HI


KKU
2024

Thank you for attention

Stability regions in the Double Pendulum Phase Space

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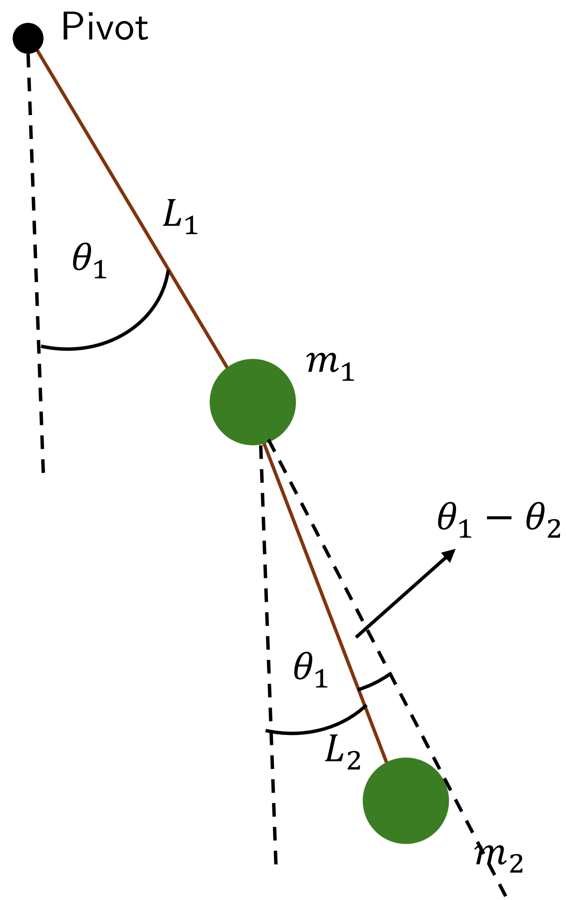
What is double pendulum?

A double pendulum is a physical system consisting of two pendulums connected end to end and a second pendulum hangs from the tip of the first, which itself swings from a fixed pivot point.

Which is our tools to investigate the chaotic motion and its phase space use to see its behavior on each initial conditions.

System Setup and Coordinate

A double pendulum consist of two rigid, massless rods.
Rod 1 of length L_1 carries mass m_1 at its tip;
Rod 2 of length L_2 hanged from m_1 and carries mass m_2.
The pivot of rod 1 is fixed at the origin



Definition (Generalised coordinate). Let $\theta_1(t)$ and $\theta_2(t)$ denote the angles each rod makes with the downward vertical, measure in radians. Then the cartisian position of the two masses are

$$\langle x_1, y_1 \rangle = \langle L_1 sin\theta_1, -L_1 cos\theta_1 \rangle \quad (1)$$
$$\langle x_1, y_1 \rangle = \langle x_1 + L_2 sin\theta_2, x_1 - L_2 cos\theta_2 \rangle \quad (2)$$

The four – dimension state vector is

$$s(t) = (\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2)$$

Where $\dot{\theta}_i$ are the angular velocities

Lagrangian mechanics

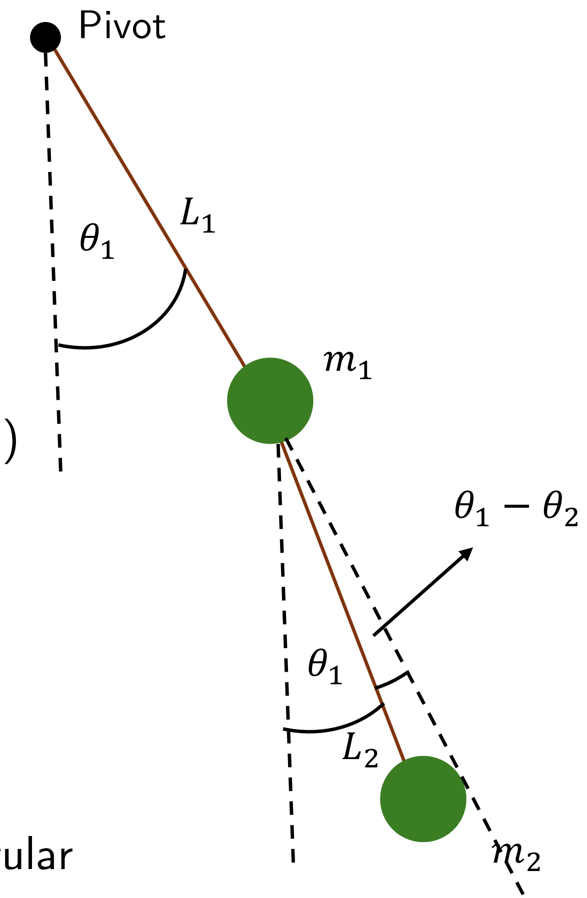
Kenetics and Potential Energy

The kenetics energy of each mass is $T_i = \frac{1}{2} m_i v_i^2$.
Computing $v_i^2 = \dot{x}_i^2 + \dot{y}_i^2$ and simplify gives

$$T = T_1 + T_2 = \frac{1}{2}(m_1 + m_2)L_1\dot{\theta}_1^2 + \frac{1}{2} m_2 L_2 \dot{\theta}_2^2 + m_2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \quad (3)$$

The potential energy (taking pivot as reference height) is

$$V = V_1 + V_2 = -(m_1 + m_2) g L_1 \cos(\theta_1) - m_2 g L_2 \cos\theta_2 \quad (4)$$



Euler - Lagrange Equations

The Lagrangian is $\mathcal{L} = T - V$. Applying $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_i}\right) - \frac{\partial \mathcal{L}}{\partial \theta_i} = 0$ and solve for angular acceleration yields:

