Review of Goldstone Bosons

1. Basic idea
2. SSB for internal symmetries
3. Stuckelberg trick
4. SSB for spacetime symmetries.

: Ghost Condensation.

1. Spontaneous Symmetry Breaking

In our previous lecture, we have learned several key concepts of the EFT, for examples

(1) Separation of scales

(2) Cutoff and derivative expansion

and so on.

But we didn't discuss much on the symmetries.

⇒ Of course, once we specify the symmetries of the EFT, we can just write down all the terms compatible to such symmetries.

* However, in many cases, we find ourselves in a situation where a background field does not respect some symmetries.

⇔ [Such symmetries are spontaneously broken]

or we say a vacuum carries non-trivial charges.

Suppose one starts from a symmetry group G

: $G \to H$ by a vacuum $|0\rangle$

$\overset{\text{unbroken}}{H}$

$$\Rightarrow \quad Q^a|0\rangle \neq 0 \quad \text{and} \quad Q^i|0\rangle = 0$$

a = index of broken generators

$= 1, \ldots, \dim G - \dim H$

i = index of unbroken generators

We refer $G \to H$ as "symmetry breaking pattern"

- Goldstone's theorem : There will appear one massless and spinless particle for each broken generator

[Nambu - Goldstone bosons]

! This only applies to global internal symmetries.

For breaking of spacetime symmetries, # of NG bosons $\neq$ # of broken generators

(Typicall # NG < # of broken generators)

<u>Note</u> Actually when we say the symmetries are spontaneously broken by ϕ_0, what we actually mean is that such broken symmetries are now <u>non-linearly realized</u>.

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<u>Example</u> A $U(1)$ complex scalar field with Mexican hat potential

$$\mathcal{L} = -\partial_\mu \phi^* \partial^\mu \phi + \mu^2 \phi^* \phi - \lambda (\phi^* \phi)^2$$

↳ not usual mass term

$\Rightarrow$ This $\mathcal{L}$ is invariant under $\phi \to e^{i\alpha} \phi$

: global $U(1)$ symmetry

Let us find the ground states from $V(\phi^*\phi)$

It is convenient to write

$$\phi(x) = \underbrace{\rho(x)}_{\text{Radial}} e^{i\pi(x)} \quad : \text{ Generic form.}$$

Angular

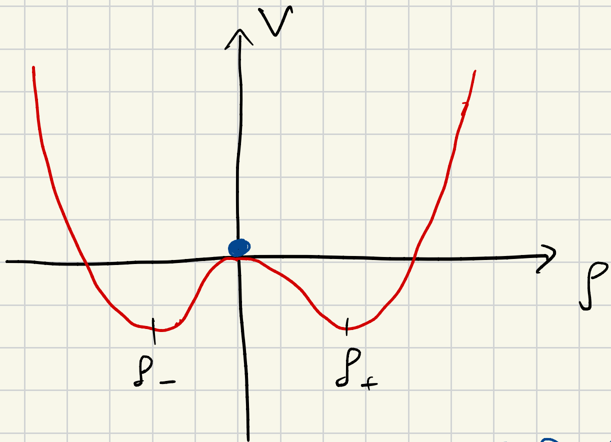
$[\pi] = 0$

$$\Rightarrow V = -\mu^2 \phi^* \phi + \lambda (\phi^* \phi)^2$$

$$V(\rho, \pi) = -\mu^2 \rho^2 + \lambda \rho^4 = -\rho^2 (\mu^2 - \lambda \rho^2)$$

$$V(\rho, \pi) = -\rho^2(\mu^2 - \lambda\rho^2)$$

For $\mu^2 > 0$ and $\lambda > 0$



$$\frac{dV}{d\rho} = -2\mu^2\rho + 4\lambda\rho^3$$

$$\left.\frac{dV}{d\rho}\right|_{\rho=\rho_0} = 0$$

$$2\rho_0(-\mu^2 + 2\lambda\rho_0^2) = 0$$

$$\Downarrow$$

$$\rho_0 = 0, \quad \rho_0 = \pm\frac{\mu}{\sqrt{2\lambda}}$$

Note $\rho_0^2 = \phi_0^* \phi_0$

$$\rho_\pm = \pm\frac{\mu}{\sqrt{2\lambda}}$$

$\Rightarrow$ For $\rho_0 = 0$, the global $U(1)$ is still OK

[symmetric or unbroken phase] : $e^{i\alpha}\phi_0 = \phi_0 = 0$

$\Rightarrow$ But for $\rho_0 = \rho_\pm$, the global $U(1)$ is spontaneously broken

