

Constrained Modified Gravity

DHOST, U-DHOST, MTMG and VCDM

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Preface

These notes are built around one question: *how can a gravitational theory differ substantially from General Relativity without paying for the modification with an unwanted propagating mode?* The four theories studied here give two sharply different answers. DHOST and U-DHOST theories engineer degenerate kinetic structures. MTMG and VCDM instead alter the constraint algebra so that only the two tensorial gravitational modes remain.

The intended reader is a master's student who has completed a first course in General Relativity, or a beginning doctoral student. Every calculation is organized so that it can be repeated with pencil and paper. Longer algebraic checks are paired with Maple scripts whose names end in `_cdx.mpl`. The conventions follow the original papers whenever possible; every change of convention is stated where it occurs.

The six chapters correspond to six 90-minute meetings. Boxes labelled “Checkpoint” isolate computations worth doing during class. Exercises are part of the narrative rather than an afterthought: several of the essential constraint counts become clear only after one performs them.

Primary sources. The DHOST chapters follow the original construction and Hamiltonian analysis of Refs. [1, 2, 3] and the review [4]. The U-DHOST discussion is based on Refs. [5, 6]. MTMG follows Refs. [7, 8], and VCDM follows Refs. [9, 10].

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CHAPTER 1

Lecture 1: Higher derivatives, constraints, and the ADM toolkit

Ninety-minute map

0–15 min	What a degree of freedom count measures, and why equations of motion can mislead.
15–35 min	Ostrogradsky’s construction for a non-degenerate higher-derivative coordinate.
35–60 min	A degenerate two-variable model: primary and secondary constraints in full detail.
60–80 min	ADM kinematics and the kinetic matrix of a scalar–tensor theory.
80–90 min	Diagnostic checklist and exercises.

The central technical lesson of this course can be stated before any gravitational model is introduced:

Key idea

The order of the Euler-Lagrange equations is not, by itself, a count of physical degrees of freedom. The reliable count comes from the rank of the kinetic map and from the complete set of constraints, including the constraints generated by time evolution.

This distinction is the reason DHOST theories can have higher-order field equations without an Ostrogradsky mode, and it is also the reason MTMG and VCDM can modify gravity while retaining only two local gravitational modes.

1. What exactly is being counted?

For a finite-dimensional Hamiltonian system with a phase space of dimension N_{ph} , let F and S denote the number of independent first-class and second-class constraints. The number of physical configuration-space degrees of freedom is

$$N_{\text{dof}} = \frac{1}{2} (N_{\text{ph}} - 2F - S) . \quad (1.1)$$

A first-class constraint removes one phase-space direction and generates a gauge orbit, so it costs two dimensions. A second-class constraint removes only one. In field theory the same formula applies point by point, provided that zero modes and boundary degrees of freedom are treated separately.

There are three recurrent sources of mistakes.

1. A variable with no time derivative is called “non-dynamical” before checking whether its equation is a constraint or determines another multiplier.

2. A constraint visible at linear order is assumed to survive nonlinearly.
3. Gauge fixing is performed before the constraint algebra is understood, and a genuine equation is then mistaken for a gauge condition.

The examples below are designed to make all three failures difficult.

2. Ostrogradsky's construction

Consider one coordinate $q(t)$ with

$$L = L(q, \dot{q}, \ddot{q}), \quad \frac{\partial^2 L}{\partial \ddot{q}^2} \neq 0. \quad (2.1)$$

The last inequality is the non-degeneracy assumption. Introduce two coordinates

$$Q_1 = q, \quad Q_2 = \dot{q}, \quad (2.2)$$

and their Ostrogradsky momenta

$$P_2 = \frac{\partial L}{\partial \ddot{q}}, \quad P_1 = \frac{\partial L}{\partial \dot{q}} - \frac{d}{dt} \frac{\partial L}{\partial \ddot{q}}. \quad (2.3)$$

These definitions are also read directly from the boundary term in the variation. Integrating by parts twice gives

$$\delta S = \int dt \mathcal{E}_q \delta q + \left[\left(\frac{\partial L}{\partial \dot{q}} - \frac{d}{dt} \frac{\partial L}{\partial \ddot{q}} \right) \delta q + \frac{\partial L}{\partial \ddot{q}} \delta \dot{q} \right]_{t_i}^{t_f}. \quad (2.4)$$

Thus the coefficients of the independent endpoint variations $(\delta q, \delta \dot{q})$ are precisely (P_1, P_2) . This gives a useful variational reason for the apparently asymmetric definitions in (2.3). Non-degeneracy allows one to invert the first relation:

$$\ddot{q} = A(Q_1, Q_2, P_2). \quad (2.5)$$

The Hamiltonian is consequently

$$\begin{aligned} H &= P_1 \dot{Q}_1 + P_2 \dot{Q}_2 - L \\ &= P_1 Q_2 + P_2 A(Q_1, Q_2, P_2) - L(Q_1, Q_2, A(Q_1, Q_2, P_2)). \end{aligned} \quad (2.6)$$

It is affine in P_1 . Holding all other canonical variables fixed and taking $P_1 \rightarrow \pm\infty$ makes H unbounded above and below. This is the Ostrogradsky instability; a modern concise review is Ref. [11].

Common pitfall

The theorem does not say that every Lagrangian containing $\ddot{q}$ is unstable. It assumes that the highest-derivative Hessian is non-singular and that no gauge symmetry or constraint removes the extra canonical pair. The theories in these notes are built precisely by violating that assumption in a controlled way.

2.1. A concrete fourth-order oscillator. Take

$$L = \frac{\varepsilon}{2} \ddot{q}^2 + \frac{1}{2} \dot{q}^2 - \frac{\omega_0^2}{2} q^2, \quad \varepsilon \neq 0. \quad (2.7)$$

Its equation of motion,

$$\varepsilon q^{(4)} - \ddot{q} - \omega_0^2 q = 0, \quad (2.8)$$

requires four initial data and therefore carries two modes. Here $P_2 = \varepsilon \ddot{q}$ and $P_1 = \dot{q} - \varepsilon q^{(3)}$, giving

$$H = P_1 Q_2 + \frac{P_2^2}{2\varepsilon} - \frac{Q_2^2}{2} + \frac{\omega_0^2 Q_1^2}{2}. \quad (2.9)$$

The linear $P_1 Q_2$ term displays the problem without solving the differential equation.

3. Degeneracy removes the extra pair

We now follow the mechanical model used in the original DHOST construction [1, 4]. Let

$$L = \frac{a}{2} \ddot{\phi}^2 + b \ddot{\phi} \dot{q} + \frac{c}{2} \dot{q}^2 + \frac{1}{2} \dot{\phi}^2 - V(\phi, q), \quad (3.1)$$

where a, b, c are constants. Replace ϕ by an independent variable Q and impose $Q = \phi$ with λ :

$$L = \frac{a}{2} \dot{Q}^2 + b \dot{Q} \dot{q} + \frac{c}{2} \dot{q}^2 + \frac{1}{2} Q^2 - V(\phi, q) - \lambda(Q - \phi). \quad (3.2)$$

The kinetic matrix for (Q, q) is

$$M = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \quad \det M = ac - b^2. \quad (3.3)$$

3.1. The non-degenerate case. Varying Q and q gives

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} \ddot{Q} \\ \ddot{q} \end{pmatrix} = \begin{pmatrix} Q - \lambda \\ -V_q \end{pmatrix}. \quad (3.4)$$

If $ac - b^2 \neq 0$, both accelerations are determined. Together with $\dot{\phi} = Q$ and $\dot{\lambda} = -V_\phi$, the initial data are $(Q, \dot{Q}, q, \dot{q}, \phi, \lambda)$: three degrees of freedom. One is the unwanted higher-derivative mode.

3.2. The degenerate case, step by step. Assume $c \neq 0$ and

$$ac - b^2 = 0. \quad (3.5)$$

Multiply the second row of (3.4) by b/c and subtract it from the first. The accelerations cancel, leaving

$$\lambda = Q + \frac{b}{c} V_q. \quad (3.6)$$

This is a Lagrangian constraint. In the Hamiltonian description it is the configuration-space form of the *secondary* constraint generated by preserving the primary momentum constraint below. Define

$$x = q + \frac{b}{c} \dot{\phi} = q + \frac{b}{c} Q. \quad (3.7)$$

The second row of (3.4) becomes

$$c \ddot{x} + V_q = 0. \quad (3.8)$$

Differentiate (3.6), use $\dot{\lambda} = -V_\phi$, substitute $q = x - (b/c)\dot{\phi}$, and obtain

$$\left(1 - \frac{b^2}{c^2} V_{qq}\right) \ddot{\phi} + \frac{b}{c} V_{qq} \dot{x} + \frac{b}{c} V_{q\phi} \dot{\phi} + V_\phi = 0. \quad (3.9)$$

Equations (3.8) and (3.9) form a second-order system for (x, ϕ) . Four initial data suffice: the system has two, not three, degrees of freedom.

The primary and secondary constraints

The canonical momenta following from (3.2) are

$$p_Q = a\dot{Q} + b\dot{q}, \quad p_q = b\dot{Q} + c\dot{q}, \quad p_\phi = \lambda, \quad p_\lambda = 0. \quad (3.10)$$

Under (3.5), the first two obey

$$\Psi_1 \equiv p_Q - \frac{b}{c}p_q \approx 0. \quad (3.11)$$

Preservation by the total Hamiltonian yields

$$\Psi_2 \equiv Q - \lambda + \frac{b}{c}V_q \approx 0, \quad \{\Psi_1, \Psi_2\} = -1 + \frac{b^2}{c^2}V_{qq}. \quad (3.12)$$

Its configuration-space form is (3.6). Away from $1 - (b^2/c^2)V_{qq} = 0$, the pair is second class and removes one canonical pair. On that exceptional locus the Dirac algorithm must be continued rather than using the generic count. The elementary constraints $p_\lambda \approx 0$ and $p_\phi - \lambda \approx 0$ only remove the auxiliary implementation of $Q = \dot{\phi}$. This is the pattern to remember: a null vector of the Hessian produces a primary constraint; its time preservation produces the secondary constraint that removes the Ostrogradsky pair.

Checkpoint

The combination $x = q + (b/c)\dot{\phi}$ contains a velocity. This is not an ordinary point transformation, yet it is a useful variable at the level of the reduced equations. Never use such a transformation to declare two actions equivalent before verifying invertibility and boundary terms.

4. ADM geometry in one page

Let spacetime be foliated by spacelike hypersurfaces Σ_t . The metric is written as

$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \quad (4.1)$$

with future-directed unit normal n^μ , lapse N , shift N^i , and induced metric $h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$. Our extrinsic-curvature convention is

$$K_{ij} = \frac{1}{2N} \left(\dot{h}_{ij} - D_i N_j - D_j N_i \right). \quad (4.2)$$

Up to a boundary term, Gauss–Codazzi gives

$$\sqrt{-g} {}^{(4)}R = N\sqrt{h} \left({}^{(3)}R + K_{ij}K^{ij} - K^2 \right) + \text{boundary}. \quad (4.3)$$

The Einstein–Hilbert momentum is therefore

$$\pi^{ij} = \frac{M_P^2 \sqrt{h}}{2} (K^{ij} - K h^{ij}). \quad (4.4)$$

The six components of h_{ij} give a 12-dimensional phase space. The Hamiltonian and three momentum constraints are first class. Equation (1.1) yields

$$N_{\text{dof}}^{\text{gr}} = \frac{1}{2}(12 - 2 \times 4) = 2. \quad (4.5)$$

Those two modes are the transverse-traceless tensor polarizations. This count is the reference point for MTMG and VCDM.

5. Where the DHOST kinetic matrix comes from

For a scalar field define an independent one-form

$$Q_\mu \equiv \nabla_\mu \phi, \quad (5.1)$$

and split it into normal and spatial pieces,

$$A_* \equiv n^\mu Q_\mu, \quad \widehat{A}_\mu \equiv h_\mu{}^\nu Q_\nu. \quad (5.2)$$

Since $h_\mu{}^\nu = \delta_\mu{}^\nu + n_\mu n^\nu$, these definitions imply

$$\widehat{A}_\mu = Q_\mu + A_* n_\mu, \quad Q_\mu = -A_* n_\mu + \widehat{A}_\mu. \quad (5.3)$$

The two pieces are orthogonal, $n^\mu \widehat{A}_\mu = 0$, while $n_\mu n^\mu = -1$. Squaring the last decomposition therefore gives, one step at a time,

$$X \equiv Q_\mu Q^\mu = A_*^2 n_\mu n^\mu - 2A_* n^\mu \widehat{A}_\mu + \widehat{A}_\mu \widehat{A}^\mu = -A_*^2 + \widehat{A}_i \widehat{A}^i. \quad (5.4)$$

To identify the velocities, write the coordinate-time vector as $t^\mu = N n^\mu + N^\mu$, define $\dot{A}_* \equiv t^\mu \nabla_\mu A_*$, and denote by $a_\mu \equiv n^\nu \nabla_\nu n_\mu$ the normal acceleration. Using $\nabla_{[\mu} Q_{\nu]} = 0$, the complete 3 + 1 decomposition is

$$\begin{aligned} \nabla_\mu Q_\nu &= D_\mu \widehat{A}_\nu - A_* K_{\mu\nu} + n_\mu (K_{\nu\rho} \widehat{A}^\rho - D_\nu A_*) + n_\nu (K_{\mu\rho} \widehat{A}^\rho - D_\mu A_*) \\ &\quad + \frac{n_\mu n_\nu}{N} (\dot{A}_* - N^i D_i A_* - N \widehat{A}_i a^i). \end{aligned} \quad (5.5)$$

This equation explains why there are only two independent kinds of velocity: $\dot{A}_*$ and K_{ij} , the latter containing $\dot{h}_{ij}$. In particular, $\widehat{A}_i$ is not independent; the integrability condition $\nabla_{[\mu} Q_{\nu]} = 0$ rewrites it in terms of spatial derivatives, A_* , the lapse, and the shift. Keeping only the terms that contain the two velocities gives

$$(\nabla_{(\mu} Q_{\nu)})_{\text{kin}} = \lambda_{\mu\nu} \dot{A}_* + \Lambda_{\mu\nu}{}^{ij} K_{ij}, \quad (5.6)$$

where

$$\lambda_{\mu\nu} = \frac{1}{N} n_\mu n_\nu, \quad \Lambda_{\mu\nu}{}^{ij} = -A_* h_{(\mu}{}^i h_{\nu)}{}^j + 2n_{(\mu} h_{\nu)}{}^i \widehat{A}^j. \quad (5.7)$$

The apparently missing pieces in (5.6) are precisely the shift, acceleration, and spatial-derivative terms displayed in (5.5); they do not contribute to the Hessian.

Now write the part of a quadratic action that contains two copies of ∇Q as

$$L_{\nabla Q} = C^{\mu\nu\rho\sigma} (\nabla_\mu Q_\nu) (\nabla_\rho Q_\sigma). \quad (5.8)$$

Substituting (5.6) and multiplying the two brackets produces exactly three terms:

$$\begin{aligned} (L_{\nabla Q})_{\text{kin}} &= \underbrace{C^{\mu\nu\rho\sigma} \lambda_{\mu\nu} \lambda_{\rho\sigma}}_{\mathcal{A}} \dot{A}_*^2 + 2 \underbrace{C^{\mu\nu\rho\sigma} \lambda_{\mu\nu} \Lambda_{\rho\sigma}{}^{ij}}_{\mathcal{B}^{ij}} \dot{A}_* K_{ij} \\ &\quad + \underbrace{C^{\mu\nu\rho\sigma} \Lambda_{\mu\nu}{}^{ij} \Lambda_{\rho\sigma}{}^{kl}}_{\mathcal{K}^{ijkl}} K_{ij} K_{kl}. \end{aligned} \quad (5.9)$$

The term $F(\phi, X)^{(4)}R$, after Gauss–Codazzi and one integration by parts, shifts $\mathcal{B}^{ij}$ and $\mathcal{K}^{ij,kl}$ but creates no new kind of velocity. Consequently the full kinetic sector always has the form

$$L_{\text{kin}} = \mathcal{A}\dot{A}_*^2 + 2\mathcal{B}^{ij}\dot{A}_*K_{ij} + \mathcal{K}^{ij,kl}\dot{K}_{ij}K_{kl}. \quad (5.10)$$

This is the field-theory analogue of (3.2).

The null-vector test

Collect the velocities into $Y = (\dot{A}_*, K_{ij})$. Up to an irrelevant overall factor, their Hessian is the block matrix

$$M_{\text{kin}} = \begin{pmatrix} \mathcal{A} & \mathcal{B}^{kl} \\ \mathcal{B}^{ij} & \mathcal{K}^{ij,kl} \end{pmatrix}. \quad (5.11)$$

The numbers v_0 and $\mathcal{V}_{ij}$ are not background fields or new perturbations: they are the scalar and symmetric-tensor components of a candidate kernel vector of this matrix. Thus a non-zero pair $(v_0, \mathcal{V}_{ij})$ must satisfy

$$v_0\mathcal{A} + \mathcal{B}^{ij}\mathcal{V}_{ij} = 0, \quad v_0\mathcal{B}^{ij} + \mathcal{K}^{ij,kl}\mathcal{V}_{kl} = 0. \quad (5.12)$$

The overall normalization of the pair is arbitrary. These relations are important because the momenta have the schematic form

$$p_* = 2\mathcal{A}\dot{A}_* + 2\mathcal{B}^{ij}\dot{K}_{ij} + \dots, \quad \pi^{ij} = 2\mathcal{B}^{ij}\dot{A}_* + 2\mathcal{K}^{ij,kl}\dot{K}_{kl} + \dots. \quad (5.13)$$

Contracting them with $(v_0, \mathcal{V}_{ij})$ cancels every velocity. The result is a primary phase-space constraint, whose time preservation supplies the secondary constraint that removes the Ostrogradsky canonical pair. A covariant DHOST theory passes this test for an arbitrary foliation; U-DHOST requires it only on the unitary foliation defined by the scalar.

6. A reusable diagnostic protocol

When presented with a modified-gravity action, use the following order.

- Step 1.** Choose a foliation and list every variable carrying a normal time derivative.
- Step 2.** Compute the Hessian before using background equations or fixing a gauge.
- Step 3.** Find its null vectors and translate them into primary constraints.
- Step 4.** Preserve every primary constraint in time. Classify all resulting constraints by their Poisson brackets.
- Step 5.** Count with (1.1).
- Step 6.** Only then expand about FLRW and test ghosts, gradients, tensor speed, and strong-coupling limits.

Common pitfall

A Hessian evaluated only on FLRW can have a smaller rank than the full Hessian. Such an “accidental” degeneracy may signal strong coupling rather than a healthy theory. DHOST degeneracy is covariant; U-DHOST degeneracy is restricted to configurations with timelike scalar gradient and comes with an elliptic boundary-value problem.

7. Exercises

1.1: Complete the Dirac count

For the degenerate mechanical model, compute the Poisson bracket of the primary constraint (3.11) with the canonical Hamiltonian and show that it reproduces (3.6). Under what condition on V_{qq} does the constraint pair cease to be independent?