From Euler-Lagrange equation by solving two variables algebra we get that

Equations of Motion (Use in simulations)

$$\ddot{\theta}_1 = \frac{-g(2m_1 + m_2)sin\theta_1 - m_2 gsin(\theta_1 - 2\theta_2) - \sin(\theta_1 - \theta_2)m_2(L_2\dot{\theta}_2^2 + L_1\dot{\theta}_1^2\cos(\theta_1 - \theta_2))}{L_1(2m_1 + m_2 - m_2\cos(2(\theta_1 - \theta_2))} \quad (5)$$

$$\ddot{\theta}_2 = \frac{2sin(\theta_1 - \theta_2)\left[L_1\dot{\theta}_1^2(m_1 + m_2) + g(m_1 + m_2)cos\theta_1 + m_2 L_2\dot{\theta}_2^2 cos(\theta_1 - \theta_2)\right]}{L_2(2m_1 + m_2 - m_2\cos(2(\theta_1 - \theta_2))} \quad (6)$$

Remark equation (5)-(6) form a system of two coupled, nonlinear second-order ODEs. Which has been proven that no single closed-form function of $(\theta_1^0, \theta_2^0, t)$ solves them in general numerical integration is therefore required

Phase plane analysis

Since our state function contains four component $s(t) = (\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2)$ to analysis for fixed static point which give $\dot{\theta}_1 = \dot{\omega}_1 = \dot{\theta}_2 = \dot{\omega}_2 = 0$
From that we define $\dot{\theta}_1 = \omega_1$, $\dot{\theta}_2 = \omega_2$, $\dot{\omega}_1 = f(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2)$, $\dot{\omega}_2 = (\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2)$
Which automatically give $\dot{\theta}_1 = \dot{\theta}_2 = 0$

Which simplify equation of motion to

$$\ddot{\theta}_1^0 = \dot{\omega}_1^0 = \frac{-g(2m_1 + m_2)sin\theta_1 - m_2 gsin(\theta_1 - 2\theta_2)}{L_1(2m_1 + m_2 - m_2\cos(2(\theta_1 - \theta_2))} = 0 \qquad (7)$$

$$\ddot{\theta}_2^0 = \dot{\omega}_2^0 = \frac{2sin(\theta_1 - \theta_2)g(m_1 + m_2)cos\theta_1}{L_2(2m_1 + m_2 - m_2\cos(2(\theta_1 - \theta_2))} = 0 \qquad (8)$$

Which dependence on each numerator independence of constant parameter

$sin(\theta_1 - \theta_2)cos\theta_1 = 0$ (9) or $-(2m_1 + m_2)sin\theta_1 - m_2 sin(\theta_1 - 2\theta_2) = 0$ (10)

Phase plane analysis

From equation (9)-(10) case that make equation (9) true is that $\theta_1 = \theta_2$ or $\theta_1 = \theta_2 \pm n\pi$
Applies that

Case A: substitute $\theta_1 = \theta_2$ in the equation

$$-(2m_1 + m_2)sin\theta_2 - m_2sin(\theta_2 - 2\theta_2) = 0$$

Thus, result is $\theta_1 = 0$ and $\theta_2 = 0$

Case B: substitute $\theta_1 = \theta_2 \pm n\pi$ or $\theta_1 \pm n\pi = \theta_2$ in the equation

$$-(2m_1 + m_2)sin\theta_1 - m_2sin(\theta_1 - 2\theta_1 \pm 2n\pi) = 0$$

Thus, result is $\theta_1 = 0\ or\ \pi$ and $\theta_2 = 0\ or\ \pi$

So it mean that all fixed point of (θ_1, θ_2) is $(0,0)$, $(\pi, 0)$, $(0, \pi)$, (π, π) and only stable fixed point are $(0,0)$ others is unstable and can be chaotic with small pertubations.

Numerical Integration: Runge – Kutta 4

To evolve forward in time the code uses the classical **fourth-order Runge-Kutta** (RK4) method.

Algorithm

Given $s_n = s(t_n)$ and step size Δt, the update $s_{n+1} \approx s(t_n + \Delta t)$ is computed as:

$$k_1 = f(s_n, t_n), \tag{11}$$

$$k_2 = f\left(s_n + \frac{\Delta t}{2} k_1, t_n + \frac{\Delta t}{2}\right), \tag{12}$$

$$k_3 = f\left(s_n + \frac{\Delta t}{2} k_2, t_n + \frac{\Delta t}{2}\right), \tag{13}$$

$$k_4 = f(s_n + \Delta t k_3, t_n + \Delta t), \tag{14}$$

$$s_{n+1} = s_n + \frac{\Delta t}{6}(k_1 + 2k_2 + 2k_3 + k_4), \tag{15}$$

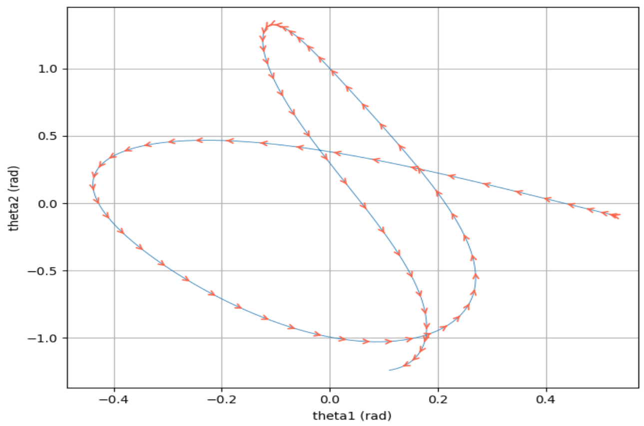
Where $f(s,t)$ is the right hand side of equation (5)-(6), and the state vector contains all four components