[Broken phase : $e^{i\alpha Q}\phi_0 = e^{i\alpha}\phi_0 \neq \phi_0$

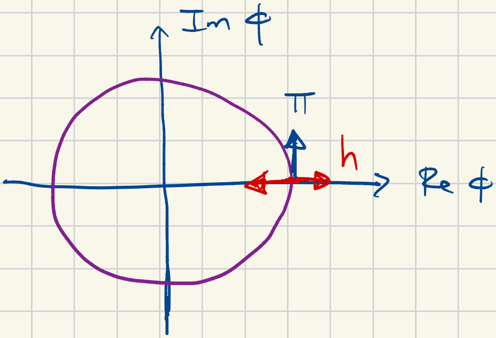
$Q\phi_0 = \phi_0 \neq 0$]

To see this explicitly, let's consider perturbations around $\rho_0 = \rho_+ = \dfrac{\mu}{\sqrt{2\lambda}}$

We write $v = \dfrac{\mu}{\sqrt{\lambda}}$

$$\Rightarrow \quad \rho_+ = \frac{v}{\sqrt{2}}$$

$\Rightarrow$ We write $\phi(x) = \dfrac{1}{\sqrt{2}}\big(v + \underbrace{h(x)}_{\text{along radial}}\big)\, e^{i \overbrace{\pi}^{\text{Angular}}}$



Now the Lagrangian around ρ_+ becomes

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu h)^2 - \mu^2 h^2 - \sqrt{\lambda}\,\mu h^3 - \lambda h^4$$
$$- \frac{1}{2}(v+h)^2 (\partial_\mu \pi)^2$$

Mass of radial mode

$$m_h^2 = \mu^2 = \lambda v^2$$

No mass term for π

$\pi(x)$ is a Goldstone boson!

Note that # of d.o.fs must be the same before and after SSB.

In our cace : $U(1) \to$ Nothing

(only 1 broken generator)

$$\phi(x) \rightsquigarrow h(x),\ \pi(x)$$

2 d.o.f. $\qquad$ 2 d.o.f.

Actually, in the broken phace the Goldstone $\pi(x)$ non-linearly realizes $U(1)$ symmetry :

$$\phi(x) = \frac{1}{\sqrt{2}}(v + h(x))\, e^{i\pi}$$

$\Rightarrow$ $\mathcal{L}$ around ρ_+ is invariant under

$\pi \to \pi + \alpha$: shift symmetry

There is no non-derivative of π's

$\Rightarrow$ Mass term is not allowed.

Top - down approach

From $\mathcal{L}(h, \pi)$

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu h)^2 - \mu^2 h^2 - \sqrt{\lambda}\,\mu h^3 - \lambda h^4$$

$$- \frac{1}{2}\underbrace{(v+h)^2}_{v^2 + 2vh + h^2}(\partial_\mu \pi)^2 \qquad -\frac{\cancel{2}}{2} v h (\partial_\mu \pi)^2 \qquad -\frac{1}{2} h^2 (\partial_\mu \pi)^2$$

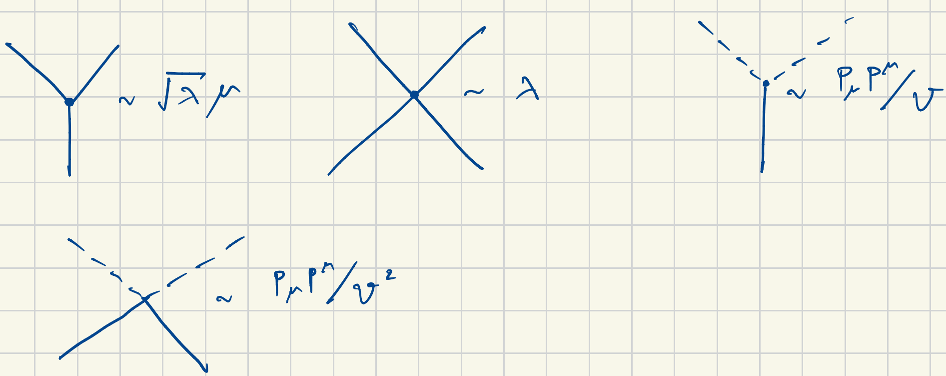
Let $\pi_c = v\pi$, we have

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu h)^2 - \mu^2 h^2 - \sqrt{\lambda}\,\mu h^3 - \lambda h^4$$

$$- \frac{1}{2}(\partial_\mu \pi_c)^2 \underbrace{- \frac{h}{v}(\partial_\mu \pi_c)^2}_{5\text{-dim}} \underbrace{- \frac{h^2}{v^2}(\partial_\mu \pi_c)^2}_{6\text{-dim}}$$

(Irrelevant operator)

- Vertice in the UV.