1.2: GR in extended phase space

Repeat the GR count while including (N, N^i) and their four conjugate momenta. Identify the eight first-class constraints and verify that the answer is still two.

1.3: A square is a null direction

Show that

$$L_{\text{kin}} = \frac{c}{2} \left(\dot{q} + \frac{b}{c} \dot{Q} \right)^2 \quad (7.1)$$

reproduces the kinetic terms of (3.2) precisely when $ac = b^2$. Read the null vector directly from this square.

Short answers

For Exercise 1.2, the extended phase space has dimension 20: twelve dimensions from (h_{ij}, π^{ij}) and eight from lapse, shift, and their momenta. The four primary constraints $\pi_N \approx 0$, $\pi_i \approx 0$ and the four secondary Hamiltonian/momentum constraints are first class, hence $(20 - 2 \times 8)/2 = 2$. For Exercise 1.3 the null vector in the $(\dot{Q}, \dot{q})$ basis is proportional to $(1, -b/c)$.

CHAPTER 2

Lecture 2: Quadratic DHOST theories, from degeneracy to cosmology

Ninety-minute map

0–15 min	The covariant basis of quadratic scalar–tensor operators.
15–45 min	The 3 + 1 kinetic matrix and the three degeneracy conditions.
45–60 min	Class Ia, Horndeski, beyond Horndeski, and disformal maps.
60–75 min	The luminal subclass selected by gravitational waves.
75–90 min	FLRW background and the quadratic action for perturbations.

Horndeski’s original criterion was elegant: require second-order field equations. Degeneracy asks a more economical question: *how many independent accelerations can the equations determine?* This change of question enlarges the theory space while preserving one scalar and two tensor modes [12, 13, 1].

1. The quadratic covariant basis

Throughout this chapter

$$\phi_\mu \equiv \nabla_\mu \phi, \quad \phi_{\mu\nu} \equiv \nabla_\mu \nabla_\nu \phi, \quad X \equiv g^{\mu\nu} \phi_\mu \phi_\nu. \quad (1.1)$$

Thus $X < 0$ on a homogeneous configuration with timelike scalar gradient. The most general action at most quadratic in $\phi_{\mu\nu}$ is

$$S = \int d^4x \sqrt{-g} \left[P(\phi, X) + Q(\phi, X) \square \phi + F(\phi, X)^{(4)} R + \sum_{I=1}^5 A_I(\phi, X) L_I^{(2)} \right], \quad (1.2)$$

where

$$\begin{aligned} L_1^{(2)} &= \phi_{\mu\nu} \phi^{\mu\nu}, & L_2^{(2)} &= (\square \phi)^2, \\ L_3^{(2)} &= (\square \phi) \phi^\mu \phi_{\mu\nu} \phi^\nu, & L_4^{(2)} &= \phi^\mu \phi_{\mu\rho} \phi^{\rho\nu} \phi_\nu, \\ L_5^{(2)} &= (\phi^\mu \phi_{\mu\nu} \phi^\nu)^2. \end{aligned} \quad (1.3)$$

The functions P and Q do not enter the highest-derivative Hessian. The degeneracy problem is therefore a set of algebraic relations among F and A_I , evaluated at each (ϕ, X) .

Why exactly five operators?

$\phi_{\mu\nu}$ is symmetric. A quadratic scalar can contract the two copies with metrics only, producing L_1 and L_2 ; with two gradients, producing L_3 and L_4 ; or with four gradients,

producing L_5 . Any other index arrangement is related to these by symmetry and relabelling of dummy indices.

1.1. Landmarks inside the basis. The quartic Horndeski sector is recovered for

$$F = G_4, \quad A_1 = -A_2 = 2G_{4X}, \quad A_3 = A_4 = A_5 = 0. \quad (1.4)$$

Quartic beyond Horndeski, also called GLPV, corresponds to

$$F = G_4, \quad A_1 = -A_2 = 2G_{4X} + XF_4, \quad A_3 = -A_4 = 2F_4, \quad A_5 = 0. \quad (1.5)$$

Both are special points in the same DHOST class. This fact is not visible from the differential order of their equations; it is visible from the null vector of their kinetic matrix.

2. Building the kinetic matrix

Use the split of [Section 4](#) and introduce Q_μ as an independent field, enforcing $Q_\mu = \nabla_\mu \phi$ with a multiplier. The kinetic part takes the universal form

$$L_{\text{kin}} = \mathcal{A} \dot{A}_*^2 + 2\mathcal{B}^{ij} \dot{A}_* K_{ij} + \mathcal{K}^{ij,kl} K_{ij} K_{kl}. \quad (2.1)$$

Because the only preferred spatial vector is $\widehat{A}_i$, a null tensor that is constructed covariantly and algebraically from the background data must have the form

$$\mathcal{V}_{ij} = v_1 h_{ij} + v_2 \widehat{A}_i \widehat{A}_j. \quad (2.2)$$

Indeed, h_{ij} and $\widehat{A}_i$ are the only spatial tensors available; introducing any other direction would make the putative null vector depend on data absent from the action. Equivalently, in a frame where $\widehat{A}_i$ points along the third axis, $\mathcal{V}_{ij} = \text{diag}(v_1, v_1, v_1 + v_2 \widehat{A}^2)$, as required by the residual rotations around that axis. Possible vector or transverse-traceless null tensors would describe an additional degeneracy of the metric sector itself. They are absent on the regular scalar-tensor branches considered here.

The same symmetry fixes the most general coefficient tensors. Introduce local shorthand b_a and k_a for their scalar coefficients, including the contribution of $F^{(4)} R$:

$$\mathcal{B}^{ij} = b_1 h^{ij} + b_2 \widehat{A}^i \widehat{A}^j, \quad (2.3)$$

$$\begin{aligned} \mathcal{K}^{ij,kl} = & k_1 h^{i(k} h^{l)j} + k_2 h^{ij} h^{kl} + \frac{k_3}{2} \left(\widehat{A}^i \widehat{A}^j h^{kl} + \widehat{A}^k \widehat{A}^l h^{ij} \right) \\ & + \frac{k_4}{2} \left(\widehat{A}^i \widehat{A}^{(k} h^{l)j} + \widehat{A}^j \widehat{A}^{(k} h^{l)i} \right) + k_5 \widehat{A}^i \widehat{A}^j \widehat{A}^k \widehat{A}^l. \end{aligned} \quad (2.4)$$

These b_a are only temporary coefficients and are unrelated to the EFT parameters β_a introduced later.

Set $s \equiv \widehat{A}_i \widehat{A}^i$. Substituting (2.2)–(2.4) into the two null-vector equations of Lecture 1 and separating the coefficients of 1, h^{ij} , and $\widehat{A}^i \widehat{A}^j$ gives

$$\underbrace{\begin{pmatrix} \mathcal{A} & 3b_1 + b_2 s & b_1 s + b_2 s^2 \\ b_1 & k_1 + 3k_2 + \frac{1}{2} k_3 s & k_2 s + \frac{1}{2} k_3 s^2 \\ b_2 & \frac{3}{2} k_3 + k_4 + k_5 s & k_1 + (\frac{1}{2} k_3 + k_4) s + k_5 s^2 \end{pmatrix}}_{M_{\text{scal}}} \begin{pmatrix} v_0 \\ v_1 \\ v_2 \end{pmatrix} = 0. \quad (2.5)$$

Thus it is the null-vector problem, rather than a choice of three physical velocities, that reduces the tensorial calculation to a 3×3 matrix. The associated velocity combinations are $(\dot{A}_*, K, K_{ij} \hat{A}^i \hat{A}^j)$.

For completeness, if m_{ab} denotes the entries of (2.5), expansion along its first row gives

$$\det M_{\text{scal}} = \mathcal{A}(m_{22}m_{33} - m_{23}m_{32}) - m_{12}(b_1m_{33} - m_{23}b_2) + m_{13}(b_1m_{32} - m_{22}b_2). \quad (2.6)$$

One now inserts the explicit b_a and k_a obtained from the five operators $L_I^{(2)}$ and from $F^{(4)}R$.

After using $s = X + A_*^2$, the higher powers cancel and the relevant scalar determinant, up to non-zero regular tensor/vector factors and an overall convention, is the polynomial

$$\det M_{\text{kin}} \propto D_0(\phi, X) + D_1(\phi, X)A_*^2 + D_2(\phi, X)A_*^4. \quad (2.7)$$

A covariant degeneracy must hold independently of the foliation, hence

$$D_0 = 0, \quad D_1 = 0, \quad D_2 = 0. \quad (2.8)$$

The first coefficient already reveals the classification:

$$D_0 = -4(A_1 + A_2) \{XF(2A_1 + XA_4 + 4F_X) - 2F^2 - 8X^2F_X^2\}. \quad (2.9)$$

The expressions for D_1 and D_2 are longer and are not needed for the classification logic at this point; they are recorded, together with the unitary-adapted c_{D1}, c_{D2}, c_U notation, in [Appendix 5](#).

Why three conditions, not one

On a particular slicing, A_* is a fixed function and one could arrange a single cancellation among D_0, D_1, D_2 . But changing the normal n^μ changes A_* while leaving the covariant action unchanged. A null direction that exists in every frame therefore requires every coefficient of the polynomial to vanish. Imposing only the value of (2.7) on the unitary foliation leads instead to U-DHOST, studied in [Chapter 3](#).

There is also a simple way to understand the factorization of D_0 . Algebraically, D_0 is the constant term of the polynomial, obtained by setting $A_* = 0$ at fixed X . At that point the explicit quadratic-DHOST coefficients obey

$$\mathcal{A}|_{A_*=0} = \frac{A_1 + A_2}{N^2}, \quad \mathcal{B}^{ij}|_{A_*=0} = 0. \quad (2.10)$$

The scalar kinetic matrix is therefore block diagonal: its determinant is $(A_1 + A_2)$ times the determinant of the scalar metric block. Including the F and F_X contributions to that metric block yields precisely the second factor in (2.9). This explains the two branches without memorizing the long expression. The substitution $A_* = 0$ is an algebraic device for extracting the polynomial coefficient; for $X < 0$ it is not a real foliation, and in particular it is not unitary gauge.

When the metric block $\mathcal{K}$ is invertible, the same statement can be written as the Schur-complement test

$$\mathcal{A} - \mathcal{B}^{ij}(\mathcal{K}^{-1})_{ijkl}\mathcal{B}^{kl} = 0. \quad (2.11)$$

Clearing the denominator and collecting in A_*^2 reproduces D_0, D_1, D_2 . This route is often the shortest symbolic derivation.

The branches of $D_0 = 0$ define three broad classes:

Class	Defining property	Comment
I	$A_1 + A_2 = 0$	contains Horndeski and GLPV
II	$F \neq 0, A_1 + A_2 \neq 0$	scalar/tensor gradient conflict in EFT
III	$F = 0$	no standard Einstein kinetic term

The phenomenologically relevant branch is class Ia, for which $F - XA_1 \neq 0$.

3. Class Ia in explicit form

Set

$$A_2 = -A_1, \quad F - XA_1 \neq 0. \quad (3.1)$$

Solving $D_1 = D_2 = 0$ for A_4 and A_5 gives

$$A_4 = \frac{1}{8(F - XA_1)^2} \left[-16XA_1^3 + 4(3F + 16XF_X)A_1^2 - X^2FA_3^2 \right. \\ \left. - (16X^2F_X - 12XF)A_1A_3 - 16F_X(3F + 4XF_X)A_1 \right. \\ \left. + 8F(XF_X - F)A_3 + 48FF_X^2 \right], \quad (3.2)$$

$$A_5 = \frac{(4F_X - 2A_1 + XA_3)(-2A_1^2 - 3XA_1A_3 + 4F_XA_1 + 4FA_3)}{8(F - XA_1)^2}. \quad (3.3)$$

Thus a quadratic class-Ia theory is specified by three arbitrary functions, which may be chosen as (F, A_1, A_3) , in addition to the lower-derivative functions (P, Q) .

The denominator is physical. The tensor kinetic coefficient is proportional to $F - XA_1$. If it vanishes, one is not allowed simply to take a limit of (3.2)–(3.3); the rank of the metric kinetic sector changes and the theory belongs to subclass Ib.

3.1. Two consistency checks. For Horndeski, insert $A_1 = 2F_X$ and $A_3 = 0$. The two numerators reduce to $A_4 = A_5 = 0$, as required by (1.4). For GLPV, insert (1.5); after simplification one obtains $A_4 = -A_3$ and $A_5 = 0$. These are valuable algebraic tests for a computer implementation: they exercise every term in (3.2).

2.1: Verify the Horndeski point

Perform the Horndeski substitution explicitly in (3.2). Factor the numerator before expanding. Why is the assumption $F - 2XF_X \neq 0$ needed for this particular parametrization but not for the definition of Horndeski theory itself?

4. Disformal maps

Consider an invertible field redefinition

$$\tilde{g}_{\mu\nu} = C(\phi, X)g_{\mu\nu} + D(\phi, X)\phi_\mu\phi_\nu. \quad (4.1)$$

For the algebraic metric map to be invertible one requires, locally,

$$C \neq 0, \quad C + DX \neq 0, \quad (4.2)$$

together with the non-vanishing Jacobian associated with the X dependence. An invertible change of variables cannot create a physical degree of freedom, even though it can turn second-order equations into higher-order ones.

Class Ia is closed under invertible conformal–disformal transformations, and in the absence of matter any class-Ia action can be mapped to a Horndeski representative [3]. This does *not* make all frames phenomenologically identical once one declares which metric minimally couples to matter. Moving the gravitational action to a simple frame generally moves complexity into the matter coupling.

Invertible versus singular maps

When the disformal Jacobian vanishes, constraint counting can change. A singular transformation is not a harmless relabelling and must be analysed as a new theory. The mimetic construction is the standard warning example.

5. The luminal quadratic subclass

On the unitary foliation, the tensor-relevant terms are

$$L_{\text{tensor}} \supset (F - X A_1) K_{ij} K^{ij} + F {}^{(3)}R. \quad (5.1)$$

Therefore

$$M^2 = 2(F - X A_1), \quad c_T^2 = \frac{F}{F - X A_1}. \quad (5.2)$$

The near-simultaneous observation of GW170817 and its electromagnetic counterpart constrains c_T extremely close to unity at the observed frequencies [14, 15, 16]. Requiring luminal propagation on arbitrary cosmological backgrounds gives

$$A_1 = A_2 = 0, \quad (5.3)$$

and, in an extension that also contains terms cubic in $\phi_{\mu\nu}$, removes those cubic DHOST operators if the requirement is imposed without background-dependent tuning. The present chapter remains restricted to the quadratic sector. Equations (3.2)–(3.3) become

$$A_4 = \frac{48F_X^2 - 8(F - XF_X)A_3 - X^2 A_3^2}{8F}, \quad (5.4)$$

$$A_5 = \frac{(4F_X + X A_3)A_3}{2F}. \quad (5.5)$$

The surviving covariant action is described by four free functions (P, Q, F, A_3) ; A_4 and A_5 are fixed by degeneracy.

A transparent one-function deformation

If $F = M_{\text{P}}^2/2$ is constant, then

$$A_4 = -A_3 - \frac{X^2 A_3^2}{4M_{\text{P}}^2}, \quad A_5 = \frac{X A_3^2}{M_{\text{P}}^2}. \quad (5.6)$$

Taking $A_3 \rightarrow 0$ returns Einstein gravity plus the kinetic-braiding sector $P + Q\Box\phi$. This family is useful for testing symbolic codes because the degeneracy relations remain nontrivial while $F_X = 0$.

The observational statement has an effective-field-theory caveat. Dark energy models often have a strong-coupling scale not far, in logarithmic terms, from interferometer frequencies. A UV completion might restore luminosity at high frequency. The clean conclusion is conditional: within the assumed EFT regime and without fine-tuned cancellations, the quadratic luminal subclass above is selected.

6. Homogeneous cosmology: seeing the square

Set

$$ds^2 = -N^2(t)dt^2 + a^2(t)\delta_{ij}dx^i dx^j, \quad \phi = \phi(t), \quad X = -\frac{\dot{\phi}^2}{N^2}. \quad (6.1)$$

For class Ia, after integrations by parts, the homogeneous action can be written as [4]

$$\begin{aligned} \bar{S} = \int dt N a^3 \Bigg\{ & -6(F - X A_1) \left[H - \mathcal{V} \frac{\dot{\phi}}{N^2} \frac{d}{dt} \left(\frac{\dot{\phi}}{N} \right) \right]^2 \\ & - 3(Q + 2F_\phi) H \frac{\dot{\phi}}{N} - \frac{Q}{N} \frac{d}{dt} \left(\frac{\dot{\phi}}{N} \right) + P \Bigg\}, \end{aligned} \quad (6.2)$$

where $H = \dot{a}/(Na)$ and

$$\mathcal{V} = \frac{4F_X + X A_3 - 2A_1}{4(F - X A_1)}. \quad (6.3)$$

The potentially dangerous $\dot{a}$ and $\ddot{\phi}$ terms occur in one square. That is the minisuperspace shadow of the full null vector; it is a useful check, but it is not a substitute for the full degeneracy calculation.

Define a new scale factor b by

$$a = e^{\lambda(\phi, X)} b, \quad \lambda_X = -\frac{1}{2}\mathcal{V}. \quad (6.4)$$

Since $\dot{X}$ contains $\ddot{\phi}$, the $\lambda_X \dot{X}$ term in $\dot{a}/a$ absorbs the second derivative inside the square. In terms of b the background action contains at most first derivatives after one final integration by parts. This construction makes the reduced second-order system explicit.

Do not vary after setting $N = 1$

The lapse equation is the Hamiltonian constraint. Because X and $d(\dot{\phi}/N)/dt$ depend on N and $\dot{N}$, fixing $N = 1$ before variation discards terms. The safe order is: keep $N(t)$, vary, simplify with the degeneracy relations, and only then choose cosmic time.

7. Cosmological perturbations in EFT language

In unitary gauge the scalar is uniform, and the action is a functional of N , K_{ij} , ${}^{(3)}R_{ij}$ and t . To quadratic order around flat FLRW, all DHOST theories can be encoded as

$$S^{(2)} = \frac{1}{2} \int dt d^3x a^3 M^2 \left\{ \delta K_{ij} \delta K^{ij} - \left(1 + \frac{2}{3} \alpha_L \right) \delta K^2 + (1 + \alpha_T) \delta_2(\sqrt{h} {}^{(3)}R) / a^3 \right. \\ \left. + H^2 \alpha_K \delta N^2 + 4H \alpha_B \delta K \delta N + (1 + \alpha_H) {}^{(3)}R \delta N \right. \\ \left. + 4\beta_1 \delta K \delta \dot{N} + \beta_2 (\delta \dot{N})^2 + \frac{\beta_3}{a^2} (\partial_i \delta N)^2 \right\}. \quad (7.1)$$

Here α_K is kineticity, α_B kinetic braiding, $\alpha_M = H^{-1} d \ln M^2 / dt$ Planck-mass running, $\alpha_T = c_T^2 - 1$, and α_H measures the beyond-Horndeski operator. The coefficients $(\alpha_L, \beta_1, \beta_2, \beta_3)$ are needed for general DHOST theories [17, 18].

