

Lecture 2: Quadratic DHOST Theories

From Degeneracy to Cosmology

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Motivation: Beyond General Relativity

- General Relativity (GR) is highly successful but leaves open questions (dark energy, cosmic acceleration).
- Modified gravity introduces additional degrees of freedom, typically a scalar field ϕ .
- How to construct the most general scalar-tensor theory without instabilities?
- A key guiding principle: avoiding the Ostrogradsky ghost.

Horndeski's Theorem

- Discovered in 1974 by G. W. Horndeski.
- The most general scalar-tensor theory with **second-order** equations of motion.
- Second-order EOMs are a sufficient condition to avoid the Ostrogradsky instability.
- For a long time, it was thought to be the most general healthy theory.

The Ostrogradsky Ghost

- Higher-derivative theories generally suffer from the Ostrogradsky ghost.
- Associated with an unbounded Hamiltonian from below.
- Creates a fatal instability (negative kinetic energy).
- Standard wisdom: equations of motion must not exceed second order.

Beyond Horndeski (GLPV)

- Gleyzes, Langlois, Piazza, and Vernizzi (GLPV) realized that second-order EOMs are sufficient, but **not necessary**.
- We can have higher-order derivatives in the action, provided they form a **degenerate kinetic matrix**.
- Degeneracy introduces constraints that eliminate the extra, ghost-like degree of freedom.
- Leads to theories known as “Beyond Horndeski” and eventually DHOST.

DHOST: A New Paradigm

- DHOST = **D**egenerate **H**igher-**O**rders **S**calar-**T**ensor theories.
- They generalize Horndeski and GLPV.
- Economical question: *how many independent accelerations can the equations determine?*
- By ensuring degeneracy, the theory propagates exactly 1 scalar and 2 tensor modes (3 DoFs).

ADM Decomposition: Foliating Spacetime

- To understand the ghost elimination mathematically, we use the Arnowitt-Deser-Misner (ADM) decomposition.
- Spacetime is foliated into 3D spatial hypersurfaces Σ_t parameterized by time t .
- Metric decomposition:

$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt)$$

- N : Lapse function, N^i : Shift vector, h_{ij} : Spatial metric.

ADM Variables and Normal Vector

- The unit normal vector to Σ_t is:

$$n_\mu = (-N, 0, 0, 0), \quad n^\mu = \left(\frac{1}{N}, -\frac{N^i}{N} \right)$$

- Satisfies $n_\mu n^\mu = -1$.
- The spatial metric can also be used as a projector: $h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$.

Extrinsic Curvature

- Extrinsic curvature K_{ij} dictates how the 3D surfaces are embedded in 4D.
- Defined as the Lie derivative of the spatial metric along n^μ :

$$K_{ij} = \frac{1}{2} \mathcal{L}_n h_{ij} = \frac{1}{2N} (\dot{h}_{ij} - D_i N_j - D_j N_i)$$

- Contains the “velocities” of the spatial metric $\dot{h}_{ij}$.

General Relativity in ADM

- Using Gauss-Codazzi relations, the GR action (R) decomposes to:

$$S = \int d^4x N \sqrt{h} \left(R^{(3)} + K_{ij} K^{ij} - K^2 \right)$$

- Note: NO time derivatives of N or N^i appear in the action.

Primary Constraints in GR

- Since $\dot{N}$ and $\dot{N}^i$ are absent, their conjugate momenta vanish:

$$\pi_N = 0, \quad \pi_i = 0$$

- These primary constraints lead to the Hamiltonian and momentum constraints.
- Crucial for GR having only 2 propagating tensor modes.

Second Derivatives of a Scalar Field

- DHOST theories involve second derivatives of the scalar: $\phi_{\mu\nu} \equiv \nabla_\mu \nabla_\nu \phi$.
- We define the kinetic term of the scalar: $X \equiv g^{\mu\nu} \phi_\mu \phi_\nu$.
- The goal is to see how $\phi_{\mu\nu}$ behaves in the ADM language.