$$s(t) = (\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2)$$

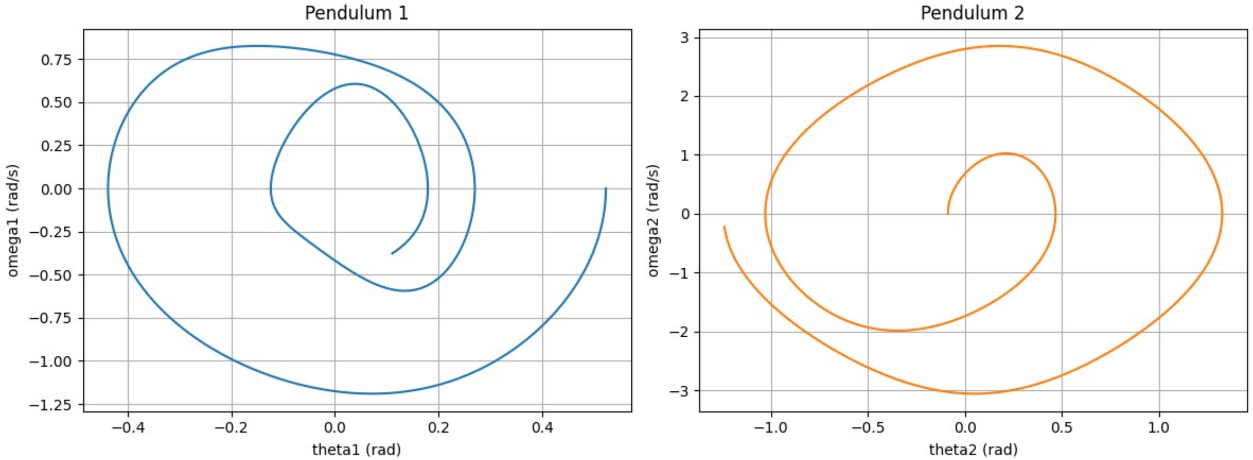
Numerical Integration: Result from Runge – Kutta 4