For class I, degeneracy becomes the compact set

$$\alpha_L = 0, \quad \beta_2 = -6\beta_1^2, \quad \beta_3 = -2\beta_1 [2(1 + \alpha_H) + \beta_1(1 + \alpha_T)]. \quad (7.2)$$

The conspicuous $(\delta \dot{N})^2$ term does not give the lapse an independent mode, because all lapse velocities combine with the curvature perturbation.

7.1. Tensor sector. For $h_{ij} = a^2(\delta_{ij} + \gamma_{ij})$ with $\partial_i \gamma_{ij} = \gamma_{ii} = 0$,

$$S_T^{(2)} = \frac{M^2}{8} \int dt d^3x a^3 \left[\dot{\gamma}_{ij}^2 - (1 + \alpha_T) \frac{(\partial_k \gamma_{ij})^2}{a^2} \right]. \quad (7.3)$$

Tensor stability requires

$$M^2 > 0, \quad 1 + \alpha_T > 0. \quad (7.4)$$

The propagation equation is

$$\ddot{\gamma}_{ij} + (3 + \alpha_M) H \dot{\gamma}_{ij} + (1 + \alpha_T) \frac{k^2}{a^2} \gamma_{ij} = 0. \quad (7.5)$$

7.2. Scalar sector and the hidden constraint. Write

$$N = 1 + \delta N, \quad N^i = \delta^{ij} \partial_j \psi, \quad h_{ij} = a^2 e^{2\zeta} \delta_{ij}. \quad (7.6)$$

For class I define

$$\tilde{\zeta} \equiv \zeta - \beta_1 \delta N. \quad (7.7)$$

The momentum constraint obtained by varying ψ is algebraic in the lapse:

$$\delta N = \frac{\dot{\tilde{\zeta}}}{H(1 + \alpha_B) - \dot{\beta}_1}. \quad (7.8)$$

After substituting this relation and integrating by parts,

$$S_\zeta^{(2)} = \frac{M^2}{2} \int dt d^3x a^3 \left[\mathcal{A}_\zeta \dot{\tilde{\zeta}}^2 - \mathcal{B}_\zeta \frac{(\partial_i \tilde{\zeta})^2}{a^2} \right], \quad (7.9)$$

where

$$\mathcal{A}_\zeta = \frac{1}{(1 + \alpha_B - \dot{\beta}_1/H)^2} \left[\alpha_K + 6\alpha_B^2 - \frac{6}{a^3 H^2 M^2} \frac{d}{dt} (a^3 H M^2 \alpha_B \beta_1) \right], \quad (7.10)$$

$$\mathcal{B}_\zeta = -2(1 + \alpha_T) + \frac{2}{a M^2} \frac{d}{dt} \left[\frac{a M^2 [1 + \alpha_H + \beta_1(1 + \alpha_T)]}{H(1 + \alpha_B) - \dot{\beta}_1} \right]. \quad (7.11)$$

The scalar no-ghost and gradient conditions are

$$M^2 \mathcal{A}_\zeta > 0, \quad c_s^2 \equiv \frac{\mathcal{B}_\zeta}{\mathcal{A}_\zeta} > 0. \quad (7.12)$$

Key idea

Degeneracy and stability are logically distinct. Degeneracy removes the extra scalar pair. The remaining scalar can still be a ghost ($\mathcal{A}_\zeta < 0$), have a gradient instability ($\mathcal{B}_\zeta < 0$), or become strongly coupled ($\mathcal{A}_\zeta \rightarrow 0$).

Class-II EFT relations force the scalar and tensor gradient coefficients to have opposite signs, so both sectors cannot be stable simultaneously. This is the perturbative reason class Ia receives most phenomenological attention.

8. Exercises

2.2: Luminal reduction

Starting from (3.2)–(3.3), impose $A_1 = 0$ and recover (5.4)–(5.5). Then set $F_X = 0$ and check the one-function deformation in the checkpoint.

2.3: Tensor normalization

Expand (5.1) to second order in a transverse-traceless γ_{ij} . Verify $M^2 = 2(F - X A_1)$ and $c_T^2 = F/(F - X A_1)$, keeping track of the factor $1/8$ in (7.3).

2.4: The GR limit of the scalar EFT

Set $\alpha_H = \beta_1 = \alpha_T = 0$ in (7.10)–(7.11). Explain why taking also $\alpha_K = \alpha_B = 0$ on a consistent vacuum GR background does not describe a propagating scalar but rather makes the scalar action vanish. What additional matter mode would remain in a dust-filled universe?

Conceptual answers

Exercise 2.4 illustrates strong coupling versus absence of a mode. In pure GR there is no gravitational scalar, so its quadratic kinetic coefficient vanishes; one must return to the constrained action rather than divide by $\mathcal{A}_\zeta$. With dust, the matter density contrast remains as the single scalar mode.

CHAPTER 3

Lecture 3: U-DHOST theories and the shadowy mode

Ninety-minute map

0–20 min	Covariant degeneracy versus degeneracy on the unitary foliation.
20–40 min	Classification and the 2×2 unitary-gauge kinetic block.
40–65 min	A solvable 1 + 1 model: fourth-order appearance, boost, and elliptic constraint.
65–80 min	Green function, boundary data, and the meaning of “shadowy”.
80–90 min	Cosmological perturbations and nonlinear interpretation.

Unitary gauge is available whenever $\nabla_\mu \phi$ is timelike: the time coordinate can be chosen so that $\phi = \phi(t)$. A DHOST theory is degenerate before this choice. A U-DHOST theory is degenerate only after it. At first sight this sounds inconsistent, because a coordinate choice should not change the number of physical modes. The resolution is one of the most instructive parts of the subject: the apparent extra mode is fixed by an elliptic equation and boundary conditions rather than by independent initial data [5].

1. The unitary-gauge degeneracy condition

Start from the quadratic action (1.2). On the unitary foliation,

$$\partial_i \phi = 0, \quad \widehat{A}_i = D_i \phi = 0, \quad A_* = \frac{\dot{\phi}}{N}, \quad X_U = -A_*^2. \quad (1.1)$$

Thus the $v_2 \widehat{A}_i \widehat{A}_j$ part of (2.2) vanishes and the null tensor reduces to $\mathcal{V}_{ij} = v_1 h_{ij}$. The scalar null-vector problem correspondingly collapses from the 3×3 system of Lecture 2 to the 2×2 block mixing the normal scalar velocity and the trace of K_{ij} .

It is essential, however, not to identify this restriction with $D_0 = 0$. On the unitary foliation $A_*^2 = -X$, so the covariant determinant polynomial (2.7) is evaluated as

$$D_U(\phi, X) \equiv D_0 - X D_1 + X^2 D_2 = 0. \quad (1.2)$$

This is one relation among the three coefficients, not the first relation by itself. Inserting their explicit expressions gives

$$\begin{aligned} 4[X(A_1 + 3A_2) + 2F] [A_1 + A_2 + X(A_3 + A_4) + X^2 A_5] \\ = 3X(2A_2 + XA_3 + 4F_X)^2. \end{aligned} \quad (1.3)$$

For comparison with the notation commonly used in the U-DHOST and scordatura literature, define [19]

$$\begin{aligned}
c_{D1} &\equiv X(A_1 + A_2), & \mathcal{G}_T &\equiv F - XA_1, \\
\mathcal{H}_D &\equiv 2\mathcal{G}_T + 3c_{D1} = 2F + X(A_1 + 3A_2), \\
\Upsilon &\equiv -\frac{X(4F_X + 2A_2 + XA_3)}{2\mathcal{H}_D}, & \tilde{\mathcal{G}}_T &\equiv (1 - \Upsilon)F - 2XF_X, \\
c_{D2} &\equiv X^2A_4 + 4X\Upsilon F_X - 2\mathcal{G}_T + 2(1 - \Upsilon)\tilde{\mathcal{G}}_T, \\
c_U &\equiv -X^3A_5 - X^2(A_3 + A_4) - c_{D1} + 3\Upsilon^2\mathcal{H}_D.
\end{aligned} \tag{1.4}$$

$$\tag{1.5}$$

The subscripts on the c 's are labels: in particular, c_{D1} is not D_1 , and c_{D2} is not D_2 . To see the relation without a lengthy expansion, move the right-hand side of (1.3) to the left and call the result C_U . Then, for the normalization of the D_i used in Lecture 2,

$$D_U = \mathcal{G}_T C_U, \quad c_U = -\frac{X}{4\mathcal{H}_D} C_U = -\frac{X}{4\mathcal{H}_D \mathcal{G}_T} D_U. \tag{1.6}$$

Thus $c_U = 0$ and $D_U = 0$ describe the same regular unitary-degeneracy locus when $X\mathcal{G}_T\mathcal{H}_D \neq 0$; they are not literally the same coefficient.

Every quadratic DHOST theory satisfies (1.3), because covariant degeneracy implies degeneracy in every gauge. In the regular class-I sector used in [19], the covariant DHOST conditions can be written

$$c_{D1} = c_{D2} = c_U = 0, \tag{1.7}$$

whereas a genuine U-DHOST theory obeys $c_U = 0$ with $(c_{D1}, c_{D2}) \neq (0, 0)$. The D_i formulation remains useful because it also keeps the other covariant DHOST branches visible. The explicit D_1, D_2 and the precise dictionary between the two parameterizations are collected in Appendix 5.

Two different substitutions

$A_* = 0$ extracts the constant coefficient D_0 at fixed X ; it describes a gradient tangent to the chosen slice when such a configuration is real. Unitary gauge instead sets $\hat{A}_i = 0$ and therefore $A_*^2 = -X \neq 0$ for a non-trivial timelike scalar. Confusing the two is the reason one might incorrectly conclude that only D_0 survives.

Definition 3.1. A higher-order scalar–tensor action is *U-degenerate* on a region of field space if its kinetic matrix is degenerate on the foliation $\phi = \phi(t)$ there. The name U-DHOST is commonly reserved for the members that are U-degenerate but not covariantly DHOST.

The definition assumes $X < 0$. If the scalar gradient becomes null or spacelike, unitary time ceases to be an admissible coordinate and the description must be reconsidered.

2. Totally U-degenerate building blocks

An especially useful object is an action whose entire kinetic sector vanishes in unitary gauge. For two arbitrary functions $F(\phi, X)$ and $\gamma(\phi, X)$, define

$$\begin{aligned}
L_{\text{tUd}}[F, \gamma] &= FR + \frac{F}{X}(L_1^{(2)} - L_2^{(2)}) + \frac{2(F - 2XF_X)}{X^2}(L_3^{(2)} - L_4^{(2)}) \\
&\quad - \gamma(XL_4^{(2)} - L_5^{(2)}).
\end{aligned} \tag{2.1}$$

Direct ADM decomposition gives

$$\mathcal{A}_U = 0, \quad \mathcal{B}_U^{ij} = 0, \quad \mathcal{K}_U^{ijkl} = 0. \quad (2.2)$$

The action need not vanish; spatial derivatives and lower-derivative terms remain. “Totally” refers only to the velocity block.

Any U-degenerate quadratic theory can be separated into a totally U-degenerate curvature-containing part and a residual quadratic action. To avoid confusing the arbitrary function in (2.1) with the five residual coefficients, the former has deliberately been called γ , whereas the latter will be denoted by α_I .

More precisely, write

$$FR + \sum_{I=1}^5 A_I L_I^{(2)} = L_{\text{tUd}}[F, \gamma] + \sum_{I=1}^5 \alpha_I L_I^{(2)}. \quad (2.3)$$

Comparing the coefficient of each operator gives the map

$$\begin{aligned} \alpha_1 &= A_1 - \frac{F}{X}, & \alpha_2 &= A_2 + \frac{F}{X}, \\ \alpha_3 &= A_3 - \frac{2(F - 2XF_X)}{X^2}, & \alpha_4 &= A_4 + \frac{2(F - 2XF_X)}{X^2} + \gamma X, \\ \alpha_5 &= A_5 - \gamma. \end{aligned} \quad (2.4)$$

Thus the symbols A_I and α_I are not interchangeable. The split is not unique because γ moves the totally U-degenerate combination $XL_4^{(2)} - L_5^{(2)}$ between the two terms of (2.3). Nevertheless, all combinations that enter the unitary kinetic determinant below are independent of γ . Only in the special case $F = 0$ with the choice $\gamma = 0$ does the map reduce to $\alpha_I = A_I$ for all five coefficients. Subtracting L_{tUd} does not change the unitary-gauge Hessian, because its complete velocity block vanishes. The simple determinant of the residual action therefore contains the whole degeneracy test.

It is convenient to use the normal velocity

$$V_* \equiv \frac{1}{N} (\dot{A}_* - N^i D_i A_*) . \quad (2.5)$$

It differs from $\dot{A}_*$ by a non-zero lapse factor and terms without an independent velocity, so either variable gives the same rank. On the unitary foliation, the velocity-containing part of the scalar Hessian is

$$(\phi_{\mu\nu})_{\text{kin}} = V_* n_\mu n_\nu - A_* K_{\mu\nu}, \quad (\Box\phi)_{\text{kin}} = -V_* - A_* K, \quad (2.6)$$

and

$$(\phi^\mu \phi_{\mu\nu} \phi^\nu)_{\text{kin}} = A_*^2 V_* = -X V_*. \quad (2.7)$$

From the five operators to one 2×2 determinant

Equations (2.6) and (2.7) give, keeping only terms quadratic in the velocities,

$$\begin{aligned} (L_1^{(2)})_{\text{kin}} &= V_*^2 - X K_{ij} K^{ij}, & (L_2^{(2)})_{\text{kin}} &= V_*^2 + 2A_* K V_* - X K^2, \\ (L_3^{(2)})_{\text{kin}} &= X V_*^2 + A_* X K V_*, & (L_4^{(2)})_{\text{kin}} &= X V_*^2, \\ (L_5^{(2)})_{\text{kin}} &= X^2 V_*^2. \end{aligned} \quad (2.8)$$

Introduce the three combinations

$$\mathbf{B} \equiv \alpha_1 + 3\alpha_2, \quad \mathbf{T} \equiv 2\alpha_2 + X\alpha_3, \quad \mathbf{S} \equiv \alpha_1 + \alpha_2 + X(\alpha_3 + \alpha_4) + X^2\alpha_5. \quad (2.9)$$

Using (2.4), these are related to the coefficients of the original covariant action by

$$\begin{aligned} \mathbf{B} &= \frac{X(A_1 + 3A_2) + 2F}{X}, & \mathbf{T} &= 2A_2 + XA_3 + 4F_X, \\ \mathbf{S} &= A_1 + A_2 + X(A_3 + A_4) + X^2A_5. \end{aligned} \quad (2.10)$$

The arbitrary splitting function γ cancels explicitly from $\mathbf{S}$, as it must. Decompose the extrinsic curvature into its trace and traceless parts,

$$K_{ij} = \tilde{K}_{ij} + \frac{1}{3}K h_{ij}, \quad h^{ij}\tilde{K}_{ij} = 0. \quad (2.11)$$

Since $K_{ij}K^{ij} = \tilde{K}_{ij}\tilde{K}^{ij} + K^2/3$, the sum of (2.8) becomes

$$L_{\text{kin}}^{\text{U}} = -X\alpha_1\tilde{K}_{ij}\tilde{K}^{ij} + \mathbf{S}V_*^2 + A_*\mathbf{T}KV_* - \frac{X}{3}\mathbf{B}K^2. \quad (2.12)$$

Thus the trace sector, in the velocity basis (V_*, K) , is represented by

$$\mathbf{M}_{\text{U}}^{\text{scal}} = \begin{pmatrix} \mathbf{S} & \frac{1}{2}A_*\mathbf{T} \\ \frac{1}{2}A_*\mathbf{T} & -\frac{X}{3}\mathbf{B} \end{pmatrix}. \quad (2.13)$$

Its determinant is

$$\det \mathbf{M}_{\text{U}}^{\text{scal}} = -\frac{X}{3}\mathbf{B}\mathbf{S} - \frac{A_*^2}{4}\mathbf{T}^2 = \frac{X}{12}(3\mathbf{T}^2 - 4\mathbf{B}\mathbf{S}), \quad (2.14)$$

where $A_*^2 = -X$ was used in the last equality. For a non-trivial timelike scalar, $X < 0$ and hence $X \neq 0$. The scalar block is degenerate precisely when

$$4(\alpha_1 + 3\alpha_2) [\alpha_1 + \alpha_2 + X(\alpha_3 + \alpha_4) + X^2\alpha_5] = 3(2\alpha_2 + X\alpha_3)^2. \quad (2.15)$$

This is the curvature-stripped form of (1.3); it is not a further condition. Indeed, substitution of (2.10) into $4\mathbf{B}\mathbf{S} = 3\mathbf{T}^2$ and multiplication by $X \neq 0$ reproduces (1.3) term by term.

The first term of (2.12) is important but does not alter (2.15). Orthogonality $h^{ij}\tilde{K}_{ij} = 0$ makes the Hessian block diagonal between the five traceless components and the two scalar velocities (V_*, K) . On the regular branch $-X\alpha_1 \neq 0$, the traceless block is invertible and the full kinetic determinant has the schematic factorization

$$\det \mathbf{M}_{\text{kin}}^{\text{U}} \propto (-X\alpha_1)^5 \det \mathbf{M}_{\text{U}}^{\text{scal}}. \quad (2.16)$$

It therefore multiplies the scalar determinant by a non-zero factor rather than producing the scalar null direction. Physically, its coefficient still controls the regularity and the sign of the traceless metric kinetic sector.

On the further regular branch $\mathbf{B} \neq 0$, imposing (2.15) allows (2.12) to be completed to a square:

$$L_{\text{kin}}^{\text{U}} = -X\alpha_1\tilde{K}_{ij}\tilde{K}^{ij} - \frac{X\mathbf{B}}{3} \left(K + \frac{3\mathbf{T}}{2A_*\mathbf{B}}V_* \right)^2. \quad (2.17)$$

Equivalently, it has the tensorial form

$$L_{\text{kin}}^{\text{U}} = \widehat{\mathcal{K}}^{ij,kl} (K_{ij} + \sigma h_{ij} V_*) (K_{kl} + \sigma h_{kl} V_*), \quad \sigma = \frac{\text{T}}{2A_*\text{B}}. \quad (2.18)$$

The square makes the null direction manifest: the simultaneous velocity variation

$$\delta V_* = \varepsilon, \quad \delta K_{ij} = -\sigma h_{ij} \varepsilon \quad (2.19)$$

leaves every bracket in (2.18) unchanged. In the notation of the null-vector test of Lecture 1 this says

$$(v_0, \mathcal{V}_{ij}) \propto (1, -\sigma h_{ij}). \quad (2.20)$$

The expression inside the square is the surviving kinetic combination; the vector in (2.20), not the square itself, is the null vector.