Then, integrating out h at tree level gives

$$\mathcal{L}_\pi = -\frac{1}{2}(\partial_\mu \pi_c)^2 + C_1 \frac{(\partial_\mu \pi)^4}{v^4} + \ldots$$

where $C_1 \equiv v^2/\mu^2 \rightarrow$ cutoff of the EFT

$\Rightarrow$ The IR theory is invariant under $\pi \rightarrow \pi + \alpha$

Broken gauge symmetries

Now we let $\alpha = \alpha(x)$. In this case we have a scalar QED:

$$\mathcal{L} = -D_\mu \phi^* D^\mu \phi - V(\phi) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

with $D_\mu = \partial_\mu + i g A_\mu$, $\phi \rightarrow e^{i\alpha(x)} \phi$

Here in the broken phase we have

$$\phi(x) = \frac{1}{\sqrt{2}}(v + h(x))\, e^{i\pi}$$

$$\Rightarrow \quad -D_\mu\phi^* D^\mu\phi = -(\partial_\mu + igA_\mu)\phi^*(\partial^\mu - igA^\mu)\phi$$

$$= -\partial_\mu\phi^*\partial^\mu\phi + \partial_\mu\phi^* igA^\mu\phi$$

$$- igA_\mu\phi^*\partial^\mu\phi$$

$$- g^2\phi^*\phi A_\mu A^\mu$$

$$-D_\mu\phi^* D^\mu\phi = -\partial_\mu\phi^*\partial^\mu\phi + igA^\mu(\phi\partial_\mu\phi^* - \phi^*\partial_\mu\phi)$$

$$- g^2\phi^*\phi A_\mu A^\mu$$

$$D_\mu\phi \to D_\mu(e^{i\alpha}\phi) = (\partial_\mu + igA_\mu)e^{i\alpha}\phi$$

$$= e^{i\alpha}\partial_\mu\phi + i\partial_\mu\alpha\, e^{i\alpha}\phi + igA_\mu e^{i\alpha}\phi$$

$$\boxed{A_\mu \to A_\mu - \partial_\mu\alpha/g}$$

$$D_\mu\phi \to e^{i\alpha}\partial_\mu\phi + \cancel{i\partial_\mu\alpha\, e^{i\alpha}\phi} + igA_\mu e^{i\alpha}\phi$$

$$\cancel{- i\partial_\mu\alpha\, e^{i\alpha}\phi}$$

$$D_\mu\phi \to e^{i\alpha}(\partial_\mu + igA_\mu)\phi = e^{i\alpha}D_\mu\phi$$

$\Rightarrow$ We have a freedom to remove $\pi(x)$

by choosing $\alpha = -\pi(x)$

[This freedom happens because we are in the local $U(1)$ gauge, similar to GR]

$\Rightarrow$ Therefore, $\pi = 0$ (in a new frame)

We call this "unitary gauge" [Goldstone boson $= 0$]

Now we see that A_μ becomes massive manifestly

$$\phi = \frac{1}{\sqrt{2}}(v + h)$$

$$\Rightarrow \quad -D_\mu \phi^* D^\mu \phi = -\frac{1}{2}(\partial_\mu h)^2 - \frac{g^2 v^2}{2} A_\mu A^\mu + \dots$$

$$\Rightarrow \quad \mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2 - \frac{m_A^2}{2} A_\mu A^\mu - \frac{1}{2}(\partial_\mu h)^2 + \dots$$

! A_μ becomes massive $=$ 3 d.o.f.