Unitary Gauge

- We choose the Unitary Gauge where $\phi = \phi(t)$ depends only on time.
- Hypersurfaces of constant ϕ coincide with spatial foliation Σ_t .
- With the convention $X = g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} < 0$, the normal is:

$$n_\mu = -\frac{\nabla_\mu \phi}{\sqrt{-X}}, \quad X = -\frac{\dot{\phi}^2}{N^2}$$

Decomposing $\phi_{\mu\nu}$

- We expand $\phi_{\mu\nu}$:

$$\phi_{\mu\nu} = \nabla_\mu(-\sqrt{-X} n_\nu) = -n_\nu \nabla_\mu \sqrt{-X} - \sqrt{-X} \nabla_\mu n_\nu$$

- Using $\nabla_\mu n_\nu = K_{\mu\nu} - n_\mu a_\nu$ (where $a_\mu = n^\lambda \nabla_\lambda n_\mu$ is acceleration).

The Appearance of $\dot{N}$

- The projections of $\phi_{\mu\nu}$ yield:

$$\phi_{ij} = -\frac{\dot{\phi}}{N} K_{ij}$$
$$n^\mu n^\nu \phi_{\mu\nu} = \frac{1}{N} \frac{d}{dt} \left(\frac{\dot{\phi}}{N} \right)$$

- Notice that $\frac{d}{dt} \left(\frac{1}{N} \right)$ contains $\dot{N}$.
- $\dot{N}$ explicitly appears in components of $\phi_{\mu\nu}$!

Consequences of $\dot{N}$

- Lagrangians with $\phi_{\mu\nu}\phi^{\mu\nu}$ or $(\square\phi)^2$ will produce $\dot{N}^2$ and $\dot{N}h_{ij}$.
- For a generic linear combination, $\pi_N = \partial\mathcal{L}/\partial\dot{N}$ is non-zero and independent.
- This would activate an additional higher-derivative scalar mode.
- DHOST avoids it when the full kinetic block is degenerate.

The Quadratic Action

- Most general action at most quadratic in $\phi_{\mu\nu}$:

$$S = \int d^4x \sqrt{-g} \left[P + Q \square \phi + F(\phi, X)^{(4)} R + \sum_{l=1}^5 A_l(\phi, X) L_l^{(2)} \right]$$

- F and A_l dictate the highest-derivative terms (the kinetic matrix).

The 5 Basis Operators

- We can form 5 independent quadratic invariants from $\phi_{\mu\nu}$:

$$L_1^{(2)} = \phi_{\mu\nu}\phi^{\mu\nu}$$

$$L_2^{(2)} = (\Box\phi)^2$$

$$L_3^{(2)} = (\Box\phi)\phi^\mu\phi_{\mu\nu}\phi^\nu$$

$$L_4^{(2)} = \phi^\mu\phi_{\mu\rho}\phi^{\rho\nu}\phi_\nu$$

$$L_5^{(2)} = (\phi^\mu\phi_{\mu\nu}\phi^\nu)^2$$

- Horndeski is recovered for specific parameter choices:

$$F = G_4, \quad A_1 = -A_2 = 2G_{4X}, \quad A_3 = A_4 = A_5 = 0$$

- Substituting these relations in the class-Ia formulas gives $A_4 = A_5 = 0$; this is a useful symbolic check.

- Quartic Beyond Horndeski (GLPV) corresponds to:

$$F = G_4$$

$$A_1 = -A_2 = 2G_{4X} + XF_4$$

$$A_3 = -A_4 = 2F_4$$

$$A_5 = 0$$

- Also a special point in the DHOST parameter space.

Universal Kinetic Form

- On an arbitrary foliation, define $A_* = n^\mu \nabla_\mu \phi$. The kinetic part takes the form:

$$L_{\text{kin}} = \mathcal{A} \dot{A}_*^2 + 2\mathcal{B}^{ij} \dot{A}_* K_{ij} + \mathcal{K}^{ij,kl} K_{ij} K_{kl}$$

- Here $\dot{A}_* = t^\mu \nabla_\mu A_*$. In unitary gauge it contains $\dot{N}$; this is useful intuition, but covariant degeneracy must hold before choosing that foliation.
- To remove the extra mode, this kinetic matrix must have a null vector.

Why the Null Tensor Has Only Two Coefficients

- The spatial coefficients contain only h_{ij} and the preferred vector $\hat{A}_i$.
- A covariantly defined scalar-sector kernel tensor must therefore be

$$\mathcal{V}_{ij} = v_1 h_{ij} + v_2 \hat{A}_i \hat{A}_j, \quad \mathcal{B}^{ij} = b_1 h^{ij} + b_2 \hat{A}^i \hat{A}^j.$$

- In a frame with $\hat{A}_i \parallel \hat{z}$, this is $\text{diag}(v_1, v_1, v_1 + v_2 \hat{A}^2)$: residual rotations around $\hat{A}_i$ force the first two entries to agree.
- Vector/TT null tensors would signal an additional degeneracy of the metric sector; they are excluded on the regular branch.

