- Fixing both vielbeins to triangular form selects a preferred Lorentz frame, enabling controlled Lorentz breaking.

The Precursor Theory

- Substitute ADM vielbeins directly into dRGT wedge-product potential:

$$\mathcal{S}_{\text{pre}} = \frac{M_{\text{P}}^2}{2} \int d^4x \left\{ N\sqrt{\gamma} \left[{}^{(3)}R + K_{ij}K^{ij} - K^2 \right] - m^2 M \mathcal{H}_1 - m^2 N \mathcal{H}_0 \right\}$$

- The mass term is linear in both lapses and independent of the physical shift.

Precursor Count

- 9 configuration variables $e^I_j \implies$ 18-dimensional phase space.
- 12 independent second-class constraints on a generic background.
- Number of degrees of freedom:

$$N_{\text{dof}}^{\text{pre}} = \frac{1}{2}(18 - 12) = 3$$

- Two vector polarizations removed, but one scalar graviton remains.

Step 2: The Minimal Constraints

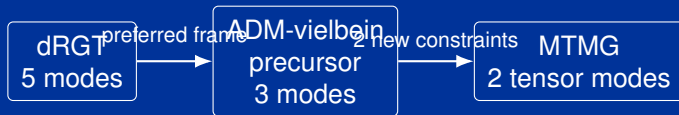
- MTMG imposes four new expressions $\mathcal{C}_0 \approx 0$ and $\mathcal{C}_i \approx 0$.
- Total Hamiltonian has the form:

$$H_{\text{MTMG}} = \int d^3x \left[-N\mathcal{R}_0^{\text{GR}} - N^i\mathcal{R}_i^{\text{GR}} + m^2(N\mathcal{H}_0 + M\mathcal{H}_1) + \lambda\mathcal{C}_0 + \lambda^i\mathcal{C}_i + \dots \right]$$

- λ, λ^i are Lagrange multipliers enforcing exactly two new independent constraints.

Nonlinear Mode Count of MTMG

- 18-dimensional vielbein phase space reduced by 14 independent second-class constraints.
- $(18 - 14)/2 = 2$ degrees of freedom.



Metric Formulation

- Compact notation with spatial square-root matrices $\mathcal{K}^i_k \mathcal{K}^k_j = \tilde{\gamma}^{ik} \gamma_{kj}$.
- MTMG Action:

$$\begin{aligned} S_{\text{MTMG}} = S_{\text{pre}} &+ \frac{M_{\text{P}}^2}{2} \int d^4x N \sqrt{\gamma} \left(\frac{m^2 M \lambda}{4N} \right)^2 (\dots) \Theta^{ij} \Theta^{kl} \\ &- \frac{M_{\text{P}}^2}{2} \int d^4x \sqrt{\gamma} \left[\lambda \bar{\mathcal{C}}_0 - (D_j \lambda^i) \mathcal{C}^j_i \right] + S_m \end{aligned}$$

FLRW Background Setup

- Ansatz: $\gamma_{ij} = a^2 \delta_{ij}$, $\tilde{\gamma}_{ij} = \tilde{a}^2 \delta_{ij}$, and $X \equiv \frac{\tilde{a}}{a}$.
- Define parameters:

$$H = \frac{\dot{a}}{Na}, \quad H_f = \frac{\dot{\tilde{a}}}{M\tilde{a}}, \quad r = \frac{M}{NX}$$

- Kinematical identity: $\frac{\dot{X}}{X} = MH_f - NH$.

Reducing Matrices

- Spatial matrices are proportional to identity: $\mathcal{K}^i_j = X^{-1}\delta^i_j$, $K^i_j = H\delta^i_j$.
- Θ^{ij} reduces to:

$$\Theta^{ij} = 2J(X)\gamma^{ij} \quad \text{with} \quad J(X) \equiv c_1 X^2 + 2c_2 X + c_3$$

- The scalar constraint reduces to:

$$\bar{\mathcal{C}}_0 = 3m^2 MJ(H - XH_f)$$

Minisuperspace Lagrangian

- Using $K_{ij}K^{ij} - K^2 = -6H^2$, the exact homogeneous Lagrangian is:

$$\begin{aligned} L_{\text{bg}} = & -\frac{3M_{\text{P}}^2 a \dot{a}^2}{N} - \frac{M_{\text{P}}^2 m^2 a^3}{2} [NU(X) + MV(X)] + L_m \\ & + \frac{3M_{\text{P}}^2 m^2 a^3}{2} M\lambda J(X)(XH_f - H) - \frac{3M_{\text{P}}^2 m^4 a^3}{16N} M^2 \lambda^2 J(X)^2 \end{aligned}$$

- Note: λ leaves essential terms in the background action!

Variations: 1. The Lapse

- Friedmann Equation:

$$E_0 \equiv 3M_{\text{P}}^2 H^2 - \rho - \rho_g - \rho_\lambda = 0$$

- Where effective energy densities are:

$$\rho_g = \frac{m^2 M_{\text{P}}^2}{2} U(X)$$

$$\rho_\lambda = -\frac{3M_{\text{P}}^2 m^4 M^2 J^2 \lambda^2}{16N^2} - \frac{3M_{\text{P}}^2 m^2 H M J \lambda}{2N}$$

Variations: 2. The Scalar Multiplier

- Variation w.r.t λ yields the branch equation:

$$E_\lambda = m^2 J(X) \left[\frac{2M}{N} (XH_f - H) - \frac{m^2 M^2}{2N^2} J(X) \lambda \right] = 0$$

- The two branches of MTMG are visible here:

- 1 $J(X) = 0$ (Self-accelerating branch).
- 2 The bracket term vanishes (Normal branch).