$\Rightarrow$ The Goldstone π has become the longitudinal mode of the massive vector A_μ

(usually we say π has been eaten by A_μ to make A_μ massive) $=$ Higgs mechanism for local $U(1)$

We see that now $\mathcal{L}$ in the unitary gauge is no longer invariant under the gauge transformation of $A_\mu \to A_\mu - \partial_\mu \alpha / g$ due to the mass term

Note before SSB : d.o.f = ϕ + A_μ

$$2 + 2$$

(massless)

After SSB : d.o.f. = h + A_μ

$$1 + 3$$ (massive)

However, in order to understand the behavior of the theory at high energy, it is useful to reintroduce π This trick of reintroducing the Goldstone bosons is Stückelberg trick

$\Rightarrow$ We have to undo the gauge fixing :

$$A_\mu \to A_\mu + \partial_\mu \pi / g$$

The mass term now becomes

$$-\frac{1}{2} m_A^2 A_\mu A^\mu \to -\frac{1}{2} m_A^2 \left(A_\mu + \frac{\partial_\mu \pi}{g}\right)\left(A^\mu + \frac{\partial_\mu \pi}{g}\right)$$

This in fact is invariant under

$$A_\mu \to A_\mu - \partial_\mu \alpha / g \quad \text{and} \quad \pi \to \pi + \alpha(x)$$

"Non-linear transformation"

$\Rightarrow$ Now we see that the gauge symmetry is restored, but the price to pay is you have to introduce a massless field $\pi(x)$

[Now A_μ has 2 d.o.f.s and π has 1]

$\hookrightarrow$ Transverse modes

$\hookrightarrow$ Longitudinal mode.

For convenience, we write

$(D_\mu = \partial_\mu + igA_\mu)$

$$A_\mu \to A_\mu + \frac{\partial_\mu \pi}{g} \equiv \frac{i}{g} U D_\mu U^\dagger \; ; \; U \equiv e^{i\pi}$$

Then, using this in the unitary-gauge action gives (up to integration by parts)

$$\mathcal{L} = \underbrace{-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}} - \frac{f_\pi^2}{2} D_\mu U^\dagger D^\mu U$$

(This term is not affected since it is already gauge invariant. or $\pi(x)$ does not appear in the gauge-invariant terms)

$$f_\pi \equiv m/g$$

(This is similar to the pion Lagrangian derived from QCD)

- At quadratic level, we have

$$\mathcal{L}_2 = -\frac{1}{4} F_{\mu\nu}^2 - \frac{1}{2} (\partial_\mu \pi_c)^2 - \frac{1}{2} m^2 A_\mu^2 + \underbrace{m \partial_\mu \pi_c A^\mu}$$

$$\pi_c = f_\pi \pi$$

(Mixing term of π_c & A_μ)

$\Rightarrow$ The decoupling limit between $\pi(x)$ and A_μ happens when $m \to 0$, $g \to 0$, but f_π = constant $\Rightarrow$ No mixing between π and A_μ

$$\frac{\mathcal{L}_{mix}}{(\partial \pi_c)^2} \sim 1 \quad \text{when} \quad E \sim m = E_{mix}$$

- We see that for $E > E_{mix} = m$

$\pi(x)$ and A_μ are decoupled

= The longitudinal mode is decoupled from the transverce modes

= The scattering of the longitudinal mode is now described by the scattering of the Goldstone bosons up to $\mathcal{O}(m/E)$ and $\mathcal{O}(g^2)$

= Goldstone boson equivalence theorem.

[Basically one can use the theory of only $\pi(x)$ to describe physics when going to sufficiently high energy]

- If one includes the self-interactions of π's, e.g.

$$\mathcal{L}_4 = \frac{1}{\tilde{\Lambda}^2} \pi_c^2 (\partial_\mu \pi_c)^2 \quad ; \quad \tilde{\Lambda} = m/g$$

$\Rightarrow$ The theory becomes strongly coupled when

$E \sim 4\pi \tilde{\Lambda}$ (Unitarity bound : $\mathcal{M}_4 \sim 1$)

$\Rightarrow$ We see that in the window

$$m \ll E \ll 4\pi\tilde{\Lambda} = \Lambda_{Uni}$$

the physics of $\pi(x)$ is weakly coupled

and it can be studied neglecting

the mixing with transverce components.

- In the next lecture, we are going to apply this procedure to the physics of inflation or dark energy with a single scalar field.

Analogy : U(1) example Vs. Inflation / DE

$$(A_\mu, \phi) \sim (g_{\mu\nu}, \phi)$$

$$U(1) \text{ gauge symm.} \sim \text{Diffeomorphism}$$

Sym. breaking pattern : $U(1) \mapsto$ Nothing by $v = \langle\phi\rangle$ $\sim$ 4d diff. $\rightarrow$ 3d diff. by $\phi_0(t)$

Unitary-gauge action : $\mathcal{L}(A_\mu)$ ↳ Massive

$$\mathcal{L}(\underbrace{g^{00}, K, {}^{(3)}R}_{\text{Transform covariantly under 3d diffeo.}}; t)$$