Now we be able to simulate phase portrait , trajectory , angles over time

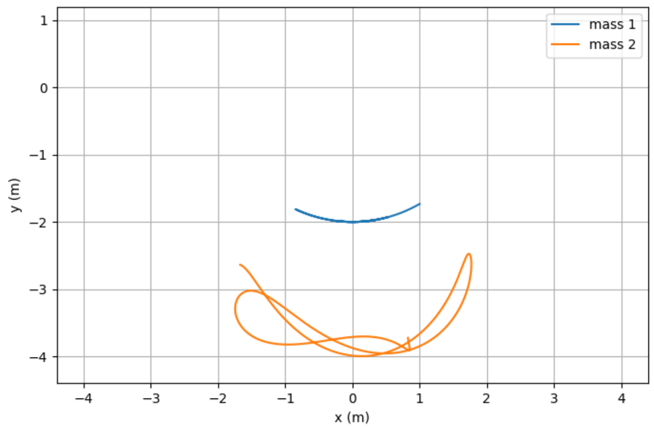
Phase space of theta1 vs theta2



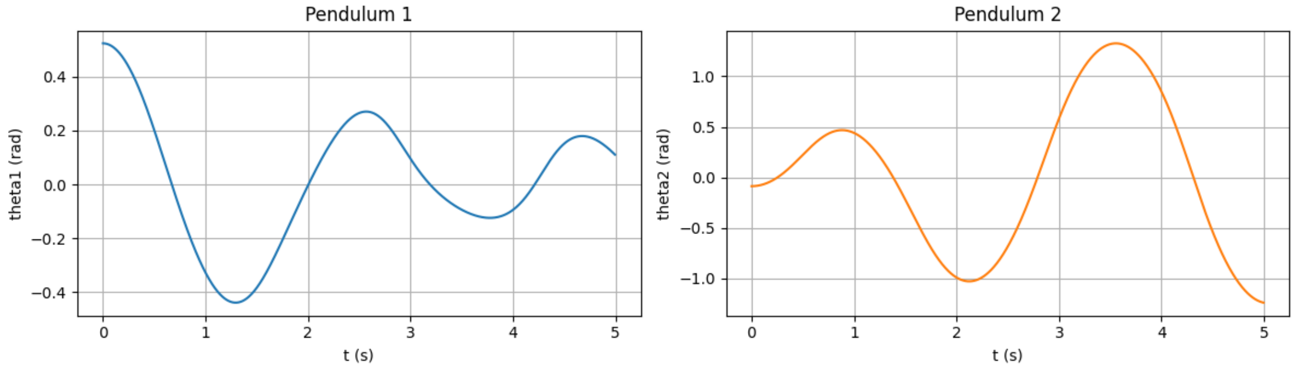
Phase space of omega 1 ,2 vs theta 1,2



Trajectory



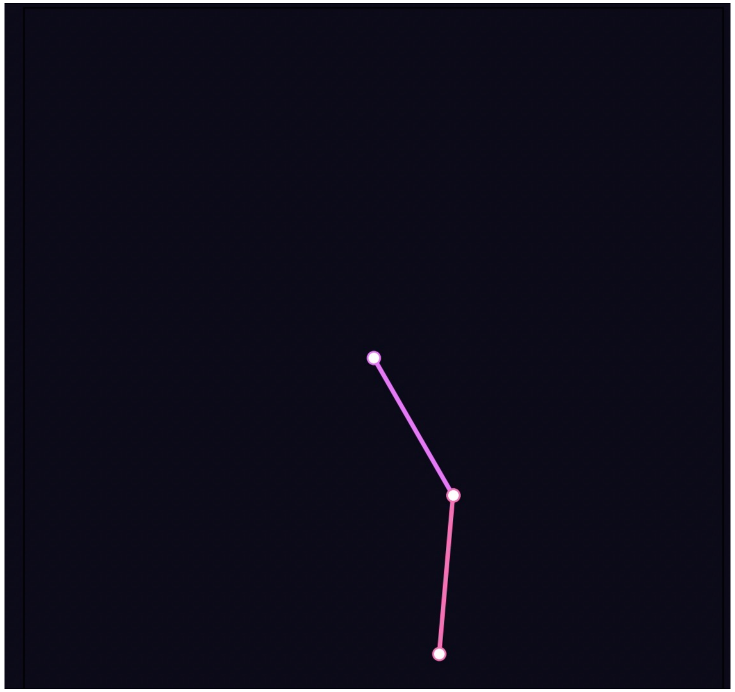
Graph of theta1 , theta 2 vs time



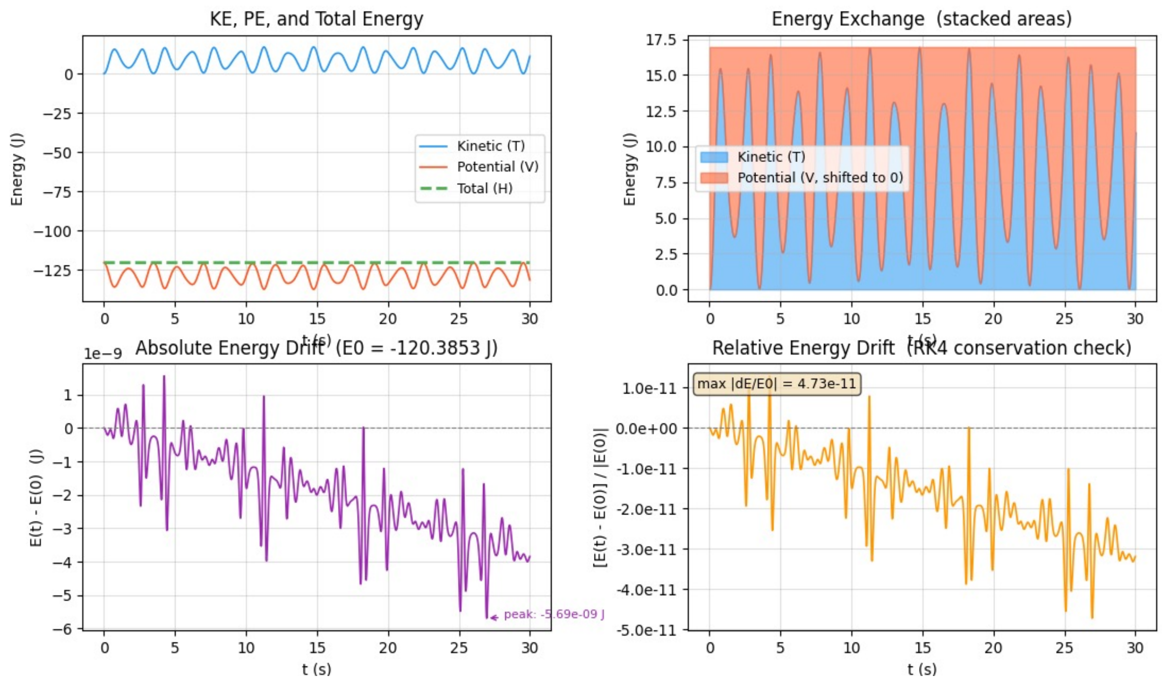
Numerical Integration: Result from Runge – Kutta 4

From numerical result we could confirm our phase analysis and create simulation to determine it chaotic behaviors or create an animation to compared it to experimental result or to confirm the assumption such as the conservation of energy and its loss from simulation error.

Simulation using python code



Energy analysis and error from simulation



Create The Chaos map to find its island of stability

from our simulation each initial condition give different result even slight pertubation could give another result, but The chaos map give us the answer to *"For every possible pair of starting angle* (θ_1^0, θ_2^0)*, How long does the pendulum stay ordered before going chaotic?"* The answer, would be encoded as colour for each pixel, produce 2-D image whose patterns reveal the geometric structure of chaos in the double pendulum's phase space

Phase Space and State Vectors

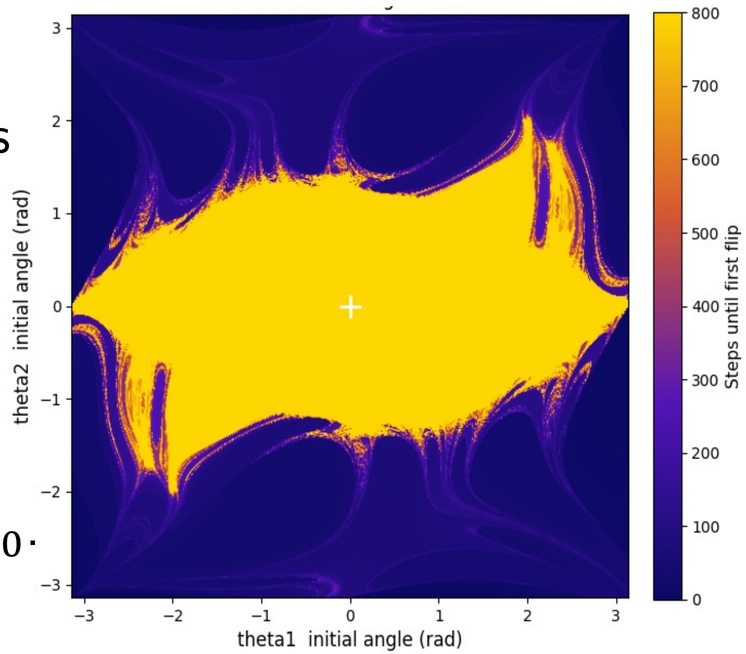
Definition The phase space of double pendulum chaos map is

$$\Omega = [-\pi, \pi]^2 \times \mathbb{R}^2,$$

With point $$s = (\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2)$$

The equation of motion define a flow vector $\phi^t : \Omega \to \Omega$ such that $s(t) = \phi^t(s_0)$ is the state at time t given initial state s_0.