The Tensor Problem Becomes 3×3

Set $s \equiv \hat{A}^2$. Separating the scalar, h^{ij} , and $\hat{A}^i \hat{A}^j$ parts of $\mathbf{M}_{\text{kin}} \mathbf{v} = 0$ gives

$$\underbrace{\begin{pmatrix} \mathcal{A} & 3b_1 + b_2 s & b_1 s + b_2 s^2 \\ b_1 & k_1 + 3k_2 + \frac{1}{2}k_3 s & k_2 s + \frac{1}{2}k_3 s^2 \\ b_2 & \frac{3}{2}k_3 + k_4 + k_5 s & k_1 + (\frac{1}{2}k_3 + k_4)s + k_5 s^2 \end{pmatrix}}_{\mathbf{M}_{\text{scal}}} \begin{pmatrix} v_0 \\ v_1 \\ v_2 \end{pmatrix} = 0.$$

- b_a, k_a are the coefficients of the allowed spatial tensor basis after including FR .
- A non-zero kernel exists iff $\det \mathbf{M}_{\text{scal}} = 0$.

Deriving the Determinant Polynomial

- Expanding the displayed matrix along its first row,

$$\det \mathbf{M}_{\text{scal}} = \mathcal{A}(m_{22}m_{33} - m_{23}m_{32}) \\ - m_{12}(b_1 m_{33} - m_{23}b_2) + m_{13}(b_1 m_{32} - m_{22}b_2).$$

- Insert the explicit b_a, k_a , use $s = X + A_*^2$, and collect in A_*^2 :

$$\det \mathbf{M}_{\text{kin}} \propto D_0(\phi, X) + D_1(\phi, X)A_*^2 + D_2(\phi, X)A_*^4$$

- Covariant degeneracy must hold for every normal n^μ , hence for every allowed A_* : all three coefficients vanish.

The Three Degeneracy Conditions

- Therefore, we need:

$$D_0 = 0, \quad D_1 = 0, \quad D_2 = 0$$

- The first coefficient gives:

$$D_0 = -4(A_1 + A_2)\{XF(2A_1 + XA_4 + 4F_X) - 2F^2 - 8X^2F_X^2\}$$

- The branches of $D_0 = 0$ define the distinct DHOST classes.

A Simple Way to Read D_0

- D_0 is the algebraic intercept of the determinant polynomial: set $A_* = 0$ at fixed X .

$$\mathcal{A}|_{A_*=0} = \frac{A_1 + A_2}{N^2}, \quad \mathcal{B}^{ij}|_{A_*=0} = 0.$$

- The scalar kinetic matrix becomes block diagonal, so

$$D_0 \propto (A_1 + A_2) \times \det(\text{scalar metric block}).$$

- The second factor in the previous slide is that metric-block determinant after the F, F_X terms are included.
- This is only a coefficient-extraction device; $A_* = 0$ is **not** unitary gauge.

DHOST Classes

Class	Defining property	Comment
I	$A_1 + A_2 = 0$	Contains Horndeski and GLPV
II	$F \neq 0, A_1 + A_2 \neq 0$	Scalar/tensor gradient conflict
III	$F = 0$	No standard Einstein kinetic term

- Phenomenologically relevant branch is Class Ia, for which $F - XA_1 \neq 0$.

Class Ia Explicit Form

- Set $A_2 = -A_1$. Solving $D_1 = D_2 = 0$ gives A_4 and A_5 in terms of (F, A_1, A_3) .
- A quadratic Class-Ia theory has **three** arbitrary functions (F, A_1, A_3) , plus lower-derivative ones (P, Q) .

Consistency Checks (Maple Practice)

- Program the A_4, A_5 relations and test distinguished points before using a general result.
- Substitute $A_1 = 2F_X$ and $A_3 = 0$ (Horndeski).
- The software automatically simplifies the numerators to yield $A_4 = A_5 = 0$.
- Substitute GLPV forms and verify $A_4 = -A_3$ and $A_5 = 0$.