Regular and exceptional branches

If $\alpha_1 = 0$, the traceless block is itself degenerate and the standard single-null-vector U-DHOST count cannot be used without a separate constraint analysis. If $\text{B} = 0$, the polynomial condition forces $\text{T} = 0$, but the formula for σ divides by B ; the trace block must again be analyzed directly. These are additional rank changes, not counterexamples to the 2×2 derivation.

Common pitfall

The sum of two degenerate Lagrangians need not be degenerate. Null vectors must align. Equation (2.1) is useful because its unitary kinetic block is exactly zero, so adding it cannot rotate the null vector of (2.18) on that foliation.

3. A solvable shadowy-mode model

Freeze metric fluctuations and work in two-dimensional Minkowski spacetime with signature $(-, +)$. Consider

$$L = -\frac{1}{2}X - \mu \left(X L_4^{(2)} - L_5^{(2)} \right), \quad \mu > 0. \quad (3.1)$$

The second term is the $F = 0$ member of (2.1). The first term supplies one ordinary scalar mode.

3.1. Background aligned with unitary time. Let

$$\bar{\phi} = t, \quad \phi = t + \chi(t, x). \quad (3.2)$$

To quadratic order,

$$\mathcal{L}_0^{(2)} = \frac{1}{2}(\dot{\chi}^2 - \chi'^2) + \mu \dot{\chi}'^2. \quad (3.3)$$

There is no $\ddot{\chi}$. For $\chi \propto \exp(-i\omega t + ikx)$,

$$(1 + 2\mu k^2)\omega^2 - k^2 = 0, \quad \omega_{\pm} = \pm \frac{k}{\sqrt{1 + 2\mu k^2}}. \quad (3.4)$$

Only one pair of frequencies is present. Notice that $\omega^2 \rightarrow 1/(2\mu)$ as $k \rightarrow \infty$: the group velocity tends to zero, and there is no high- k gradient instability for $\mu > 0$.

3.2. A tilted background. Now take

$$\bar{\phi} = t + \alpha x, \quad |\alpha| < 1, \quad (3.5)$$

so the gradient is still timelike but is not aligned with the chosen t . Expanding $\phi = \bar{\phi} + \chi$ gives

$$\begin{aligned} \mathcal{L}_\alpha^{(2)} = & \frac{1}{2}(\dot{\chi}^2 - \chi'^2) + \mu \left[\alpha^2(\ddot{\chi}^2 + \chi''^2) - 2\alpha(1 + \alpha^2)(\ddot{\chi}\dot{\chi}' + \chi''\dot{\chi}') \right. \\ & \left. + (1 + 4\alpha^2 + \alpha^4)\dot{\chi}'^2 \right]. \end{aligned} \quad (3.6)$$

The plane-wave polynomial is now quartic in ω :

$$\begin{aligned} 0 = & 2\alpha^2\mu\omega^4 + 4\alpha(1 + \alpha^2)k\mu\omega^3 + [1 + 2(\alpha^4 + 4\alpha^2 + 1)k^2\mu]\omega^2 \\ & + 4\alpha(1 + \alpha^2)k^3\mu\omega + 2\alpha^2k^4\mu - k^2. \end{aligned} \quad (3.7)$$

Two roots are real and two are generally a complex-conjugate pair. If one interprets all four as propagating modes with arbitrary initial amplitudes, one concludes that the theory is unstable. That conclusion misses the boundary-value part of the problem.

4. The boost that exposes the elliptic equation

Because $|\alpha| < 1$, define coordinates aligned with the scalar gradient,

$$\tilde{t} = \frac{t + \alpha x}{\sqrt{1 - \alpha^2}}, \quad \tilde{x} = \frac{x + \alpha t}{\sqrt{1 - \alpha^2}}. \quad (4.1)$$

Then $\bar{\phi} = \sqrt{1 - \alpha^2}\tilde{t}$ and the equation for χ is

$$[1 - 2\mu(1 - \alpha^2)^2\partial_{\tilde{x}}^2]\partial_{\tilde{t}}^2\chi - \partial_{\tilde{x}}^2\chi = 0. \quad (4.2)$$

It is second order in the physical time $\tilde{t}$.

Introduce an auxiliary field ψ and

$$\kappa \equiv \frac{1}{\sqrt{2\mu}(1 - \alpha^2)}. \quad (4.3)$$

Equation (4.2) is equivalent to

$$(\partial_{\tilde{t}}^2 - \partial_{\tilde{x}}^2)\chi = \psi, \quad (4.4)$$

$$(\partial_{\tilde{x}}^2 - \kappa^2)\psi = -\partial_{\tilde{x}}^4\chi. \quad (4.5)$$

The first equation is hyperbolic and evolves one scalar. The second is elliptic on each constant- $\tilde{t}$ hypersurface. It does not ask for $\dot{\psi}$ as initial data; it asks for spatial boundary conditions.

Green function and the length of the shadow

Demand boundedness as $|\tilde{x}| \rightarrow \infty$. The Green function of the elliptic operator is

$$(\partial_{\tilde{x}}^2 - \kappa^2)G(\tilde{x}, \tilde{y}) = \delta(\tilde{x} - \tilde{y}), \quad G = -\frac{e^{-\kappa|\tilde{x} - \tilde{y}|}}{2\kappa}. \quad (4.6)$$

Hence

$$\psi(\tilde{t}, \tilde{x}) = \frac{1}{2\kappa} \int d\tilde{y} \partial_{\tilde{y}}^4 \chi(\tilde{t}, \tilde{y}) e^{-\kappa|\tilde{x} - \tilde{y}|}. \quad (4.7)$$

The field ψ is determined instantaneously on the preferred slice. Its kernel decays over $\ell_{\text{sh}} = \kappa^{-1}$, the “length of the shadow”. At wavelengths $L \gg \ell_{\text{sh}}$ its effect is derivative-suppressed.

What became of the complex roots of (3.7)? After the boost, their mode functions contain

$$\exp\left(\pm \frac{\alpha \operatorname{Im} \omega}{\sqrt{1 - \alpha^2}} \tilde{x}\right). \quad (4.8)$$

Each diverges at one spatial end. The boundary condition used to invert (4.5) sets their amplitudes to zero. The fourth-order equation written in a tilted coordinate mixes a hyperbolic initial-value problem with an elliptic boundary-value problem; treating all four formal solutions as Cauchy data overcounts the physics.

What “shadowy” means

The shadowy mode is not an additional wave carrying independent initial data. It is a spatially nonlocal response fixed on each preferred-time slice by an elliptic constraint and boundary conditions. Calling it “instantaneous” describes the equation type, not a signal propagating with an infinite Lorentz-invariant speed.

5. A cosmological U-DHOST model

A simple four-dimensional model that isolates the same structure is [6]

$$S = \int d^4x \sqrt{-g} \left[\frac{M_{\text{P}}^2}{2} R + P(X) + Q(X) \square \phi - \xi \left(X \phi^\mu \phi_{\mu\nu} \phi^{\nu\lambda} \phi_\lambda - (\phi^\mu \phi_{\mu\nu} \phi^\nu)^2 \right) \right]. \quad (5.1)$$

It has

$$A_1 = A_2 = A_3 = 0, \quad A_4 = -\xi X, \quad A_5 = \xi, \quad (5.2)$$

so (1.3) holds, but the covariant DHOST conditions do not hold for $\xi \neq 0$.

On homogeneous FLRW, the ξ operator vanishes. With $X = -\dot{\phi}^2/N^2$,

$$3M_{\text{P}}^2 H^2 + P - 2X P_X + 6H \frac{\dot{\phi}}{N} X Q_X = 0, \quad (5.3)$$

$$M_{\text{P}}^2 \left(3H^2 + 2 \frac{\dot{H}}{N} \right) + P + \frac{\dot{\phi} \ddot{X}}{N^2} Q_X = 0. \quad (5.4)$$

Thus the background is identical to kinetic gravity braiding. The U-DHOST character first appears in inhomogeneous perturbations.

Choose unitary spatial gauge, $\delta\phi = 0$, and

$$N \rightarrow N(1 + \alpha), \quad N_i = N \partial_i \chi, \quad h_{ij} = a^2(1 + 2\xi) \delta_{ij}. \quad (5.5)$$

After using the background equations, the scalar quadratic Lagrangian is

$$\mathcal{L}^{(2)} = Na^3 \left[-3M_{\text{P}}^2 \frac{\dot{\zeta}^2}{N^2} - \frac{M_{\text{P}}^2}{a^2} \zeta \partial^2 \zeta + \alpha \left(\Sigma - \xi \frac{\dot{\phi}^6}{N^6} \frac{\partial^2}{a^2} \right) \alpha \right. \\ \left. - \frac{2\Theta}{a^2} \alpha \partial^2 \chi + 6\Theta \alpha \frac{\dot{\zeta}}{N} - \frac{2M_{\text{P}}^2}{a^2} \alpha \partial^2 \zeta + \frac{2M_{\text{P}}^2}{a^2} \frac{\dot{\zeta}}{N} \partial^2 \chi \right], \quad (5.6)$$

with

$$\Sigma = -3M_{\text{P}}^2 H^2 + X P_X + 2X^2 P_{XX} - 6H \frac{\dot{\phi}}{N} (2X Q_X + X^2 Q_{XX}), \quad (5.7)$$

$$\Theta = M_{\text{P}}^2 H - \frac{\dot{\phi}^3}{N^3} Q_X. \quad (5.8)$$

The momenta of α and χ vanish, yielding two primary constraints. Their preservation produces two secondary constraints. All four are second class, and the scalar phase-space count is

$$N_{\text{dof}}^{\text{scalar}} = \frac{1}{2}(6 - 4) = 1, \quad (5.9)$$

where the six variables are the three canonical pairs associated with (α, χ, ζ) .

The lapse secondary constraint has the revealing form

$$\widehat{L}\alpha \equiv \left[2a\xi \frac{\dot{\phi}^6}{N^6} \partial^2 - 2a^3 \left(\Sigma + \frac{3\Theta^2}{M_{\text{P}}^2} \right) \right] \alpha \\ \approx -2M_{\text{P}}^2 a \partial^2 \zeta - \frac{\Theta}{M_{\text{P}}^2} p_\zeta. \quad (5.10)$$

For $\xi = 0$ the lapse is fixed algebraically, as in DHOST. For $\xi \neq 0$ it is fixed by a Poisson-type equation. The homogeneous solution of $\widehat{L}\alpha = 0$, selected by spatial boundary data, is the cosmological shadowy mode.

5.1. Reducing the quadratic worksheet. The last part of `Maple_code/udhost_opt.mpl` carries this reduction one step beyond (5.6). The worksheet treats the general shift-symmetric quadratic action, derives the background equations from the linear action, fixes $\delta\phi = 0$, verifies the U-degeneracy of the $(\dot{\zeta}, \dot{\alpha})$ block, and introduces

$$\zeta = \zeta_2 - \frac{K_{12}}{K_{11}} \alpha. \quad (5.11)$$

On the U-degenerate branch this isolates the one kinetic direction and removes $\dot{\alpha}^2$. After the integrations by parts, the object called `az5b` has the transparent coefficient form

$$L^{(2)} = B_1(\partial^2 \chi)^2 + B_2(\partial_i \alpha)^2 + B_3 \alpha \partial^2 \zeta_2 + B_4 \alpha \partial^2 \chi + B_5 \alpha \dot{\zeta}_2 + B_6(\partial^2 \chi) \dot{\zeta}_2 \\ + B_7(\partial^2 \chi) \zeta_2 + B_8 \alpha^2 + B_9 \dot{\zeta}_2^2 + B_{10}(\partial_i \zeta_2)^2 + B_{11}(\partial_i \chi)^2. \quad (5.12)$$

The explicit zero tests in the worksheet establish, on the chosen background branch,

$$B_7 = 0, \quad B_{11} = 0, \quad B_4 = \frac{B_5 B_6}{2B_9}. \quad (5.13)$$

These are not assumptions to enter by hand: they are checked before the corresponding substitutions are made.

For a cosine Fourier mode, the code maps

$$\alpha \mapsto \text{AL}, \quad \chi \mapsto \text{CH}, \quad \zeta_2 \mapsto \text{ZZ2}, \quad \partial^2 \mapsto -k^2. \quad (5.14)$$

Neither AL nor CH has a time derivative. Their Euler-Lagrange equations are therefore constraint equations for each k ; the worksheet solves first for AL, then for CH, and substitutes both into the action. A final integration by parts removes the $\zeta_2 \dot{\zeta}_2$ term. The result, called az7, can be written in the ultraviolet as

$$L_{\text{red}}^{(2)}(k) = \frac{a^3}{N} Q_s \dot{\zeta}_k^2 - N a Q_s c_s^2 k^2 \zeta_k^2 + \dots. \quad (5.15)$$

Consequently the last four Maple lines implement precisely

$$Q_s^{\text{UV}} = \lim_{k \rightarrow \infty} \frac{N}{a^3} \text{coeff} \left(L_{\text{red}}^{(2)}, \dot{\zeta}_k^2 \right), \quad (5.16)$$

$$(c_s^2)^{\text{UV}} = \lim_{k \rightarrow \infty} \left[- \frac{\text{coeff}(L_{\text{red}}^{(2)}, \zeta_k^2)}{\text{coeff}(L_{\text{red}}^{(2)}, \dot{\zeta}_k^2)} \frac{a^2}{N^2 k^2} \right]. \quad (5.17)$$

The no-ghost and short-wavelength gradient-stability tests are $Q_s^{\text{UV}} > 0$ and $(c_s^2)^{\text{UV}} > 0$, respectively. These tests must be supplemented by checking that the denominators generated while solving the constraints do not vanish. At finite k , the reduced coefficients can be rational functions of k^2 : in position space this is the inverse elliptic operator and hence the same spatial non-locality associated with the shadowy constraint. A pole is not a propagation speed; it signals a singular constraint branch, a boundary-value zero mode, or strong coupling that must be analysed separately.

Background blindness

Because the ξ term vanishes on homogeneous FLRW, a background-only code cannot distinguish (5.1) from its $\xi = 0$ DHOST limit. This is not a bug: the distinction is spatial and appears in the elliptic part of the perturbation constraints.

6. Exercises

3.1: Derive the unitary dispersion relation

Vary (3.3), Fourier transform, and derive (3.4). Compute the phase and group velocities at $k\sqrt{\mu} \ll 1$ and $k\sqrt{\mu} \gg 1$.

3.2: Factor the boosted operator

Eliminate ψ from (4.4) and (4.5); recover (4.2). Keep the definition of κ explicit to verify every factor of $1 - \alpha^2$.

3.3: Long-distance expansion

Use (4.7) for a Fourier mode of wavenumber k . Show that $\psi_k = k^4 \chi_k / (k^2 + \kappa^2)$ and expand for $k/\kappa \ll 1$. Which local higher-spatial-derivative correction appears first in the wave equation?

3.4: Why the background term vanishes

For homogeneous $\phi(t)$, show directly that $X\phi^\mu\phi_{\mu\nu}\phi^{\nu\lambda}\phi_\lambda - (\phi^\mu\phi_{\mu\nu}\phi^\nu)^2 = 0$. Use the fact that all vectors made from ϕ_μ and one contraction of $\phi_{\mu\nu}$ are parallel to the unit normal on FLRW.

Answer to Exercise 3.3

Fourier transforming (4.5) gives $-(k^2 + \kappa^2)\psi_k = -k^4\chi_k$. Hence $\psi_k = k^4\chi_k/(k^2 + \kappa^2)$ with the Fourier convention used in (4.4); reversing the sign convention for ∂_x^2 reverses both displayed intermediate signs but not the final dispersion relation. At long wavelength, $\psi_k = (k^4/\kappa^2)\chi_k + O(k^6/\kappa^4)$, so the first correction is a ∂_x^4/κ^2 operator.

CHAPTER 4

Lecture 4: The minimal theory of massive gravity

Ninety-minute map

0–15 min	From Fierz–Pauli and dRGT to a Lorentz-breaking two-mode theory.
15–40 min	The precursor theory in ADM vielbeins and its three degrees of freedom.
40–55 min	Two additional constraints and the nonlinear MTMG count.
55–80 min	FLRW reduction, variations, and the two background branches.
80–90 min	Scalar, vector, and tensor perturbations.

A Lorentz-invariant massive spin-2 particle carries five polarizations. The linear Fierz–Pauli tuning removes a sixth ghostlike polarization, and the dRGT potential preserves five modes nonlinearly [20, 21]. MTMG takes a different path: Lorentz invariance is broken in the gravitational sector, and nonlinear constraints remove the scalar and vector graviton polarizations. Only two massive tensor modes remain [7, 8].

Key idea

DHOST removes an Ostrogradsky pair by making a velocity Hessian singular. MTMG starts from a three-mode precursor and deliberately adds two independent second-class constraints. The final number of local gravitational degrees of freedom is two on a generic nonlinear background, not merely on FLRW.

1. Removed mode or vanishing kinetic term?

The distinction is central to massive gravity. At quadratic order around Minkowski space, the most general Lorentz-invariant algebraic mass term can be written as

$$\mathcal{L}_m = -\frac{m^2 M_{\text{P}}^2}{8} \left(h_{\mu\nu} h^{\mu\nu} - \alpha h^2 \right), \quad h \equiv \eta^{\mu\nu} h_{\mu\nu}. \quad (1.1)$$

Taking a divergence of the linear field equation and using the linearized Bianchi identity gives four relations,

$$\partial^\mu h_{\mu\nu} = \alpha \partial_\nu h. \quad (1.2)$$

Only at the Fierz–Pauli value $\alpha = 1$ does the trace equation become an additional algebraic constraint, $h = 0$; otherwise the trace is a sixth, ghostlike mode. The tuned theory therefore has

$$10 \text{ components} - 4 \text{ vector constraints} - 1 \text{ scalar constraint} = 5 \quad (1.3)$$

massive polarizations [20]. The dRGT potential preserves this five-mode count nonlinearly [21].

On some highly symmetric dRGT cosmologies, the quadratic kinetic coefficients of scalar or vector perturbations vanish. That does *not* turn the five-mode nonlinear theory into a two-mode

theory: the modes still exist away from that background, so the vanishing coefficient instead signals loss of perturbative control. MTMG changes the generic nonlinear constraint algebra first and only then specializes to FLRW. Its absent scalar and vector gravitons are removed variables, not zero-kinetic variables.

A zero in the quadratic action is not a degree-of-freedom count

The two statements “the coefficient of $\dot{q}^2$ vanishes on this solution” and “a primary-secondary constraint pair removes (q, p_q) ” are logically different. The first may indicate strong coupling; the second changes the dimension of the reduced phase space. The nonlinear Dirac count below is what licenses the two-mode interpretation of MTMG.