The chaos map plot only varies the angular components of s_0.



Create The Chaos map to find its island of stability

Sensitive Dependence on Initial condition

Define Consider a map $F: X \to X$ on matric space (X, d) exhibit sensitive dependence on initial conditions if there exist $\delta > 0$ such that for every $x \in X$ and every neighborhood $U \ni x$, there exist $y \in U$ and $n \geq 0$ with $d\big(F^n(x), F^n(y)\big) > \delta$ or in plain language: no matter how precisely you specify the initial angle eventually produces a completely different motion.

How we define "Chaos"

Define Consider a map $F: X \to X$ on matric space (X, d) exhibit sensitive dependence on initial conditions if there exist $\delta > 0$ such that for every $x \in X$ and every neighborhood $U \ni x$, there exist $y \in U$ and $n \geq 0$ with $d\big(F^n(x), F^n(y)\big) > \delta$ or in plain language: no matter how precisely you specify the initial angle eventually produces a completely different motion.

(i) **Transitive:** it has a dense orbit (there exists $x \in X$ such that $\overline{\{F^n(x) : n \geq 0\}} = X$);

(ii) **Sensitive:** it has sensitive dependence on initial condition

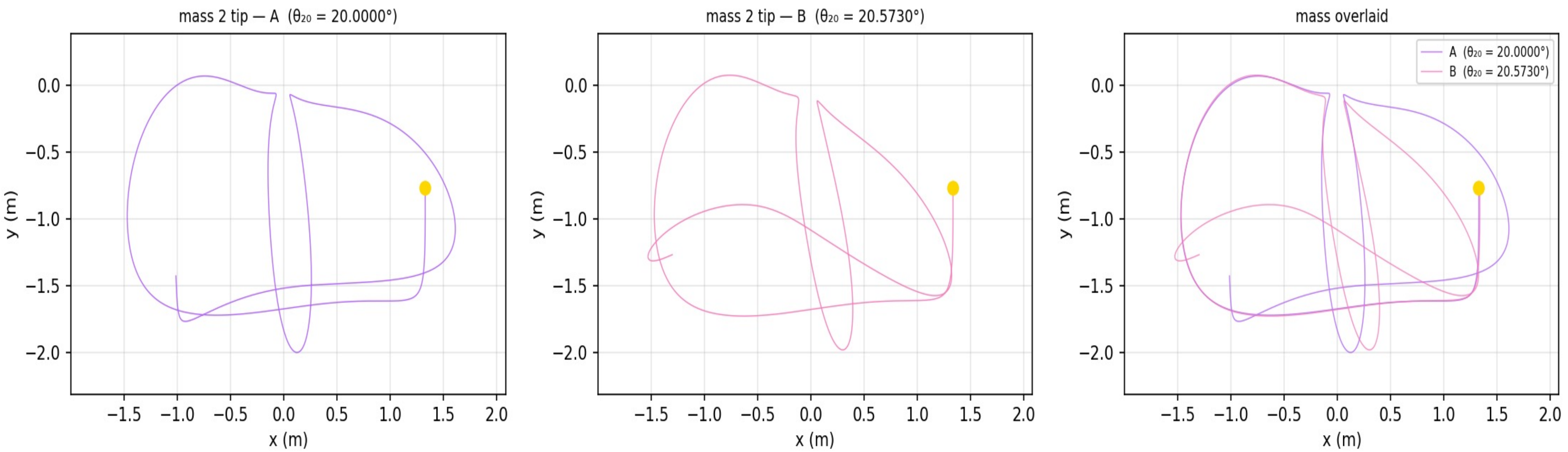
(iii) **Dense periodic orbits:** the set of periodic points of F is dense in X

These were called **Devaney's Definition**, Which are central idea for Chaotic dynamical systems

Create The Chaos map to find its island of stability

Bank et al. (1992) proved that conditions (i) and (iii) together imply condition (ii), so sensitivity is actually consequence of transitivity and dense periodic orbits. Nevertheless, sensitivity is most physically meaningful property and is the one we measure by coding.

To show how sensitivity looks like we simulate the twin divergence of double pendulum using small different in initial condition it could gave total different results in long simulation time **(called Butterfly effect).**



A Measurable Proxy for Chaos

Verify Devaney's definition requires analysising the entire orbit structure, which is computationally intractable for a continuous system.
Instead, we use practical proxy: the time until a qualitative change in behaviour occurs

Defining the Flip

We say arm i has flipped at time t if $|\theta_i(t)| > \pi$. A flip correspoonds to the arm passing over the pivot point

<u>**Remark**</u> In the continuous system angles are taken modulo 2π, so $|\theta_i| > \pi$ is the equivalent to the arm having rotated past the straight-up position, In the numerical integration we do not wrap angles; the condition $|\theta_i| > \pi$ therefore fires as soon as either arm crosses vertical for the first time.

Given The Flip-Time function by initial angles (θ_1^0, θ_2^0) and zero intitial verocities , the flip time is

$$T_{flip}(\theta_1^0, \theta_2^0) = \inf\{\, t \geq 0 : |\theta_1(t)| > \pi \; or \; |\theta_2(t)| > \pi \,\} \tag{16}$$

If neither arm ever flips within the observation window $[0, T_{max}]$, we set $T_{flip} = T_{max}$.

