Exercise: Effective Field Theory

SHEAP2026

1. Consider a $U(1)$ complex scalar field $\Phi(x)$ with a quartic Mexican-hat potential in the absence of gravity,

$$\mathcal{L}_{\rm UV} = -|\partial\Phi|^2 - V(|\Phi|) \ , \qquad V(|\Phi|) = \lambda(|\Phi|^2 - v^2)^2 \ , \tag{1}$$

where λ and v are constants. In the broken phase, $\langle\Phi\rangle = v$, one has a massless field (Goldstone boson) $\pi(x)$ and a heavy field (Higgs) $h(x)$. We are now going to integrate out the Higgs field to obtain a low-energy effective theory for $\pi(x)$.

   (a) Assume that $\Phi = h_0 \exp(i\pi)$ with h_0 being constant. Show that in the case where $X = (\partial\pi)^2 = const.$ the classical solution for h_0 is given by

$$h_0^2 = -\frac{X}{2\lambda} + v^2 = \frac{1}{4\lambda}(\mu^2 - 2X) \ , \tag{2}$$

   where $\mu^2 \equiv 4\lambda v^2$.

   (b) Use Eq. (2) in the Lagrangian (1) to show that the low-energy theory at tree-level for $\pi(x)$ is

$$\mathcal{L}_{\rm IR}(\pi) \simeq -\frac{1}{4\lambda} X(\mu^2 - X) \ . \tag{3}$$

   Notice that this EFT stops at quadratic order in X.

   (c) (*Optional*) Take the Lagrangian (3) and perform $\pi = \hat{\pi} + \delta\pi$ with $\partial_\mu\hat{\pi}$ being timelike. Show that the system is stable when $\hat{X} < \mu^2/6$, i.e. no ghost/gradient instabilities, and $\hat{X} = (\partial_\mu\hat{\pi})^2$. *Hint*: Extract the quadratic Lagrangian for $\delta\pi$ and impose the conditions under which both the time and spatial kinetic terms have the correct sign.

2. Consider the following EFT of inflation in the unitary gauge,

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{M_{\rm Pl}^2}{2}R - M_{\rm Pl}^2(3H^2 + \dot{H}) + M_{\rm Pl}^2\dot{H}\,g^{00}\right] \ , \tag{4}$$

where $H = \dot{a}/a$ and dot refers to a time derivative. The background metric is $\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2\mathrm{d}\vec{x}^2$. Note that this EFT corresponds to a canonical single field model.

   (a) Perform the Stueckelberg trick to the action (4) to show that, in the decoupling limit, the π action is

$$S_\pi = \int \mathrm{d}^4x\, a^3 M_{\rm Pl}^2\big[-3H(t+\pi)^2 - 2(1+\dot{\pi})\dot{H}(t+\pi) + \dot{H}(t+\pi)(\partial_\mu\pi)^2\big] \ . \tag{5}$$

   Convince yourself that the above action vanishes at linear order in $\pi(x)$. Recall that $R = 6(\dot{H} + 2H^2)$.

(b) Usually, in the decoupling limit, we expect that $\pi(x)$ becomes time-independent outside the horizon since $\zeta = -H\pi$ and ζ freezes outside the horizon. However, this is not manifest in the action (5) due to the polynomial terms in $\pi(x)$ that generally induce evolution of $\pi(x)$ outside the horizon. Show that the action (5) can be rewritten as

$$S_\pi = \int \mathrm{d}^4x \, a(t)^3 M_{\rm Pl}^2 \dot{H}(t+\pi)(\partial_\mu \pi)^2 \,, \tag{6}$$

where we have neglected terms of order ϵ^2, where $\epsilon \equiv -\dot{H}/H^2$. The resulting action shows explicitly that $\pi = \mathit{const.}$ is a solution. Note interestingly that all nonlinearities in π in the decoupling limit of any single field models are encoded in a single function, $\dot{H}(t+\pi)$.