2. Physical and fiducial ADM vielbeins

The dynamical variables are the lapse N , shift N^i , and spatial vielbein e^I_j . The theory also contains prescribed fiducial fields (M, M^i, E^I_j) . In triangular ADM form,

$$e^{\mathcal{A}}_{\mu} = \begin{pmatrix} N & 0 \\ N^k e^I_k & e^I_j \end{pmatrix}, \quad E^{\mathcal{A}}_{\mu} = \begin{pmatrix} M & 0 \\ M^k E^I_k & E^I_j \end{pmatrix}. \quad (2.1)$$

They define

$$\gamma_{ij} = \delta_{IJ} e^I_i e^J_j, \quad \tilde{\gamma}_{ij} = \delta_{IJ} E^I_i E^J_j, \quad (2.2)$$

and the mutually inverse spatial matrices

$$X_I^J = e_I^J E^J_j, \quad Y_I^J = E_I^J e^J_j. \quad (2.3)$$

Fixing both vielbeins to triangular form selects a preferred Lorentz frame. This is the controlled Lorentz breaking that allows two massive tensor modes without completing them into a five-polarization multiplet.

3. Step 1: the precursor theory

Substituting the ADM vielbeins directly into the dRGT wedge-product potential gives

$$\mathcal{S}_{\text{pre}} = \frac{M_{\text{P}}^2}{2} \int d^4x \left\{ N \sqrt{\gamma} [{}^{(3)}R + K_{ij} K^{ij} - K^2] - m^2 M \mathcal{H}_1 - m^2 N \mathcal{H}_0 \right\}, \quad (3.1)$$

up to a convention-dependent overall sign in $\mathcal{H}_{0,1}$. A useful explicit choice is

$$\mathcal{H}_0 = \sqrt{\gamma} (c_1 + c_2 [Y]) + \sqrt{\gamma} (c_3 [X] + c_4), \quad (3.2)$$

$$\mathcal{H}_1 = \sqrt{\gamma} \left(c_1 [Y] + \frac{c_2}{2} \{ [Y]^2 - [Y^2] \} \right) + c_3 \sqrt{\gamma}. \quad (3.3)$$

Here $[X] = X_I^I$ and similarly for Y . The crucial structural facts are that the mass term is linear in both lapses and independent of the physical shift.

The momentum conjugate to e^I_j is

$$\Pi_I^j = 2\pi^{jk} \delta_{IJ} e^J_k, \quad \pi^{ij} = \frac{M_{\text{P}}^2 \sqrt{\gamma}}{2} (K^{ij} - K \gamma^{ij}). \quad (3.4)$$

Symmetry of K_{ij} produces three primary constraints

$$\mathcal{P}_{[IJ]} \equiv \Pi_{[I}^k \delta_{J]K} e^K_k \approx 0. \quad (3.5)$$

Their preservation gives three symmetry constraints $Y^{[IJ]} \approx 0$. In the reduced vielbein phase space used here, the lapse and shift impose four constraints $\mathcal{R}_0 \approx 0$, $\mathcal{R}_i \approx 0$, often called primary constraints in the MTMG literature. In an extended phase space containing lapse, shift, and their conjugate momenta, instead $\pi_N \approx 0$, $\pi_i \approx 0$ are primary and $\mathcal{R}_0 \approx 0$, $\mathcal{R}_i \approx 0$ are secondary. Their 4×4 Poisson matrix has rank two: preservation fixes two combinations of lapse/shift multipliers, while its two null combinations generate two additional independent constraints.

Precursor count

There are nine configuration variables e^I_j , hence an 18-dimensional phase space. On a generic background the following twelve constraints are independent and second class:

$$3 \mathcal{P}_{[IJ]} + 3 Y^{[IJ]} + 1 \mathcal{R}_0 + 3 \mathcal{R}_i + 2 \tilde{C}_\tau. \quad (3.6)$$

Therefore

$$N_{\text{dof}}^{\text{pre}} = \frac{1}{2}(18 - 12) = 3. \quad (3.7)$$

The preferred-frame construction has already removed the two vector polarizations of dRGT, but one scalar graviton remains.

4. Step 2: the minimal constraints

Let

$$H_1 = \int d^3x m^2 M \mathcal{H}_1. \quad (4.1)$$

MTMG imposes the four expressions

$$C_0 \approx 0, \quad C_i \approx 0, \quad (4.2)$$

constructed schematically as

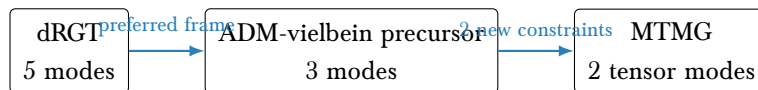
$$C_0 \sim \{\mathcal{R}_0^{\text{gr}}, H_1\} - m^2 \partial_i \mathcal{H}_0, \quad C_i \sim \{\mathcal{R}_i^{\text{gr}}, H_1\}. \quad (4.3)$$

Two linear combinations of C_i were already present as $\tilde{C}_\tau$ in the precursor. Thus (4.2) contributes exactly two new independent constraints. The total Hamiltonian has the form

$$H_{\text{mtmg}} = \int d^3x \left[-N \mathcal{R}_0^{\text{gr}} - N^i \mathcal{R}_i^{\text{gr}} + m^2 (N \mathcal{H}_0 + M \mathcal{H}_1) + \lambda C_0 + \lambda^i C_i + \alpha_{IJ} \mathcal{P}^{[IJ]} + \beta_{IJ} Y^{[IJ]} \right]. \quad (4.4)$$

Theorem 4.1 (Nonlinear mode count of MTMG). *On a generic background where the 14-constraint Poisson matrix is non-singular, MTMG propagates exactly two local gravitational degrees of freedom.*

Proof. The 18-dimensional vielbein phase space is reduced by 14 independent second-class constraints, so it has at most $(18 - 14)/2 = 2$ degrees of freedom. The transverse-traceless quadratic action on FLRW, derived below, contains two regular tensor modes, so the count is at least two. Both inequalities are saturated. $\square$



Common pitfall

The MTMG action is not Einstein–Hilbert plus an ordinary potential. After Legendre transforming, the new constraints contain extrinsic curvature and modify the kinetic structure. Dropping those terms returns the precursor, not MTMG.

5. Metric formulation in compact notation

Define spatial square-root matrices

$$\mathcal{K}^i{}_k \mathcal{K}^k{}_j = \tilde{\gamma}^{ik} \gamma_{kj}, \quad \mathfrak{K} = \mathcal{K}^{-1}, \quad (5.1)$$

and encode the prescribed time derivative of the fiducial vielbein in

$$\tilde{\zeta}^i{}_j \equiv \frac{1}{M} E_L{}^i \dot{E}^L{}_j, \quad \tilde{\zeta} \equiv \tilde{\zeta}^i{}_i. \quad (5.2)$$

The tensor obtained by differentiating the mass potential with respect to the physical spatial metric is

$$\begin{aligned} \Theta^{ij} = \frac{\sqrt{\tilde{\gamma}}}{\sqrt{\gamma}} & \left\{ c_1 (\gamma^{il} \mathcal{K}^j{}_l + \gamma^{jl} \mathcal{K}^i{}_l) \right. \\ & \left. + c_2 [(\mathcal{K})(\gamma^{il} \mathcal{K}^j{}_l + \gamma^{jl} \mathcal{K}^i{}_l) - 2\tilde{\gamma}^{ij}] \right\} + 2c_3 \gamma^{ij}. \end{aligned} \quad (5.3)$$

The scalar constraint that appears after the Legendre transform is

$$\begin{aligned} \bar{C}_0 = \frac{1}{2} m^2 M K_{ij} \Theta^{ij} - m^2 M & \left\{ \frac{\sqrt{\tilde{\gamma}}}{\sqrt{\gamma}} \left[c_1 \tilde{\zeta} + c_2 ([\mathcal{K}] \tilde{\zeta} - \mathcal{K}^m{}_n \tilde{\zeta}^n{}_m) \right] \right. \\ & \left. + c_3 \mathfrak{K}^m{}_n \tilde{\zeta}^n{}_m \right\}. \end{aligned} \quad (5.4)$$

The explicit tensor $C^j{}_i$ will not be needed below because its homogeneous contribution is a spatial divergence. With these definitions, the metric action can be organized as

$$\begin{aligned} S_{\text{mtmg}} = S_{\text{pre}} + \frac{M_{\text{P}}^2}{2} \int d^4x N \sqrt{\gamma} & \left(\frac{m^2 M \lambda}{4N} \right)^2 \left(\gamma_{ik} \gamma_{jl} - \frac{1}{2} \gamma_{ij} \gamma_{kl} \right) \Theta^{ij} \Theta^{kl} \\ & - \frac{M_{\text{P}}^2}{2} \int d^4x \sqrt{\gamma} \left[\lambda \bar{C}_0 - (D_j \lambda^i) C^j{}_i \right] + S_m. \end{aligned} \quad (5.5)$$

The multipliers (λ, λ^i) impose precisely the minimal constraints. Equation (5.5) is most useful for cosmological expansion, while (4.4) makes the degree of freedom count transparent.

6. FLRW background from the action, step by step

The branch equation is short, but obtaining it without losing a sign requires some care. We therefore reduce the metric action before varying it. This is also the cleanest place to see why the statement “the MTMG constraints vanish on FLRW” is incorrect: the vector constraint is trivial by homogeneity, but the scalar multiplier λ leaves essential linear and quadratic terms in the background action.

Take

$$\gamma_{ij} = a^2 \delta_{ij}, \quad \tilde{\gamma}_{ij} = \tilde{a}^2 \delta_{ij}, \quad X \equiv \frac{\tilde{a}}{a}, \quad (6.1)$$

and define

$$H = \frac{\dot{a}}{Na}, \quad H_f = \frac{\dot{\tilde{a}}}{M\tilde{a}}, \quad r = \frac{M}{NX}. \quad (6.2)$$

Before imposing any equation of motion, differentiation of $X = \tilde{a}/a$ gives the kinematical identity

$$\frac{\dot{X}}{X} = MH_f - NH. \quad (6.3)$$

It is useful to package the potential into three polynomials,

$$U(X) \equiv c_4 + 3c_3X + 3c_2X^2 + c_1X^3, \quad (6.4)$$

$$V(X) \equiv c_3 + 3c_2X + 3c_1X^2, \quad (6.5)$$

$$J(X) \equiv c_1X^2 + 2c_2X + c_3 = \frac{1}{3}U'(X). \quad (6.6)$$

6.1. Reducing every matrix in the action. On the ansatz (6.1), all spatial matrices are proportional to the identity:

$$\mathcal{K}^i{}_j = X^{-1}\delta^i{}_j, \quad \mathfrak{K}^i{}_j = X\delta^i{}_j, \quad K^i{}_j = H\delta^i{}_j, \quad \tilde{\zeta}^i{}_j = H_f\delta^i{}_j. \quad (6.7)$$

Consequently

$$[\mathcal{K}] = \frac{3}{X}, \quad \tilde{\gamma}^{ij} = X^{-2}\gamma^{ij}, \quad \frac{\sqrt{\tilde{\gamma}}}{\sqrt{\gamma}} = X^3. \quad (6.8)$$

Substitution in (5.3) gives

$$\begin{aligned} \Theta^{ij} &= X^3 \left[c_1 \frac{2}{X} \gamma^{ij} + c_2 \left(\frac{6}{X^2} - \frac{2}{X^2} \right) \gamma^{ij} \right] + 2c_3 \gamma^{ij} \\ &= 2 \left(c_1 X^2 + 2c_2 X + c_3 \right) \gamma^{ij} = 2J \gamma^{ij}. \end{aligned} \quad (6.9)$$

The apparently awkward DeWitt contraction in (5.5) is now just arithmetic:

$$\begin{aligned} &\left(\gamma_{ik} \gamma_{jl} - \frac{1}{2} \gamma_{ij} \gamma_{kl} \right) \Theta^{ij} \Theta^{kl} \\ &= 4J^2 \left(3 - \frac{1}{2} 3 \cdot 3 \right) = -6J^2. \end{aligned} \quad (6.10)$$

The remaining scalar constraint deserves one more line. Its two pieces reduce as

$$\frac{1}{2} m^2 M K_{ij} \Theta^{ij} = 3m^2 M H J, \quad (6.11)$$

$$m^2 M \left\{ X^3 [3c_1 H_f + 6c_2 H_f / X] + 3c_3 X H_f \right\} = 3m^2 M X H_f J. \quad (6.12)$$

Hence

$$\bar{C}_0 = 3m^2 M J (H - X H_f). \quad (6.13)$$

Equations (6.9)–(6.13) are the three algebraic reductions on which the entire background calculation rests.

For completeness, the precursor potentials in (3.2)–(3.3) become

$$\mathcal{H}_0 = a^3 U(X), \quad (6.14)$$

$$\mathcal{H}_1 = a^3 V(X). \quad (6.15)$$

For example,

$$\mathcal{H}_1 = a^3 X^3 \left[\frac{3c_1}{X} + \frac{c_2}{2} \left(\frac{9}{X^2} - \frac{3}{X^2} \right) \right] + c_3 a^3 = a^3 V. \quad (6.16)$$

6.2. The minisuperspace Lagrangian. Set the fiducial comoving spatial volume to one. Using $K_{ij}K^{ij} - K^2 = -6H^2$ and the reductions above, the exact homogeneous Lagrangian is

$$\begin{aligned} L_{\text{bg}} = & -\frac{3M_{\text{P}}^2 a \dot{a}^2}{N} - \frac{M_{\text{P}}^2 m^2 a^3}{2} [NU(X) + MV(X)] + L_m \\ & + \frac{3M_{\text{P}}^2 m^2 a^3}{2} M\lambda J(X)(XH_f - H) - \frac{3M_{\text{P}}^2 m^4 a^3}{16N} M^2 \lambda^2 J(X)^2. \end{aligned} \quad (6.17)$$

This formula is worth checking term by term. The term linear in λ is $-(M_{\text{P}}^2/2)a^3\lambda\bar{C}_0$; the last term follows from $-6J^2$ in (6.10). The homogeneous λ^i term is a spatial divergence and vanishes. No lapse has been set to one, because doing so before variation would erase the Friedmann equation.

For a minimally coupled perfect fluid we use

$$\frac{\partial L_m}{\partial N} = -a^3 \rho, \quad \frac{\delta L_m}{\delta a} = 3Na^2 P. \quad (6.18)$$

The second relation is an Euler-Lagrange derivative, not merely a partial derivative. Dust is recovered by setting $P = 0$.

6.3. Three variations.

1. The lapse. Since $H = \dot{a}/(Na)$, $\partial H/\partial N = -H/N$. Direct differentiation gives

$$\frac{1}{a^3} \frac{\partial L_{\text{bg}}}{\partial N} = 3M_{\text{P}}^2 H^2 - \rho - \rho_g - \rho_\lambda, \quad (6.19)$$

where

$$\rho_g = \frac{m^2 M_{\text{P}}^2}{2} U(X), \quad (6.20)$$

$$\rho_\lambda = -\frac{3M_{\text{P}}^2 m^4 M^2 J^2 \lambda^2}{16N^2} - \frac{3M_{\text{P}}^2 m^2 H M J \lambda}{2N}. \quad (6.21)$$

Thus the Friedmann equation is

$$E_0 \equiv 3M_{\text{P}}^2 H^2 - \rho - \rho_g - \rho_\lambda = 0. \quad (6.22)$$

Notice that ρ_λ is not zero off shell.

2. The scalar multiplier. The last two terms of (6.17) give

$$\begin{aligned} \frac{\partial L_{\text{bg}}}{\partial \lambda} &= \frac{3}{2} M_{\text{P}}^2 a^3 m^2 M J (XH_f - H) - \frac{3}{8} M_{\text{P}}^2 a^3 \frac{m^4 M^2 J^2 \lambda}{N} \\ &= \frac{3}{4} M_{\text{P}}^2 a^3 N E_\lambda. \end{aligned} \quad (6.23)$$

Therefore

$$E_\lambda = m^2 J(X) \left[\frac{2M}{N} (XH_f - H) - \frac{m^2 M^2}{2N^2} J(X) \lambda \right] = 0. \quad (6.24)$$

The two branches are already visible, but the second factor alone does not yet finish the normal-branch calculation.

3. The scale factor. Define

$$\mathcal{E}_a[L] \equiv \frac{\partial L}{\partial a} - \frac{d}{dt} \frac{\partial L}{\partial \dot{a}}. \quad (6.25)$$

The Einstein–Hilbert term gives

$$\mathcal{E}_a[L_{\text{EH}}] = 3Na^2 M_{\text{P}}^2 \left(\frac{2\dot{H}}{N} + 3H^2 \right). \quad (6.26)$$

For the mass potential, $\partial X/\partial a = -X/a$ and

$$3U - XU' = 3(c_4 + 2c_3X + c_2X^2), \quad (6.27)$$

$$3V - XV' = 3J. \quad (6.28)$$

It follows without integrations by parts that

$$\frac{\mathcal{E}_a[L_{\text{mass}}]}{3Na^2} = P_g = -\frac{m^2 M_{\text{P}}^2}{2N} [MJ + N(c_2X^2 + 2c_3X + c_4)]. \quad (6.29)$$

The multiplier pressure is longer. Split $L_\lambda = L_{\lambda,1} + L_{\lambda,2}$ into its terms linear and quadratic in λ . For the quadratic term one obtains

$$\begin{aligned} \frac{\mathcal{E}_a[L_{\lambda,2}]}{3Na^2} &= -\frac{m^4 M_{\text{P}}^2 M^2 \lambda^2}{16N^2} J(3J - 2XJ') \\ &= \frac{m^4 M_{\text{P}}^2 M^2 \lambda^2}{16N^2} J(c_1X^2 - 2c_2X - 3c_3). \end{aligned} \quad (6.30)$$

For the linear term, first keep the total derivative unexpanded:

$$\begin{aligned} \frac{\mathcal{E}_a[L_{\lambda,1}]}{3Na^2} &= \frac{m^2 M_{\text{P}}^2}{2N} \left\{ M\lambda(2J - XJ')(XH_f - H) + 2HM\lambda J \right. \\ &\quad \left. + \frac{d}{dt} \left(\frac{M\lambda J}{N} \right) \right\}. \end{aligned} \quad (6.31)$$

Expanding the last derivative and using (6.3) produces

$$\begin{aligned} P_\lambda &= \frac{m^2 M_{\text{P}}^2 MJ}{2N^2} \dot{\lambda} + \frac{m^4 M_{\text{P}}^2 M^2 J}{16N^2} (c_1X^2 - 2c_2X - 3c_3)\lambda^2 \\ &\quad + m^2 M_{\text{P}}^2 \lambda \left[\frac{XMH_f}{N^2} \{c_1MX + c_2NX + c_2M + c_3N\} \right. \\ &\quad \left. + \frac{J(N\dot{M} - M\dot{N})}{2N^3} \right]. \end{aligned} \quad (6.32)$$

This expanded expression is cumbersome, but it is precisely what is needed for an off-shell consistency check. Combining all pieces gives

$$E_1 \equiv \frac{2\dot{H}}{N} + 3H^2 + \frac{P + P_g + P_\lambda}{M_{\text{P}}^2} = 0. \quad (6.33)$$

Reproduce the background equations

Starting only from (6.17), verify (6.22), (6.24), and (6.33) in that order. The supplied script `Maple_code/mtmg_bg_only_cdx.mpl` varies the vacuum homogeneous action with respect to N , a , and λ , and factors the multiplier equation. It is a partial background

check, not an implementation of the full consistency identity or of the perturbation sector. Keeping N , M , and λ time dependent is an effective way to expose premature gauge fixing.