Physical intuition

- **Large** T_{flip} (Bright pixel): the pendulum oscillates near equilbrium for long time, indicating a "stable" or slowly chaotic region
- **Small** T_{flip} (Dark pixel): the pendulum quickly enters a spinning regime, indicating chaos at that initial condition.

The Chaos Map as a Function

Formal Definition

Definition Chaos Map. Fix $N \in N$, maximum steps $M \in N$ and time step $\Delta t > 0$
Define the grid

$$G_N = \{(\theta_1^{(i)}, \theta_2^{(j)} \colon \theta_k^{(m)} = -\pi + \frac{2\pi m}{N}, \; i, j \in \{0,1,\dots,N-1\} \tag{17}$$

The chaos map $C : G_N \to \{0,1,\dots,M\}$ is

$$C(\theta_1^0, \theta_2^0) = \min\{\, n \in N : \; |\theta_1(n\Delta t)| > \; \pi \;\; or \; |\theta_2(n\Delta t)| > \; \pi\}, \tag{18}$$

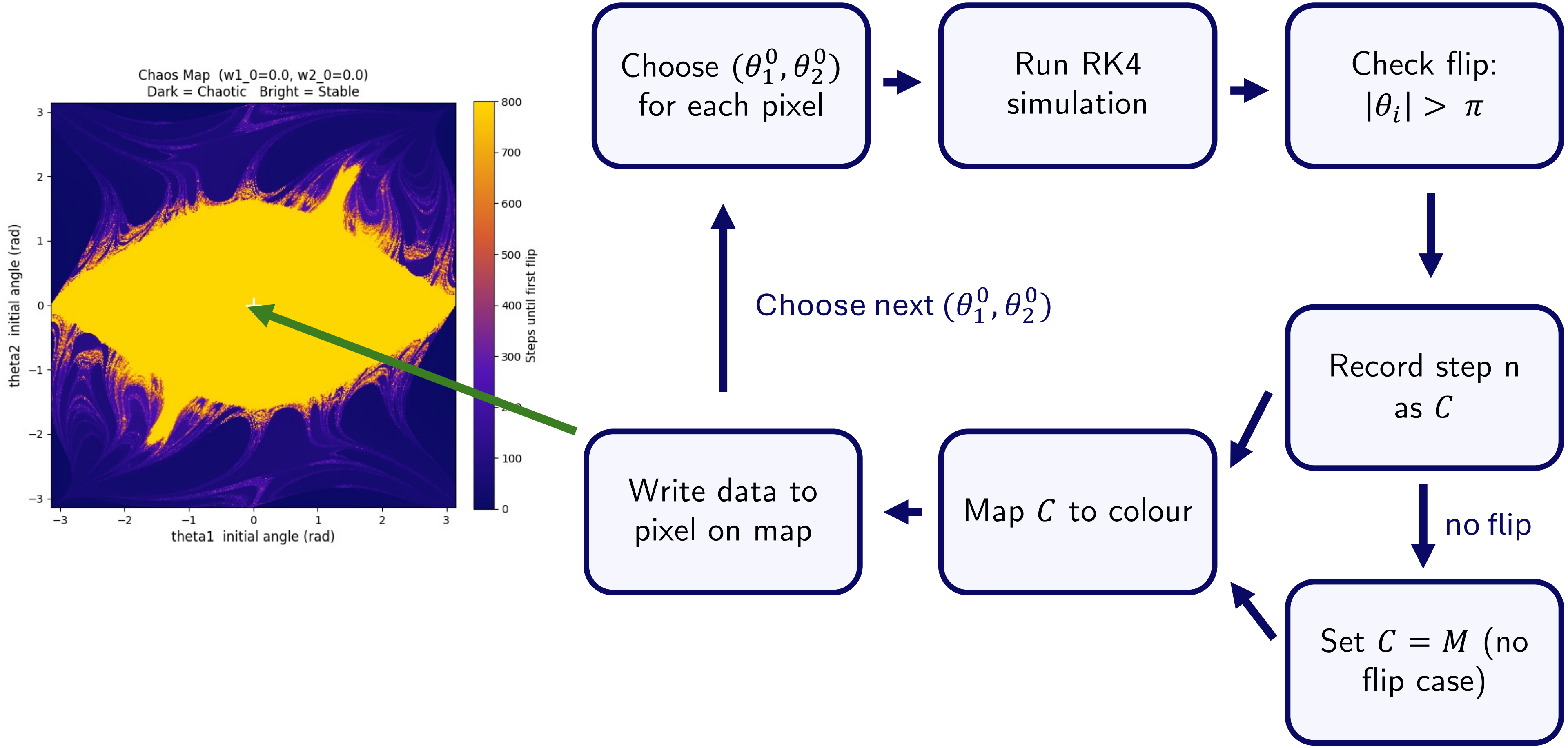
Where the minimum is taken to be M if no flip occurs before step M.