Variations: 3. The Scale Factor

- Evaluating $\mathcal{E}_a[L] \equiv \frac{\partial L}{\partial a} - \frac{d}{dt} \frac{\partial L}{\partial \dot{a}}$ gives the acceleration equation:

$$E_1 \equiv \frac{2\dot{H}}{N} + 3H^2 + \frac{P + P_g + P_\lambda}{M_{\text{P}}^2} = 0$$

- P_g and P_λ are effective pressures from the mass potential and the constraints.

Consistency Identity

- Combine equations $E_B \sim \dot{E}_0 + 3HE_0 + \dots = 0$.
- Yields a cubic polynomial in λ :

$$E_B = \lambda(\zeta_1 + \zeta_2\lambda + \zeta_3\lambda^2) = 0$$

- The cancellation of time derivatives $(\dot{H}, \dot{\rho}, \dot{\lambda}, \dot{M})$ is a stringent consistency check.

Self-Accelerating Branch

- On a continuous branch associated with an isolated root, $J(X) = 0$ implies $X = \text{const.}$
- Eliminating c_3 gives:

$$\rho_g = \frac{m^2 M_{\text{P}}^2}{2}(c_4 - 3c_2 X^2 - 2c_1 X^3) = \text{const}, \quad P_g = -\rho_g$$

- On the generic self-accelerating branch, consistency fixes $\lambda = 0$, so $\rho_\lambda = P_\lambda = 0$.
- Recovers GR with a cosmological constant!
- Self-acceleration originates from graviton potential.

Normal Branch

- For $J \neq 0$, bracket term implies $\lambda \propto (XH_f - H)$.
- Substituting into $E_B = 0$ yields a factorization where the regular normal branch is:

$$H = XH_f, \quad \lambda = 0, \quad \rho_\lambda = P_\lambda = 0$$

- Evolution of X is entirely kinematical:

$$\frac{\dot{X}}{NX} = H \left(\frac{M}{NX} - 1 \right) = H(r - 1)$$

- No propagating scalar, but equation of state w_X can differ from Λ CDM.

Scalar Perturbations

- Dust supplies one scalar mode: gauge-invariant density contrast δ_m .
- **Self-accelerating branch**: exactly the GR growth equation.
- **Normal branch**:

$$S_{\delta}^{(2)} = M_{\text{P}}^2 \int dt d^3k N a^3 Q \left[\frac{\dot{\delta}_m^2}{N^2} + 4\pi G_{\text{eff}}(t, k) \rho_m \delta_m^2 \right]$$

- Friction and G_{eff} can differ from GR. Can realize weaker gravity.

Tensor Sector and Gravitational Waves

- Second-order action for tensor modes:

$$S_T^{(2)} = \frac{M_P^2}{8} \sum_{s=+, \times} \int dt d^3x N a^3 \left[\frac{\dot{h}_s^2}{N^2} - \frac{(\partial_i h_s)^2}{a^2} - \mu_T^2 h_s^2 \right]$$

- **Speed of Gravity:** $c_T^2 = 1$; the kinetic and gradient terms have the GR signs.
- **Massive Dispersion:** $\omega^2 = k^2/a^2 + \mu_T^2$ in the adiabatic approximation.
- Avoiding rapid tachyonic growth additionally constrains the sign and time scale of μ_T^2 .

Maple Practice: Background Equations

- The background equations derived are complex and prone to algebraic errors.
- We can automate this using Maple.
- Define L_{bg} in terms of $a(t)$, $N(t)$, $\lambda(t)$ and the prescribed fiducial functions, with $X = \tilde{a}/a$.
- Setup the polynomials $U(X)$, $V(X)$, $J(X)$.

Maple Practice: Euler-Lagrange Operations

- Use the Euler-Lagrange operator to compute variations automatically.
- E.g., $\mathcal{E}_a[L] = \frac{\partial L}{\partial a} - \frac{d}{dt} \frac{\partial L}{\partial \dot{a}}$.
- The supplied worksheet prints the three raw variations and the factored multiplier equation.
- A separate comparison with hand-derived expressions is still needed before calling an equation verified.

Maple Practice: The Consistency Polynomial

- Deriving $E_B = \lambda(\zeta_1 + \zeta_2\lambda + \zeta_3\lambda^2)$ involves cancelling all $\dot{H}, \dot{\rho}, \dots$
- In Maple:
 - Construct the linear combination of $E_0, \dot{E}_0, E_1, E_\lambda, E_\rho$.
 - Call `simplify()` or `collect(expr, lambda)`.
 - Verify that time-derivative terms identically sum to zero.
- The supplied `../Maple_code/mtmg-bg-only_cdx.mpl` does *not* run this full consistency-polynomial check; it stops at the vacuum background variations.

Summary

- A generic nonlinear mass term carries the BD ghost; dRGT removes it but retains five modes and difficult cosmological branches.
- MTMG breaks Lorentz/diffeomorphism invariance to remove dangerous modes.
- Features exactly 2 tensor modes; their mass parameters must still satisfy stability bounds.
- Yields a rich background cosmology (Self-accelerating and Normal branches).
- Has $c_T^2 = 1$ and a potentially testable massive tensor dispersion relation.