6.4. The consistency identity and why there are two branches. Matter conservation reads

$$E_\rho \equiv \frac{\dot{\rho}}{N} + 3H(\rho + P) = 0. \quad (6.34)$$

A useful combination of equations is

$$E_B \equiv \frac{\dot{E}_0}{N} + 3HE_0 - \frac{3M_P^2}{4N}(4NH + m^2 M \lambda J)E_1 + \frac{3M_P^2}{4}E_\lambda + E_\rho. \quad (6.35)$$

After inserting (6.22)–(6.34), all $\dot{H}$, $\dot{\rho}$, $\dot{\lambda}$, $\dot{M}$, and $\dot{N}$ terms cancel. What remains is a cubic polynomial with no constant term:

$$E_B = \lambda(\zeta_1 + \zeta_2\lambda + \zeta_3\lambda^2), \quad (6.36)$$

where

$$\zeta_3 = -\frac{3M_P^2 M^3 m^6 J^2}{64N^3}(c_1 X^2 - 2c_2 X - 3c_3), \quad (6.37)$$

$$\zeta_2 = -\frac{3M_P^2 M^2 m^4 J}{8N^2} [H(c_1 X^2 - 2c_2 X - 3c_3) + 2XH_f(c_2 X + c_3)], \quad (6.38)$$

$$\begin{aligned} \zeta_1 = & \frac{3m^2 M J}{8N} [M_P^2 m^2 (c_2 X^2 + 2c_3 X + c_4) - 2P] \\ & - \frac{3M_P^2 X M m^2}{N} (c_2 X + c_3) H H_f - \frac{3M_P^2 M m^2}{4N} (c_1 X^2 - 2c_2 X - 3c_3) H^2. \end{aligned} \quad (6.39)$$

This calculation is long rather than conceptually difficult: differentiate E_0 , replace $\dot{X}$ using (6.3), group by powers of λ , and only then factor. The cancellation of every time derivative is a stringent check on (6.32).

6.5. Self-accelerating branch. The first factor of (6.24) gives

$$J(X) = 0. \quad (6.40)$$

For constant c_n and an isolated root, X is constant. Eliminating $c_3 = -c_1 X^2 - 2c_2 X$ gives

$$\rho_g = \frac{m^2 M_P^2}{2} (c_4 - 3c_2 X^2 - 2c_1 X^3) = \text{constant}, \quad P_g = -\rho_g. \quad (6.41)$$

Moreover $\rho_\lambda = 0$, whereas the scale-factor equation still contains

$$P_\lambda|_{J=0} = \frac{XM}{N}(c_1 X + c_2) \left(\frac{M}{N} - X \right) m^2 M_P^2 \lambda H_f. \quad (6.42)$$

The consistency polynomial reduces to

$$E_B|_{J=0} = 3m^2 M_P^2 \lambda H(c_2 X + c_3) \frac{M}{N} (H - X H_f). \quad (6.43)$$

On a generic background this fixes $\lambda = 0$, hence $P_\lambda = 0$. Vanishing of another factor represents a special, potentially rank-changing background and must be reanalysed rather than silently divided out.

For dust, the two independent equations are exactly those of GR with a cosmological constant:

$$3M_{\text{P}}^2 H^2 = \rho_m + \rho_g, \quad \frac{\dot{H}}{N} = -\frac{\rho_m}{2M_{\text{P}}^2}. \quad (6.44)$$

Self-acceleration can therefore originate from the graviton potential rather than a fundamental cosmological constant.

6.6. Normal branch. For $J \neq 0$, the second factor of (6.24) first gives

$$\lambda = \frac{4N(XH_f - H)}{m^2 M J}. \quad (6.45)$$

Substituting this result, not yet setting it to zero, into the consistency identity yields the factorization

$$E_B = -\frac{3}{2} \left[M_{\text{P}}^2 m^2 (c_2 X^2 + 2c_3 X + c_4) - 2M_{\text{P}}^2 X^2 H_f^2 - 2P \right] (H - XH_f). \quad (6.46)$$

Why select the second factor? Suppose instead that the square bracket vanished while $H \neq XH_f$. The original equations would then imply

$$3M_{\text{P}}^2 E_1 - \frac{3M}{XN} E_0 + \frac{NX - M}{NX(XH_f - H)} E_B = \frac{3}{NX} [2M_{\text{P}}^2 X^2 \dot{H}_f + M(P + \rho)] = 0. \quad (6.47)$$

This is an additional equation for the prescribed fiducial geometry and the matter source; it is not satisfied for generic external functions $(M, \tilde{a})$. The regular normal branch is therefore

$$H = XH_f, \quad \lambda = 0, \quad \rho_\lambda = P_\lambda = 0. \quad (6.48)$$

Specially tuned external data that satisfy the first factor require a separate constraint-rank analysis and are not part of this generic branch.

The evolution of X is now entirely kinematical. Combining (6.3) with (6.48),

$$\frac{\dot{X}}{NX} = \frac{MH_f}{N} - H = H \left(\frac{M}{NX} - 1 \right) = H(r - 1). \quad (6.49)$$

Thus a prescribed fiducial lapse and scale factor can make X time dependent. There is no propagating scalar hidden in this time dependence: X is built from an external field and the physical scale factor.

As a useful check, (6.20), (6.29), and (6.49) imply

$$\frac{\dot{\rho}_g}{N} = \frac{3m^2 M_{\text{P}}^2}{2} J \frac{\dot{X}}{N} = \frac{3m^2 M_{\text{P}}^2}{2} H J X (r - 1), \quad (6.50)$$

$$\rho_g + P_g = \frac{m^2 M_{\text{P}}^2}{2} X J (1 - r), \quad (6.51)$$

and hence $\dot{\rho}_g/N + 3H(\rho_g + P_g) = 0$.

Finally split

$$\rho_g = \rho_\Lambda + \rho_X, \quad \rho_\Lambda = \frac{m^2 M_{\text{P}}^2 c_4}{2}, \quad \rho_X = \frac{m^2 M_{\text{P}}^2}{2} (3c_3 X + 3c_2 X^2 + c_1 X^3). \quad (6.52)$$

The effective equation of state of the dynamical part can be obtained in two equivalent ways:

$$\begin{aligned} w_X &= -1 - \frac{\dot{\rho}_X}{3NH\rho_X} \\ &= -\frac{M(c_1X^2 + 2c_2X + c_3) + N(c_2X^2 + 2c_3X)}{N(3c_3X + 3c_2X^2 + c_1X^3)}. \end{aligned} \quad (6.53)$$

Unlike the self-accelerating branch, the normal branch can differ from Λ CDM already at background level, despite having no scalar gravitational degree of freedom.

7. Perturbations: no scalar graviton reappears

The absence of a gravitational scalar does not mean the absence of scalar cosmological perturbations. A dust fluid supplies one scalar mode, the gauge-invariant density contrast δ_m .

7.1. Self-accelerating branch. After solving lapse, shift, matter-velocity, and MTMG multiplier constraints, the dust action reduces to

$$S_\delta^{(2)} = \int dt d^3k \left[\frac{a^5 \rho_m}{2Nk^2} \dot{\delta}_m^2 + \frac{Na^5 \rho_m^2}{4M_P^2 k^2} \delta_m^2 \right]. \quad (7.1)$$

Its equation is exactly the GR growth equation,

$$\frac{1}{N} \frac{d}{dt} \left(\frac{\dot{\delta}_m}{N} \right) + 2H \frac{\dot{\delta}_m}{N} - 4\pi G_N \rho_m \delta_m = 0. \quad (7.2)$$

The Bardeen potentials satisfy $\Psi = \Phi$ and the standard Poisson equation. Vector perturbations also coincide with GR.

7.2. Normal branch. There is still only the matter scalar, but its reduced action takes the form

$$S_\delta^{(2)} = M_P^2 \int dt d^3k N a^3 Q \left[\frac{\dot{\delta}_m^2}{N^2} + 4\pi G_{\text{eff}}(t, k) \rho_m \delta_m^2 \right]. \quad (7.3)$$

The no-ghost condition is $Q > 0$. Both the friction term and G_{eff} can differ from GR. A compact illustration occurs in the subhorizon regime with $r = 1$ and subdominant ρ_g :

$$\frac{\bar{G}_{\text{eff}}}{G_N} = \frac{2\rho_m^2 + 15M_P^2 m^2 \Gamma_1 \rho_m}{2(\rho_m + 3M_P^2 m^2 \Gamma_1)^2}, \quad \Gamma_1 = -\frac{X}{4} J(X). \quad (7.4)$$

This can realize weaker gravity while the tensor mass remains positive.

7.3. Tensor sector. For both branches,

$$S_T^{(2)} = \frac{M_P^2}{8} \sum_{s=+, \times} \int dt d^3x N a^3 \left[\frac{\dot{h}_s^2}{N^2} - \frac{(\partial_i h_s)^2}{a^2} - \mu_T^2 h_s^2 \right], \quad (7.5)$$

with

$$\mu_T^2 = \frac{m^2 X}{2} [c_2 X + c_3 + rX(c_1 X + c_2)]. \quad (7.6)$$

The kinetic normalization and propagation speed are those of GR. One needs $M_P^2 > 0$ for the absence of tensor ghosts, while $\mu_T^2 \geq 0$ is the simple sufficient condition excluding a tachyonic tensor instability. A mildly negative μ_T^2 would instead have to be assessed against the relevant cosmological time scale. The tensor mass is the clean signature of the self-accelerating branch,

whose scalar and vector linear sectors otherwise match Λ CDM. Current cosmological analyses of the normal branch constrain a mass scale of order H_0 [22].

Common pitfall

Do not infer the nonlinear degree-of-freedom count from the FLRW quadratic action alone. A missing kinetic term on a symmetric background can mean strong coupling. MTMG is safe because the Hamiltonian constraint count was performed before choosing the background.

8. Exercises

4.1: Constraint bookkeeping

List the twelve precursor constraints and the two genuinely new MTMG constraints. Verify (3.7) and the final two-mode count. What would the count be if one of the two new constraints became dependent on a special background?

4.2: Self-accelerating energy density

Use $J(X) = 0$ to reduce (6.20) to (6.41). Show explicitly that its time derivative vanishes.

4.3: Normal-branch kinematics

Starting from $X = \tilde{a}/a$, derive $\dot{X}/(NX) = H(r - 1)$ under $H = XH_f$. Interpret $r = 1$.

4.4: Massive tensor dispersion

In a time interval short compared with H^{-1} , treat a and μ_T as constant and derive $\omega^2 = k^2/a^2 + \mu_T^2$. Compute the group velocity and explain why $c_T = 1$ refers to the high-frequency characteristic speed, not to the group velocity of a massive wave packet at finite k .

Answer to Exercise 4.1

If one new constraint became dependent, the generic 14 second-class constraints would fall to 13. An odd-dimensional reduced phase space signals that the classification of first- and second-class constraints must also change; one cannot simply report 5/2 degrees of freedom. The full Poisson matrix must be recomputed on that special branch. This is precisely why the generic nonlinear analysis is indispensable.

Lecture 5: VCDM and type-II minimally modified gravity

Ninety-minute map

0–15 min	Type-I versus type-II minimally modified gravity.
15–35 min	Canonical transformation, gauge fixing, and the VCDM Hamiltonian.
35–50 min	Lagrangian formulation and nonlinear degree-of-freedom count.
50–70 min	FLRW equations derived from minisuperspace.
70–82 min	Reconstruction of $V(\phi)$ for an arbitrary expansion history.
82–90 min	Tensor and matter perturbations.

Minimally modified gravity denotes theories with the same two local gravitational degrees of freedom as GR, but with different dynamics [23, 24]. Type-I theories possess an Einstein frame: after a change of variables, the gravitational Hamiltonian is that of GR and the modification resides in matter couplings. Type-II theories possess no such frame [25]. VCDM is a particularly simple type-II theory, controlled by one potential of a non-propagating auxiliary field [9].

1. Construction by a canonical detour

The construction is easier to remember as an algorithm.

1. Start from the ADM Hamiltonian of GR.
2. Perform a canonical transformation to variables $(\mathfrak{R}, \mathfrak{R}^i, \Gamma_{ij})$.
3. Add a cosmological constant in that transformed frame.
4. Add a gauge-fixing constraint so that no scalar graviton is activated.
5. Transform back to the original ADM variables.
6. Couple ordinary matter minimally in the original frame.

The cosmological constant introduced in step 3 becomes a potential $V(\phi)$ of an auxiliary field after the inverse transformation. The gauge-fixing term makes the result inequivalent to GR when $V_\phi \neq 0$.

The final Hamiltonian, after convenient field redefinitions, is

$$H_{\text{vcdm}} = \int d^3x \left[N \mathcal{H}_0^{\text{gr}} + N^i \mathcal{H}_i^{\text{gr}} + \sqrt{\gamma} \lambda_C \left(M_{\text{P}}^2 \phi - \frac{\pi^{ij} \gamma_{ij}}{\sqrt{\gamma}} \right) + \lambda_\phi \pi_\phi + \sqrt{\gamma} M_{\text{P}}^2 \lambda_{\text{gf}}^i \partial_i \phi + N \sqrt{\gamma} M_{\text{P}}^2 V(\phi) \right] + H_m. \quad (1.1)$$

The three auxiliary multiplier terms impose, respectively, the trace-momentum relation, $\pi_\phi \approx 0$, and spatial homogeneity of ϕ on each preferred slice. Hence ϕ is not a new propagating scalar.

Key idea

The symbol V in VCDM emphasizes that a function $V(\phi)$ replaces the constant vacuum term of Λ CDM, while the gravitational sector still has only two local tensor modes. The auxiliary field changes the constraint surface, not the dimension of the physical phase space.

2. Lagrangian form and constraint count

Legendre transforming (1.1) and redefining $\lambda_C = N\lambda$ gives

$$\mathcal{L}_{\text{vcdm}} = N\sqrt{\gamma} \left[\frac{M_{\text{P}}^2}{2} \left({}^{(3)}R + K_{ij}K^{ij} - K^2 - 2V(\phi) \right) - \frac{M_{\text{P}}^2}{N} \lambda_{\text{gf}}^i \partial_i \phi - \frac{3M_{\text{P}}^2}{4} \lambda^2 - M_{\text{P}}^2 \lambda (K + \phi) \right] + \mathcal{L}_m. \quad (2.1)$$

When $V_\phi = 0$, the auxiliary equations reduce the theory to GR with a cosmological constant. For $V_\phi \neq 0$, the preferred slicing remains and the theory is genuinely type II.

2.1. Nonlinear count. The canonical pairs (γ_{ij}, π^{ij}) and (ϕ, π_ϕ) give $12 + 2 = 14$ phase-space variables. Three extended momentum constraints are first class and generate spatial diffeomorphisms. Four independent constraints are second class: the Hamiltonian-type constraint, π_ϕ , the trace-momentum relation, and the scalar part of $\partial_i \phi = 0$. Therefore

$$N_{\text{dof}}^{\text{vcdm}} = \frac{1}{2}(14 - 2 \times 3 - 4) = 2. \quad (2.2)$$

Although λ_{gf}^i has three components, its transverse part drops out after integration by parts; it imposes one independent scalar constraint.

Common pitfall

The condition $\partial_i \phi = 0$ is part of the definition of the theory, not a coordinate gauge chosen after variation. Treating it as a removable gauge choice changes the constraint count and incorrectly introduces a scalar mode.

3. Deriving the FLRW equations

Take

$$N = N(t), \quad N^i = 0, \quad \gamma_{ij} = a^2(t)\delta_{ij}, \quad H = \frac{\dot{a}}{Na}. \quad (3.1)$$

Then

$${}^{(3)}R = 0, \quad K_{ij} = H\gamma_{ij}, \quad K = 3H, \quad K_{ij}K^{ij} - K^2 = -6H^2. \quad (3.2)$$

The gauge-fixing term vanishes on the homogeneous ansatz, and the gravitational minisuperspace Lagrangian per comoving volume is

$$L_{\text{mini}} = Na^3 M_{\text{P}}^2 \left[-3H^2 - V(\phi) - \frac{3}{4} \lambda^2 - \lambda(3H + \phi) \right] + L_m. \quad (3.3)$$

3.1. Matter off shell: the Schutz–Sorkin representative. For deriving the matter equations as well as the gravitational equations, a particularly economical homogeneous perfect-fluid action is

$$L_m = -Na^3 \left[\rho(n) + \frac{n}{N} \dot{\ell} \right] = -Na^3 \rho(n) - a^3 n \dot{\ell}. \quad (3.4)$$

Here n is the physical particle-number density and ℓ is a velocity potential. This is the convention used in the detailed VCDM Maple file, where the symbols are $n = J0(\tau)$ and $\ell = \text{varphi}(\tau)$. The two matter variations give

$$\frac{d}{dt}(a^3 n) = 0, \quad \dot{\ell} = -N \rho_{,n}. \quad (3.5)$$

Defining

$$P = n \rho_{,n} - \rho, \quad (3.6)$$

the lapse variation is already $\partial L_m / \partial N = -a^3 \rho$. The scale-factor variation, after using the n equation, is

$$\frac{\partial L_m}{\partial a} = -3N a^2 \rho - 3a^2 n \dot{\ell} = 3N a^2 P. \quad (3.7)$$

Therefore the often-used response formula

$$\delta S_m = \int dt N a^3 \left(-\rho \frac{\delta N}{N} + 3P \frac{\delta a}{a} \right) \quad (3.8)$$

is the on-shell metric variation of the simple off-shell action (3.4); the two statements are consistent but serve different purposes. Use the action when the fluid variables are to be varied, and the response formula when the perfect-fluid equations are already understood.

Which argument of ρ ?

The supplied detailed worksheet literally uses $\rho(J0(\tau))$, so its $J0$ denotes n . If one instead parametrizes the same energy density by $y = n^2$, the chain rule changes (3.6) to $P = 2y \rho_y - \rho$; mixing the two parametrizations without this chain rule creates an apparent factor-of-two discrepancy.

3.2. Multiplier and auxiliary equations. Varying λ immediately gives

$$\lambda = -2H - \frac{2}{3} \phi. \quad (3.9)$$

The lapse equation, after using (3.9), is

$$\phi^2 = 3V + \frac{3\rho}{M_P^2}. \quad (3.10)$$

Varying ϕ gives

$$\phi = \frac{3}{2} V_\phi - 3H. \quad (3.11)$$

Finally, a combination of the scale-factor and lapse equations yields

$$\frac{\dot{\phi}}{N} = \frac{3}{2} \frac{\rho + P}{M_P^2}, \quad (3.12)$$

while minimal matter coupling gives

$$\frac{\dot{\rho}}{N} + 3H(\rho + P) = 0. \quad (3.13)$$

Raychaudhuri equation

Differentiate (3.11):

$$\frac{\dot{\phi}}{N} = \frac{3}{2}V_{\phi\phi}\frac{\dot{\phi}}{N} - 3\frac{\dot{H}}{N}. \quad (3.14)$$

Solving for $\dot{H}/N$ and using (3.12) gives

$$\frac{\dot{H}}{N} = \frac{(\rho + P)(3V_{\phi\phi} - 2)}{4M_{\text{P}}^2}. \quad (3.15)$$

For $V_{\phi\phi} = 0$ this reduces to the GR result $\dot{H}/N = -(\rho + P)/(2M_{\text{P}}^2)$.