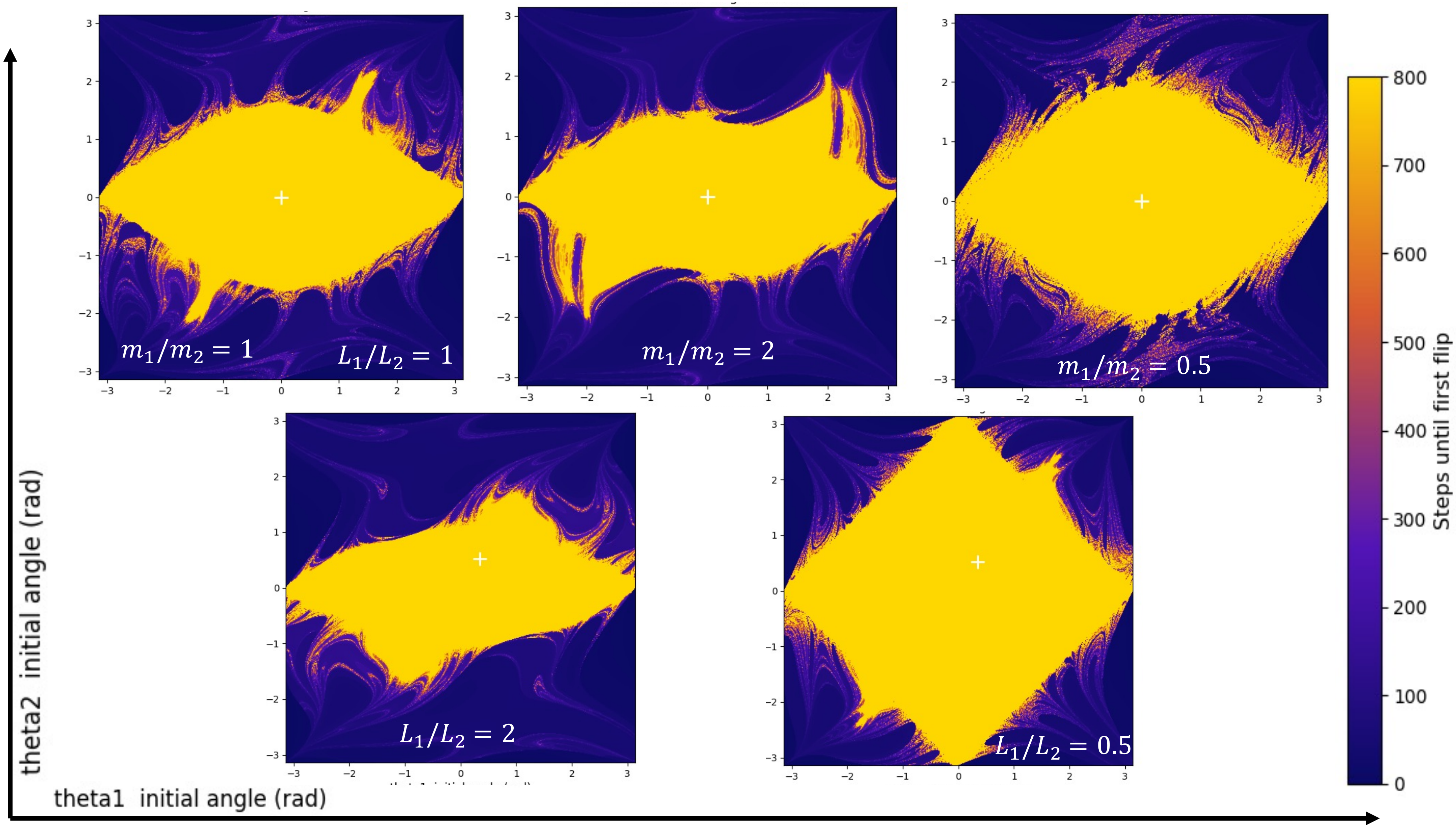
The integer $C(\theta_1^0, \theta_2^0)$ is mapped to a colour via perceptually-motivated scheme. Let $r = C/M \in \; [0,1]$ to be normalised flip time. The RGB colour is

$$Color(r) = \begin{cases} \text{dark blue}(\#0a0a64\,) \to \text{purple}(\#5314b7) & r \in [0, 0.5) \\ \text{purple}(\#5314b7) \to \text{orange}(\#d85a30) & \text{r} \in [0.5, 0.75) \\ \text{orange}(\#d85a30) \to \text{yellow}(\#ffd700) & r \in [0.75, 1] \end{cases} \tag{19}$$