Disformal Maps

- Field redefinition:

$$\tilde{g}_{\mu\nu} = C(\phi, X)g_{\mu\nu} + D(\phi, X)\phi_\mu\phi_\nu$$

- Class Ia is closed under invertible disformal transformations.
- In the absence of matter, any Class Ia action can be mapped to a Horndeski representative; choosing a minimally coupled matter metric makes the frame physically relevant.
- Caution: Singular maps can change degrees of freedom (e.g., Mimetic gravity).

The Luminal Subclass (Tensor Speed)

- The tensor kinetic term gives the tensor speed of sound c_T :

$$c_T^2 = \frac{F}{F - XA_1}$$

- Observation of GW170817 and GRB170817A constrains $c_T \approx 1$.
- Requiring luminality on arbitrary cosmological backgrounds, without background-dependent tuning, gives $A_1 = 0$. Class-I degeneracy then gives $A_2 = 0$.

- Setting $A_1 = A_2 = 0$, the remaining relations become:

$$A_4 = \frac{48F_X^2 - 8(F - XF_X)A_3 - X^2A_3^2}{8F}$$
$$A_5 = \frac{(4F_X + XA_3)A_3}{2F}$$

- Surviving action characterized by (P, Q, F, A_3) .
- The supplied `../Maple_code/dhost_bg_only_cdx.mpl` stores these class-Ia relations and checks the homogeneous action variationally; it does not perform the full covariant classification.

Homogeneous Cosmology: Minisuperspace

- Evaluate action on FLRW: $ds^2 = -N^2(t)dt^2 + a^2(t)d\mathbf{x}^2$ and $\phi = \phi(t)$.
- The potentially dangerous $\ddot{\phi}$ and $\dot{a}$ terms combine into a perfect square:

$$\propto -6(F - XA_1) \left[H - \nu \frac{\dot{\phi}}{N^2} \frac{d}{dt} \left(\frac{\dot{\phi}}{N} \right) \right]^2$$

- “Seeing the square” is a valuable background consistency check, but it does not replace the full covariant degeneracy test.

Cosmological Perturbations: EFT Action

- Linear perturbations in Unitary Gauge can be captured by an Effective Field Theory (EFT) framework.
- DHOST introduces additional EFT parameters $(\alpha_L, \beta_1, \beta_2, \beta_3)$.

Class I EFT Degeneracy

- In the EFT language, the Class I degeneracy conditions translate to:

$$\alpha_L = 0, \quad \beta_2 = -6\beta_1^2, \quad \beta_3 = -2\beta_1 [2(1 + \alpha_H) + \beta_1(1 + \alpha_T)]$$

- Eliminates the independent lapse perturbation mode.

Tensor Sector Stability

- The tensor action:

$$S_T^{(2)} = \frac{M^2}{8} \int dt d^3x a^3 \left[\dot{\gamma}_{ij}^2 - (1 + \alpha_T) \frac{(\partial_k \gamma_{ij})^2}{a^2} \right]$$

- Tensor stability requires $M^2 > 0$ and $1 + \alpha_T > 0$.

Scalar Sector and Hidden Constraint

- The EFT action can contain $\delta\dot{N}$, but the class-I degeneracy relations make lapse velocities enter only in a constrained combination.
- The momentum constraint ties δN to the curvature perturbation, and the propagating variable is $\tilde{\zeta} = \zeta - \beta_1 \delta N$.
- After eliminating auxiliaries, only one scalar mode remains.

Scalar Stability Conditions

- Stability requires positivity of the kinetic term ($\mathcal{A}_\zeta$) and sound speed ($\mathcal{B}_\zeta/\mathcal{A}_\zeta$):

$$M^2 \mathcal{A}_\zeta > 0, \quad c_s^2 \equiv \frac{\mathcal{B}_\zeta}{\mathcal{A}_\zeta} > 0$$

- Degeneracy is logically distinct from stability. Degeneracy removes the ghost. Stability ensures the remaining mode is healthy.

Summary

- Quadratic DHOST theories give the general degenerate scalar–tensor action at most quadratic in $\phi_{\mu\nu}$.
- The ADM decomposition reveals the problematic $\dot{N}$ term, neutralized by imposing $D_0 = D_1 = D_2 = 0$.
- Class Ia contains Horndeski and GLPV, and maps between them via disformal transformations.
- GW170817 limits the parameter space to the luminal subclass.
- The supplied Maple background worksheet is a partial algebraic check, not a substitute for the covariant constraint analysis.