Equations (3.10) and (3.11) can be combined into the familiar Friedmann form

$$3M_{\text{P}}^2 H^2 = \rho + \rho_{\phi}, \quad (3.16)$$

where

$$\rho_{\phi} = M_{\text{P}}^2(V - \phi V_{\phi}) + \frac{3}{4}M_{\text{P}}^2 V_{\phi}^2. \quad (3.17)$$

Defining $\dot{\rho}_{\phi} + 3NH(\rho_{\phi} + P_{\phi}) = 0$ gives

$$P_{\phi} = -\rho_{\phi} - \frac{3}{2}(\rho + P)V_{\phi\phi}. \quad (3.18)$$

The effective component can cross $w_{\phi} = -1$ without a propagating wrong-sign scalar, because ϕ is auxiliary.

4. Reconstruction theorem

Let $\mathcal{N} = \ln(a/a_0)$ and suppose an expanding background $H(\mathcal{N}) > 0$ is specified together with minimally coupled matter species satisfying

$$\rho'_i = -3(\rho_i + P_i), \quad ' \equiv \frac{d}{d\mathcal{N}}. \quad (4.1)$$

Equation (3.12) becomes

$$\phi' = \frac{3}{2} \frac{\rho + P}{M_{\text{P}}^2 H}. \quad (4.2)$$

Therefore

$$\phi(\mathcal{N}) = \phi_0 + \int_{\mathcal{N}_0}^{\mathcal{N}} \frac{3}{2} \frac{\rho(\nu) + P(\nu)}{M_{\text{P}}^2 H(\nu)} d\nu. \quad (4.3)$$

If $\rho + P > 0$, this function is monotonic and has a local inverse $\mathcal{N} = \mathcal{N}(\phi)$. The potential is then

$$V(\phi) = \frac{1}{3}\phi^2 - \frac{\rho(\mathcal{N}(\phi))}{M_{\text{P}}^2}. \quad (4.4)$$

Proposition 5.1 (Background reconstruction). *For any prescribed expanding FLRW history $H(\mathcal{N})$ and ordinary matter history with $\rho + P > 0$, equations (4.3) and (4.4) construct a local Λ CDM potential realizing that history.*

Proof. Equation (4.2) enforces the auxiliary evolution and monotonicity permits inversion. Equation (4.4) enforces the lapse equation (3.10). Differentiate it with respect to $\mathcal{N}$, use $\rho' = -3(\rho + P)$ and (4.2), and divide by ϕ' :

$$V_{\phi} = \frac{2}{3}\phi + 2H. \quad (4.5)$$

This is equivalent to (3.11); hence no independent background equation remains unsatisfied. $\square$

A reconstruction is not yet a stable model

The theorem is local in field space and concerns the homogeneous background. One must still test invertibility when $\rho + P$ vanishes, regularity of $V(\phi)$, scalar perturbations, and the range of validity of the preferred foliation.

4.1. Worked example: the entire linear-potential family gives Λ CDM. Suppose the target obeys GR with

$$3M_{\text{P}}^2 H^2 = \rho + M_{\text{P}}^2 \Lambda, \quad H' = -\frac{\rho + P}{2M_{\text{P}}^2 H}. \quad (4.6)$$

Comparing with (4.2) gives

$$\phi' = -3H', \quad \phi = C - 3H. \quad (4.7)$$

Substitution in (4.4) yields

$$V(\phi) = \frac{2C}{3}\phi - \frac{C^2}{3} + \Lambda. \quad (4.8)$$

Although the potential is linear for $C \neq 0$, its effective energy density is

$$\rho_\phi = M_{\text{P}}^2 \Lambda. \quad (4.9)$$

Indeed, the C -dependent terms cancel between the three pieces in (3.17). The familiar constant potential is the $C = 0$ representative of a larger background-equivalent family.

5. Perturbations

The tensor action is exactly that of GR:

$$S_T^{(2)} = \frac{M_{\text{P}}^2}{8} \sum_{s=+, \times} \int dt d^3x N a^3 \left[\frac{\dot{h}_s^2}{N^2} - \frac{(\partial_i h_s)^2}{a^2} \right]. \quad (5.1)$$

Thus VCDM tensors are massless and luminal, unlike MTMG tensors.

In the scalar sector the only propagating modes are those supplied by matter. For one barotropic perfect fluid, the high- k sound speed and no-ghost coefficient are the GR ones,

$$c_s^2 = \left(\frac{\partial P}{\partial \rho} \right)_s, \quad Q_s = \frac{a^2 \rho^2}{2k^2(\rho + P)} > 0. \quad (5.2)$$

The displayed positivity assumes the usual matter condition $\rho + P > 0$; the sound-speed condition must be imposed separately. For dust, after integrating all auxiliary perturbations, one obtains

$$S_\delta^{(2)} = \frac{1}{2} \int d^4x N a^3 \left[\frac{\left(\frac{3}{2}\rho_m + \frac{k^2 M_{\text{P}}^2}{a^2} \right) a^2 \rho_m \dot{\delta}_m^2}{\left(\frac{3}{2}\rho_m + \frac{k^2 M_{\text{P}}^2}{a^2} - \frac{9}{4}\rho_m V_{\phi\phi} \right) k^2 N^2} + \frac{a^2 \rho_m^2}{2k^2 M_{\text{P}}^2} \delta_m^2 \right]. \quad (5.3)$$

The potential curvature changes the kinetic normalization and therefore the friction and effective gravitational coupling. At $k^2/a^2 \gg H^2|V_{\phi\phi}|$, the modification is suppressed and GR is recovered locally. This scale dependence permits cosmological departures without spoiling short-distance weak-gravity tests.

VCDM has been used to reconstruct rapid late-time transitions and bouncing histories [10, 26]. Such applications should always be read together with the perturbative regularity conditions, not only with a fitted $H(z)$.

6. Exercises

5.1: Minisuperspace variation

Starting from (3.3), derive (3.9) and (3.11). Keep $N(t)$ until after variation. Add a matter term $-Na^3\rho(a)$ and use $a\rho_{,a} = -3(\rho + P)$.

5.2: Reconstruct a constant- w target

Take dust plus a target dark component $\rho_X = \rho_{X0}e^{-3(1+w_X)N}$ with constant $w_X > -1$. Write the two quadratures needed for $\phi(N)$ and $V(\phi)$. An analytic inversion is not required; state precisely which monotonicity condition is needed.

5.3: Linear potentials

Insert $V = V_0 + \beta\phi$ into (3.17) and show that $\rho_\phi = M_P^2(V_0 + 3\beta^2/4)$ is constant. Relate (V_0, β) to (Λ, C) in (4.8).

5.4: No-ghost denominator

From (5.3), determine the finite- k condition for a positive kinetic term. Show explicitly that it is automatically satisfied at sufficiently large k for $\rho_m > 0$ and finite $V_{\phi\phi}$.

Answer to Exercise 5.4

Besides $\rho_m > 0$, the denominator requires

$$\frac{k^2 M_P^2}{a^2} + \frac{3}{2}\rho_m - \frac{9}{4}\rho_m V_{\phi\phi} > 0. \quad (6.1)$$

The first term dominates as $k/a \rightarrow \infty$. A violation can therefore occur only over a finite range of long wavelengths, where the full constraint system and not a quasistatic approximation must be used.

CHAPTER 6

Lecture 6: Comparative cosmology and a reproducible Maple workshop

Ninety-minute map

0–15 min	Compare the four mechanisms and their observable modes.
15–35 min	A universal FLRW and perturbation workflow.
35–60 min	Read the four Maple scripts and reproduce one background calculation.
60–75 min	Constraint elimination, integration by parts, and UV checks.
75–90 min	Capstone problems and a research-grade verification ladder.

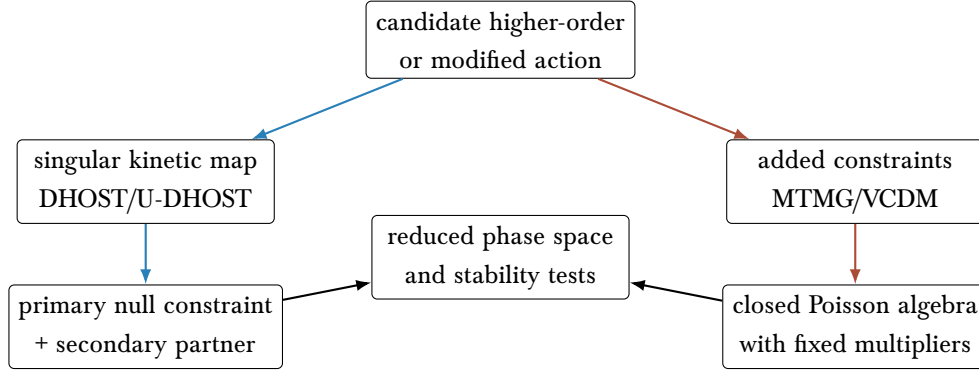
The final lecture turns the conceptual distinctions into a working calculation protocol. The goal is not to ask a computer algebra system to “find the physics”. It is to make the computer verify a sequence of physics decisions whose meaning is already understood.

1. Four theories, two mechanisms

Theory	Defining mechanism	Local gravity modes	Preferred structure	Cosmological discriminator
DHOST	Covariant kinetic degeneracy	$2T + 1S$	none beyond scalar configuration	braiding, scalar sound speed, effective gravity
U-DHOST	Degeneracy on scalar foliation	$2T + 1S$ plus elliptic response	timelike ϕ foliation	scale-dependent shadowy constraint
MTMG	Two added second-class constraints	$2T$ massive	fiducial spatial vielbein/metric	tensor mass; normal-branch growth
VCDM	Canonical construction plus gauge fixing	$2T$ massless	preferred constant- ϕ slicing	reconstructed $H(z)$ and matter growth

The phrase “three degrees of freedom” means different things in the first two rows. DHOST has three hyperbolic local modes after constraints. U-DHOST has the same number of modes carrying Cauchy data, but also an elliptic response fixed by boundary conditions. MTMG and VCDM have no gravitational scalar at all; any scalar cosmological mode comes from matter.

1.1. A constraint-flow view.



2. A universal background workflow

For any ADM action, use

$$N = N(t), \quad N^i = 0, \quad h_{ij} = a^2(t)\delta_{ij}. \quad (2.1)$$

Keep the lapse arbitrary and define $H = \dot{a}/(Na)$. For a homogeneous perfect fluid there are two equivalent levels of description. If the matter variables are retained off shell, use the Schutz–Sorkin minisuperspace representative derived in Lecture 5,

$$L_m = -Na^3 \left[\rho(n) + \frac{n}{N} \dot{\ell} \right]. \quad (2.2)$$

Its ℓ and n equations give $d(a^3 n)/dt = 0$, $\dot{\ell} = -N\rho_{,n}$, and $P = n\rho_{,n} - \rho$. Once these matter equations are used, its metric variation becomes the compact response convention

$$\delta S_m = \int dt Na^3 \left(-\rho \frac{\delta N}{N} + 3P \frac{\delta a}{a} \right). \quad (2.3)$$

Thus (2.3) is not a different matter action: it is the on-shell metric response of (2.2). The detailed VCDM worksheet uses the former action explicitly; the response formula is convenient when only the gravitational background equations are being derived. Then the background algorithm is:

- B1.** Substitute the homogeneous ansatz into the action, but do not set $N = 1$.
- B2.** Remove only genuine total derivatives. Record which boundary term was discarded.
- B3.** Vary N , a , all homogeneous fields, and every multiplier.
- B4.** Solve multiplier constraints first. Substitute them into the remaining equations, not prematurely into the action unless the elimination is known to be legitimate.
- B5.** Verify the Noether identity: one background equation must follow from the others plus matter conservation.
- B6.** Only now set $N = 1$ or change to e-fold time.

With $\mathcal{N} = \ln a$, the conversion rules are

$$\frac{1}{N} \frac{df}{dt} = Hf', \quad \frac{1}{N} \frac{dH}{dt} = HH', \quad \frac{1}{N^2} \frac{d^2 f}{dt^2} = H^2 f'' + HH' f' + \frac{\dot{N}}{N^3} \dot{f}, \quad (2.4)$$

where the final lapse term disappears in cosmic time only after variation.

Three background checks

For a complete background calculation, supplement the algebraic residuals in the supplied scripts with the following checks:

- (i) the GR or lower-theory limit;
- (ii) the matter continuity identity;
- (iii) dimensional homogeneity of every Friedmann term.

3. A universal scalar-perturbation workflow

For scalar modes use the decomposition

$$ds^2 = -N^2(1 + 2\epsilon\alpha)dt^2 + 2N\epsilon\partial_i\chi dt dx^i + [a^2(1 + 2\epsilon\zeta)\delta_{ij} + 2\epsilon\partial_i\partial_j E] dx^i dx^j, \quad (3.1)$$

$$\phi(t, \mathbf{x}) = \bar{\phi}(t) + \epsilon\delta\phi(t, \mathbf{x}). \quad (3.2)$$

The bookkeeping parameter ϵ must survive until the action has been expanded. The coefficients of ϵ^0 , ϵ^1 , and ϵ^2 play different roles:

- $S^{(0)}$ yields the background equations.
- $S^{(1)}$ must vanish after those equations are imposed. It is a strong test of signs and integrations by parts.
- $S^{(2)}$ contains constraints and physical kinetic/gradient matrices.

For a single Fourier mode one may use

$$f(t, \mathbf{x}) = \frac{f_k(t)}{\sqrt{\pi}} \cos(k\mathbf{x}), \quad \int_{-\pi}^{\pi} \cos^2(k\mathbf{x}) d\mathbf{x} = \pi \quad (k \in \mathbb{N}_{>0}). \quad (3.3)$$

This is the convention used by the Maple material that motivated the scripts. It turns spatial derivatives into powers of k while keeping all numerical normalizations explicit.

3.1. The correct elimination order. At quadratic order, separate fields into three groups.

1. **Gauge variables:** fix them only after checking that the gauge is complete and admissible. Unitary gauge requires $\dot{\phi} \neq 0$.
2. **Auxiliary variables:** lapse, shift, and theory-specific multipliers. Vary first and solve their constraint equations.
3. **Dynamical variables:** compute the reduced kinetic matrix only after all auxiliary fields have been eliminated.

An auxiliary equation may be differential in space, as in U-DHOST. In that case “solve” means invert an elliptic operator with stated boundary conditions; replacing it by an algebraic division loses the shadowy mode.

4. The supplied Maple scripts

The directory `Maple_code/` contains three short background scripts, three more detailed variants whose names end in `_cdx.mpl`, and one long U-DHOST perturbation worksheet. The main files are:

File	Main reproducible calculation
dhost_bg_only_cdx.mpl	Constructs the quadratic DHOST invariants on flat FLRW directly from the covariant action, compares two homogeneous Lagrangians variationally, and prints the N , a , and ϕ equations.
udhost_opt.mpl	Expands a general shift-symmetric quadratic scalar-tensor action to second order in scalar perturbations, performs a sequence of integrations by parts and auxiliary eliminations, constructs the reduced one-field action, and extracts the ultra-violet no-ghost coefficient Q_s and propagation speed c_s^2 . It also writes an intermediate snapshot to <code>actiogen1.txt</code> ; it does not implement the toy shadowy-mode dispersion relation of Lecture 3.
mtmg_bg_only_cdx.mpl	Constructs the homogeneous spatial square root and potential polynomials, varies the vacuum action in N , a , and λ , and factors the multiplier constraint. Matter, the full consistency identity, and perturbations are not included.
vcdm_bg_only_cdx.mpl	Varies the VCDM homogeneous action coupled to a Schutz-type fluid, prints the auxiliary and background equations, and checks the fluid continuity residual. It does not derive the reconstruction quadratures.

The files `dhost_bg_only.mpl`, `mtmg_bg_only.mpl`, and `vcdm_bg_only.mpl` are shorter companions that print only a subset of the corresponding equations. Keep the lapse explicit and perform functional variation before gauge fixing. Also note that `udhost_opt.mpl` overwrites `actiogen1.txt`; run it on a copy if that snapshot must be preserved.

The two VCDM files intentionally have different scope. The detailed `vcdm_bg_only_cdx.mpl` varies the genuine fluid action (2.2) with $n = J_0(t)$. The short companion writes $-N a^3 \rho(a)$ and varies only λ and ϕ ; it is a minimal check of the auxiliary VCDM sector, not a complete perfect-fluid background derivation.

4.1. Fundiff and small reusable procedures. The detailed worksheets use Maple's functional derivative as their primary Euler-Lagrange route. The recurring pattern is

```

1 mini_action := Physics[Intc](Lmini,t):
2 eom_q := simplify(subs(tt1=t,
3   convert(Physics[Fundiff](mini_action,q(tt1)),D))):

```

It automatically includes every derivative order present in the functional. The procedure equation in `udhost_opt.mpl` is simply a small wrapper around this same `Fundiff` call; the worksheet also uses it to verify that two expressions related by integration by parts have identical Euler-Lagrange derivatives.

The short companion files instead use a transparent local implementation. For a minisuperspace Lagrangian containing first derivatives, its simplest form is

```

1 EL := proc(L::algebraic, q::algebraic, rt::name)
2   local t;

```

```

3   t := rt;
4   simplify(diff(L, q) - diff(diff(L, diff(q,t)), t));
5 end proc:

```

If higher derivatives are present, use the generalized expression

$$E_q = \frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} + \frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{q}}. \quad (4.1)$$

The actual short scripts call this generalized helper EL2 and use temporary placeholders for q , $\dot{q}$, and $\ddot{q}$ before differentiating. There is therefore no discrepancy: Fundiff is the main route in the long calculations, while EL/EL2 is a dependency-light, inspectable route for short examples and an independent cross-check. For a research result, agreement of the two routes is more valuable than choosing one of them exclusively.

An integration-by-parts replacement must be checked at the equation level. For example,

```

1 IBP_D2_D1 := proc(L::algebraic, q::algebraic, rt::name)
2   local t, C;
3   t := rt;
4   C := coeff(coeff(expand(L), diff(q,t,t)), diff(q,t));
5   expand(L - C*diff(q,t,t)*diff(q,t)
6         - 1/2*diff(C,t)*diff(q,t)^2);
7 end proc:

```

The difference between the old and new expressions should have identically vanishing Euler-Lagrange derivative, not merely vanish after choosing a particular solution.

Expression swell versus physics

Commands such as `expand` and `simplify` can turn a one-line factorized constraint into thousands of terms. Preserve the factors that encode branches, Hessian rank, or stability. Expand only to extract a coefficient, and immediately factor the result again.

5. A verification ladder

For a research calculation, passing one check is not enough. Use the following ladder, stopping at the first failure.

- V1. Kinematics:** inverse metric times metric is the identity; determinant agrees with the ADM determinant.
- V2. Background:** $S^{(1)}$ vanishes on the background equations.
- V3. Gauge:** pure-gauge perturbations give a boundary term.
- V4. Constraints:** the number and differential type of auxiliary equations agree with the Hamiltonian count.
- V5. Reduction:** substituting auxiliary solutions back into the equations agrees with varying the reduced action.
- V6. UV:** the high- k kinetic and gradient matrices reproduce known limits.
- V7. Theory limit:** DHOST $\rightarrow$ Horndeski/GR, U-DHOST $\rightarrow$ DHOST, MTMG $m \rightarrow 0$, and Λ CDM $V_{\phi\phi} \rightarrow 0$.

V8. Independent route: reproduce one key equation by a second method, for example ADM versus covariant expansion.

6. Capstone problems

6.1: A luminal DHOST model

Choose $F = M_{\text{p}}^2/2$, $Q = 0$, and $A_3 = \eta/\Lambda^6$ constant. Construct A_4 and A_5 from (5.4)–(5.5). Check their mass dimensions if $[\phi] = 1$ and $[X] = 4$. Evaluate the homogeneous operator on $\phi = \phi(t)$ and identify the highest time derivative before using degeneracy.

6.2: Shadow kernel on a periodic box

Replace the real line in (4.5) by a periodic interval of length L . Construct the Fourier-series Green function and discuss the $k = 0$ mode. Which boundary datum replaces decay at spatial infinity?

6.3: MTMG branch scanner

For chosen (c_1, c_2, c_3, c_4) , solve $J(X) = 0$, retain positive real roots, and test $\rho_g > 0$ and $\mu_T^2 \geq 0$. Explain why the fiducial ratio r is still needed to evaluate the tensor mass even when X is fixed.

6.4: VCDM reconstruction from tabulated $H(z)$

Given a smooth table of $H(z)$ and standard matter densities, outline a numerically stable pipeline for equations (4.3)–(4.4). Use monotone interpolation for $\phi(\mathcal{N})$ and state how you would test $V_{\phi\phi}$ without amplifying interpolation noise.

6.5: Compare two theories with identical $H(z)$

Reconstruct VCDM to match a normal-branch MTMG background. List three linear observables that can still separate the theories. Your list must include one tensor observable and one scalar-growth observable.

Discussion of Exercise 6.5

The backgrounds can be identical, but MTMG tensors obey a massive dispersion relation whereas VCDM tensors are massless. The normal MTMG branch can alter G_{eff} through the graviton potential; VCDM alters dust friction and clustering through $V_{\phi\phi}$ with GR recovery at high k . Additional discriminants include gravitational slip, the scale dependence of growth, standard-siren propagation, and the integrated Sachs–Wolfe effect. Matching $H(z)$ never proves equivalence of constrained theories.

7. Reading the original papers efficiently

The following order minimizes notation changes.

1. Read the mechanical degeneracy sections of Refs. [1, 4].

2. Reproduce class Ia from the quadratic classification [3]; postpone cubic operators until the quadratic null-vector calculation is comfortable.
3. Read the solvable U-DHOST example before its general classification [5]; then read the nonlinear Hamiltonian treatment [6].
4. For MTMG, read the constraint count in the original short paper, then use the longer phenomenology paper for the metric action and both FLRW branches [7, 8].
5. For Λ CDM, derive the four background equations before reading the reconstruction section [9].

The common reward is that equations which initially look theory-specific collapse into the same small collection of ideas: null directions, secondary constraints, elliptic boundary data, and reduced kinetic matrices.

APPENDIX A

Conventions and variation identities

1. Geometric conventions

We use signature $(-, +, +, +)$ and

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma}, \quad R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}. \quad (1.1)$$

The reduced Planck mass satisfies $M_{\text{p}}^{-2} = 8\pi G_N$. The scalar kinetic variable is $X = g^{\mu\nu} \phi_\mu \phi_\nu$ rather than the alternative $X_{\text{alt}} = -\phi_\mu \phi^\mu / 2$. To translate a function $F(X_{\text{alt}})$ into our convention, replace its argument by $-X/2$ and transform every X derivative accordingly.

The ADM extrinsic curvature is (4.2). Reversing the sign of K_{ij} changes terms odd in K , including several braiding and multiplier terms, but leaves $K_{ij} K^{ij} - K^2$ invariant.

2. Euler-Lagrange operators

For a one-dimensional Lagrangian $L(q, \dot{q}, \ddot{q}; t)$,

$$\mathcal{E}_q[L] = \frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} + \frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{q}}. \quad (2.1)$$

Two Lagrangians differ by a total derivative if their Euler-Lagrange operators vanish identically for every field. Checking only that their difference vanishes on a background solution is weaker and can hide an incorrect integration by parts.

Useful identities are

$$C(t) \dot{q} \ddot{q} \simeq -\frac{1}{2} \dot{C} \dot{q}^2, \quad (2.2)$$

$$C(t) q \ddot{q} \simeq -C \dot{q}^2 + \frac{1}{2} \ddot{C} q^2, \quad (2.3)$$

$$C(t) q \dot{q} \simeq -\frac{1}{2} \dot{C} q^2, \quad (2.4)$$

where $\simeq$ means equality up to a total time derivative.

3. Perturbative matrix identities

For a matrix $\mathbf{g} = \bar{\mathbf{g}} + \epsilon \mathbf{h}$, define $\mathbf{X} = \bar{\mathbf{g}}^{-1} \mathbf{h}$. Then

$$\mathbf{g}^{-1} = \bar{\mathbf{g}}^{-1} - \epsilon \bar{\mathbf{g}}^{-1} \mathbf{h} \bar{\mathbf{g}}^{-1} + \epsilon^2 \bar{\mathbf{g}}^{-1} \mathbf{h} \bar{\mathbf{g}}^{-1} \mathbf{h} \bar{\mathbf{g}}^{-1} + \mathcal{O}(\epsilon^3), \quad (3.1)$$

$$\sqrt{-\mathbf{g}} = \sqrt{-\bar{\mathbf{g}}} \left[1 + \frac{\epsilon}{2} \text{Tr } \mathbf{X} + \frac{\epsilon^2}{8} \left((\text{Tr } \mathbf{X})^2 - 2 \text{Tr } \mathbf{X}^2 \right) + \mathcal{O}(\epsilon^3) \right]. \quad (3.2)$$

These two identities are fast independent checks on the output of a tensor package.

4. Fourier and reality conventions

For a real perturbation,

$$f(t, \mathbf{x}) = \int \frac{d^3 k}{(2\pi)^3} f_{\mathbf{k}}(t) e^{i\mathbf{k} \cdot \mathbf{x}}, \quad f_{-\mathbf{k}} = f_{\mathbf{k}}^*. \quad (4.1)$$

Then $\partial^2 \rightarrow -k^2$. A quadratic action is normally written as an integral over all $\mathbf{k}$ with $f_{\mathbf{k}} f_{-\mathbf{k}}$, or over half of Fourier space with an additional factor of two. Mixing these conventions is a common source of incorrect kinetic normalizations.

5. Two degeneracy dictionaries: D_i versus $c_{\text{Di}}, c_{\text{U}}$

The notation is potentially misleading because the subscripts look similar, but the two sets of quantities have different definitions and different jobs. The coefficients D_0, D_1, D_2 are defined by the covariant determinant polynomial (2.7). With the operator basis (1.3) and the normalization adopted in these notes, D_0 is given in (2.9), while the remaining two coefficients are [1]

$$\begin{aligned} D_1 = & 4 \left[X^2 A_1 (A_1 + 3A_2) - 2F^2 - 4XF A_2 \right] A_4 + 4X^2 F (A_1 + A_2) A_5 \\ & + 8X A_1^3 - 4(F + 4XF_X - 6XA_2) A_1^2 - 16(F + 5XF_X) A_1 A_2 \\ & + 4X(3F - 4XF_X) A_1 A_3 - X^2 F A_3^2 + 32F_X (F + 2XF_X) A_2 \\ & - 16FF_X A_1 - 8F(F - XF_X) A_3 + 48FF_X^2, \end{aligned} \quad (5.1)$$

$$\begin{aligned} D_2 = & 4 \left[2F^2 + 4XF A_2 - X^2 A_1 (A_1 + 3A_2) \right] A_5 + 4A_1^3 \\ & + 4(2A_2 - XA_3 - 4F_X) A_1^2 + 3X^2 A_1 A_3^2 - 4XF A_3^2 \\ & + 8(F + XF_X) A_1 A_3 - 32F_X A_1 A_2 + 16F_X^2 A_1 + 32F_X^2 A_2 - 16FF_X A_3. \end{aligned} \quad (5.2)$$

In the original notation of [1], f and α_I in these formulae correspond to our F and A_I . Those α_I are therefore not the residual coefficients α_I introduced in (2.3).

The unitary-adapted quantities of [19] are instead

$$\begin{aligned} c_{\text{D1}} &= X(A_1 + A_2), & \mathcal{G}_T &= F - XA_1, \\ \mathcal{H}_D &= 2\mathcal{G}_T + 3c_{\text{D1}}, \\ \Upsilon &= -\frac{X(4F_X + 2A_2 + XA_3)}{2\mathcal{H}_D}, & \tilde{\mathcal{G}}_T &= (1 - \Upsilon)F - 2XF_X, \\ c_{\text{D2}} &= X^2 A_4 + 4X\Upsilon F_X - 2\mathcal{G}_T + 2(1 - \Upsilon)\tilde{\mathcal{G}}_T, \\ c_{\text{U}} &= -X^3 A_5 - X^2(A_3 + A_4) - c_{\text{D1}} + 3\Upsilon^2 \mathcal{H}_D. \end{aligned} \quad (5.3)$$

They are not obtained by relabelling D_0, D_1, D_2 . For example,

$$D_0 = -\frac{4c_{\text{D1}}}{X} \left\{ XF(2A_1 + XA_4 + 4F_X) - 2F^2 - 8X^2 F_X^2 \right\}. \quad (5.4)$$

Consequently, $c_{\text{D1}} = 0$ selects the class-I factor of $D_0 = 0$; the converse is false on the second branch of $D_0 = 0$.

On the regular class-I branch, the equivalence becomes especially transparent. Imposing the conditions in the displayed order gives

$$\begin{aligned}
c_{D1} = 0 &\implies A_2 = -A_1, & D_0 = 0, \\
D_1 = -\frac{8\mathcal{G}_T^2}{X^2}c_{D2}, & & (c_{D1} = 0), \\
D_2 = -\frac{8\mathcal{G}_T^2}{X^3}c_U, & & (c_{D1} = c_{D2} = 0).
\end{aligned} \tag{5.5}$$

Thus, for $X\mathcal{G}_T \neq 0$, the three equations $c_{D1} = c_{D2} = c_U = 0$ reproduce the class-I covariant conditions $D_0 = D_1 = D_2 = 0$. For U-DHOST one keeps only $c_U = 0$; the two generally non-zero coefficients c_{D1}, c_{D2} then measure departures from the covariant DHOST subclass and multiply spatial operators in the unitary-gauge perturbation action. This is why the c -parameterization is preferable in Lecture 3, whereas the D_i remain preferable for the full covariant classification in Lecture 2.

APPENDIX B

Constraint ledgers

1. First-class and second-class constraints

A constraint $C_A \approx 0$ is first class if its Poisson bracket with every constraint vanishes on the constraint surface. First-class constraints generate gauge transformations. The matrix

$$\Delta_{AB}(\mathbf{x}, \mathbf{y}) = \{C_A(\mathbf{x}), C_B(\mathbf{y})\} \quad (1.1)$$

is invertible on the second-class sector. The symbol $\approx$ means equality after constraints are imposed; replacing it by an ordinary equality inside a Poisson bracket before computing the bracket is generally wrong [27].

2. Counts used in these notes

System	Phase-space dimension	First-class	Second-class	Local modes
GR, reduced ADM	12 from (h_{ij}, π^{ij})	4	0	2
GR, extended ADM	20 including lapse and shift pairs	8	0	2
Non-degenerate toy model	6 physical dimensions after trivial auxiliaries	0	0	3
Degenerate toy model	same initial phase space	0	2 additional	2
Generic quadratic DHOST	metric plus scalar, covariant constraint algebra	diffeomorphism set	Ostrogradsky-removing pair	$2T + 1S$
U-DHOST scalar perturbations	6 from (α, χ, ζ) and momenta	0 after gauge fixing	4	1 scalar
MTMG precursor	18 from $e^I{}_j$	0 in the chosen reduced description	12	3
MTMG	18 from $e^I{}_j$	0 in the chosen reduced description	14	2
VCDM	14 from $(\gamma_{ij}, \pi^{ij}; \phi, \pi_\phi)$	3	4	2

Counts in different rows may use different phase-space conventions. For example, lapse and shift pairs can be retained or removed at the outset. A valid comparison must include both their momenta and their associated primary constraints, or neither.

3. Rank changes on special backgrounds

Let Δ_{AB} be the constraint Poisson matrix. If $\det \Delta \neq 0$ generically but vanishes on a special background, three possibilities must be distinguished:

1. a new gauge symmetry appears and a second-class pair becomes first class;
2. a multiplier is no longer determined and a tertiary constraint appears;
3. the quadratic action loses a kinetic term while the nonlinear mode remains, signalling strong coupling.

Only a fresh Dirac analysis decides which occurs.

APPENDIX C

Selected worked checks

1. Null vector of the mechanical Hessian

When $ac = b^2$, the matrix (3.3) satisfies

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} 1 \\ -b/c \end{pmatrix} = \begin{pmatrix} a - b^2/c \\ 0 \end{pmatrix} = 0. \quad (1.1)$$

Contracting the acceleration equations with this vector removes both $\ddot{Q}$ and $\ddot{q}$. Contracting the momentum definitions gives the primary constraint $p_Q - (b/c)p_q = 0$. The Lagrangian and Hamiltonian null vectors are therefore the same algebraic object.

2. The luminal class-Ia reduction

Set $A_1 = A_2 = 0$ in (3.2). Its denominator becomes $8F^2$ and its numerator becomes

$$-X^2 F A_3^2 + 8F(XF_X - F)A_3 + 48FF_X^2. \quad (2.1)$$

Dividing by $8F^2$ gives (5.4). Similarly,

$$A_5 = \frac{(4F_X + XA_3)(4FA_3)}{8F^2} = \frac{(4F_X + XA_3)A_3}{2F}. \quad (2.2)$$

This two-line calculation is a useful test that sign conventions for X have not changed halfway through an implementation.

3. Fourier solution of the shadow constraint

Fourier transform (4.5) with $\partial_x^2 \rightarrow -k^2$. One obtains

$$-(k^2 + \kappa^2)\psi_k = -k^4\chi_k, \quad \psi_k = \frac{k^4}{k^2 + \kappa^2}\chi_k. \quad (3.1)$$

Substitution in (4.4) gives

$$\omega^2 = k^2 - \frac{k^4}{k^2 + \kappa^2} = \frac{\kappa^2 k^2}{k^2 + \kappa^2}. \quad (3.2)$$

Since $\kappa^{-2} = 2\mu(1 - \alpha^2)^2$, this is the boosted form of (3.4). At $k \ll \kappa$,

$$\omega^2 = k^2 - \frac{k^4}{\kappa^2} + O(k^6/\kappa^4), \quad (3.3)$$

which displays the emergent relativistic light cone at long distance.

4. MTMG self-accelerating reduction

From $J(X) = c_1 X^2 + 2c_2 X + c_3 = 0$,

$$c_3 X = -c_1 X^3 - 2c_2 X^2. \quad (4.1)$$

Therefore

$$c_4 + 3c_3X + 3c_2X^2 + c_1X^3 = c_4 - 3c_1X^3 - 6c_2X^2 + 3c_2X^2 + c_1X^3 \quad (4.2)$$

$$= c_4 - 2c_1X^3 - 3c_2X^2, \quad (4.3)$$

which proves (6.41).

5. Λ CDM linear-potential cancellation

For $V = V_0 + \beta\phi$,

$$\rho_\phi = M_{\text{P}}^2(V_0 + \beta\phi - \beta\phi) + \frac{3}{4}M_{\text{P}}^2\beta^2 = M_{\text{P}}^2\left(V_0 + \frac{3}{4}\beta^2\right). \quad (5.1)$$

Taking $\beta = 2C/3$ and $V_0 = \Lambda - C^2/3$ gives $\rho_\phi = M_{\text{P}}^2\Lambda$. Linear V is therefore a useful internal consistency check, not merely a special cosmological model.

APPENDIX D

Computer-algebra use and expected checks

1. Running the scripts

The detailed scripts require a standard Maple kernel together with the Physics, DifferentialGeometry, and Tensor packages where imported. Run each in a fresh kernel because assignments to names such as N , X , or H persist globally. From a Maple worksheet one may use

```
1 restart:
2 currentdir("/path/to/lectures/Maple_code"):
3 read "vcdm_bg_only_cdx.mpl":
```

or invoke the installed command-line interpreter directly:

```
1 /path/to/maple -q mtmg_bg_only_cdx.mpl
```

Most scripts only print their results. The exception is `udhost_opt.mpl`, which overwrites `actiogen1.txt` with an intermediate perturbation snapshot.

2. Expected zero residuals

The detailed files expose the following checks or diagnostic outputs:

- `dhost_bg_only_cdx.mpl`: covariant FLRW invariants, the variational equivalence of two homogeneous Lagrangians, and the three background Euler-Lagrange equations. All three entries in the Boolean equivalence output should be true.
- `udhost_opt.mpl`: intermediate background and perturbation residuals for a shift-symmetric quadratic action. The worksheet has no final named pass/fail summary; each displayed simplification must be inspected, and assumptions introduced by `solve` must be tracked.
- `mtmg_bg_only_cdx.mpl`: the identities $U' - 3J = 0$ and $3V - XV' - 3J = 0$, the homogeneous variations, and the factored λ constraint. It does not check the full cubic consistency identity or tensor perturbations.
- `vcdm_bg_only_cdx.mpl`: the homogeneous gravitational and fluid equations, the conserved charge, and the fluid continuity residual. It does not check reconstruction or perturbations.

3. What the scripts deliberately do not automate

They do not choose physical boundary conditions, decide whether a singular branch is acceptable, or replace a nonlinear Dirac analysis. In particular:

- a zero FLRW Hessian does not prove covariant DHOST degeneracy;

- a quartic frequency polynomial does not identify which roots satisfy the U-DHOST boundary problem;
- a factorized MTMG background equation must not be divided by a factor before branches are classified;
- a reconstructed VCDM potential must still be tested for perturbative stability.

Computer algebra is strongest when asked to prove a sharply formulated identity and weakest when asked to infer the physical problem that should have been posed.

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