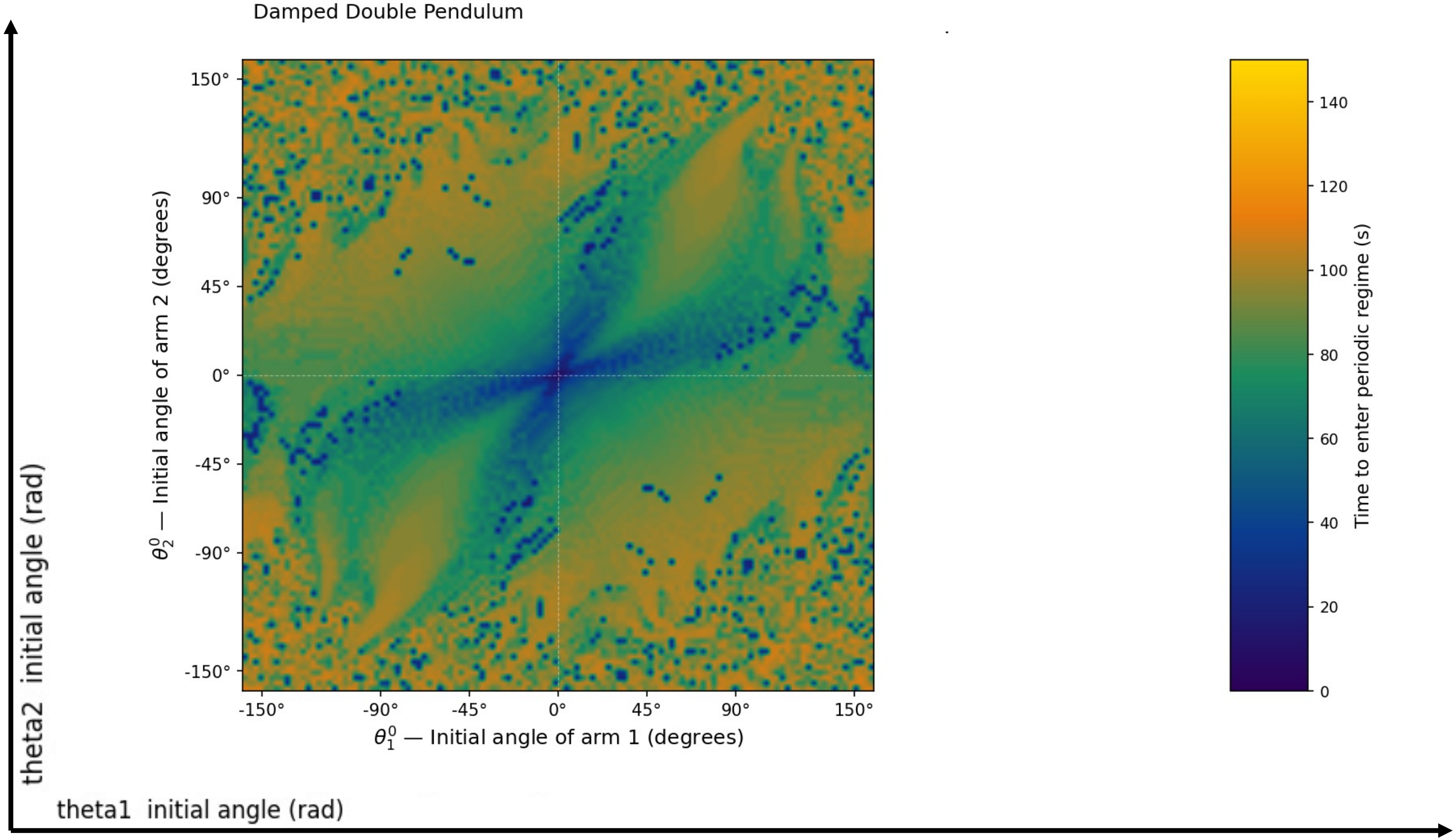
Interpolated linearly within each band. The four anchor colours correspond to the physical regime:
stable(yellow) → slowly unstable(orange) → transitional(purple) → rapidly chaotic(dark blue).

Visual Diagram of The method





$m_1/m_2 = 1$
$L_1/L_2 = 1$
$m_1/m_2 = 2$
$m_1/m_2 = 0.5$
$L_1/L_2 = 2$
$L_1/L_2 = 0.5$
theta2 initial angle (rad)
theta1 initial angle (rad)
Steps until first flip



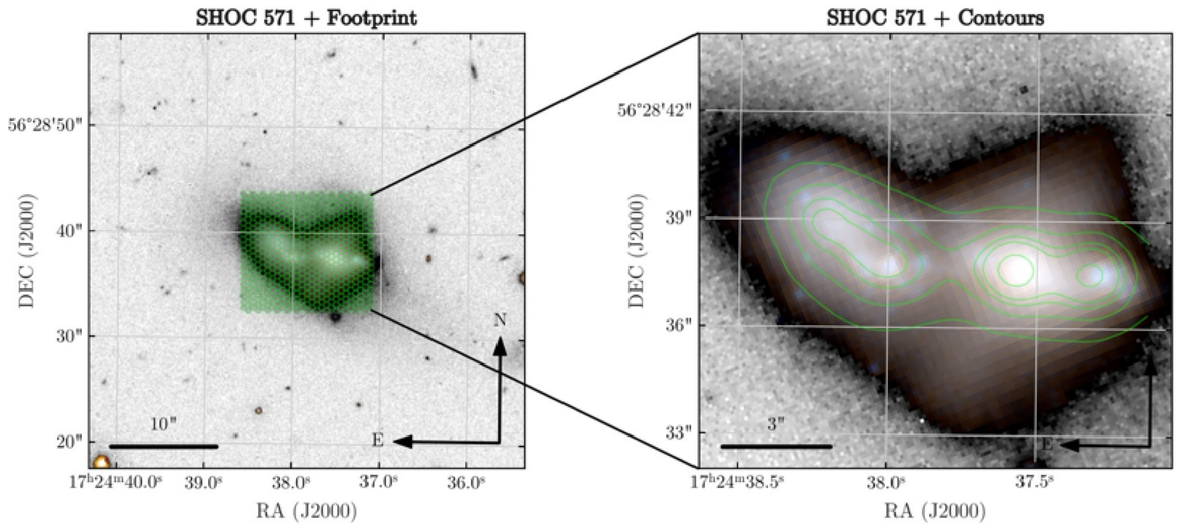
Damped Double Pendulum
150°
90°
45°
0°
-45°
-90°
-150°
-150°
-90°
-45°
0°
45°
90°
150°
θ_1^0 — Initial angle of arm 1 (degrees)
θ_2^0 — Initial angle of arm 2 (degrees)
Time to enter periodic regime (s)
0
20
40
60
80
100
120
140
theta2 initial angle (rad)
theta1 initial angle (rad)

Second Parameters in the H II Galaxy $L-\sigma$ Relation

What Are We Even Looking At?

HII GALAXY

An HII galaxy is a dwarf galaxy caught mid-starburst, its light dominated almost entirely by one giant star-forming region. The result is a spectrum with sharp, unmistakable emission lines .Hβ chief among them. instead of the smooth stellar continuum of an ordinary galaxy.



SHOC 571 (Euclid VIS/Y/J): a compact starburst dwarf, with the MEGARA IFU footprint and Hα contours.

$L(H\beta) \propto \sigma^5$

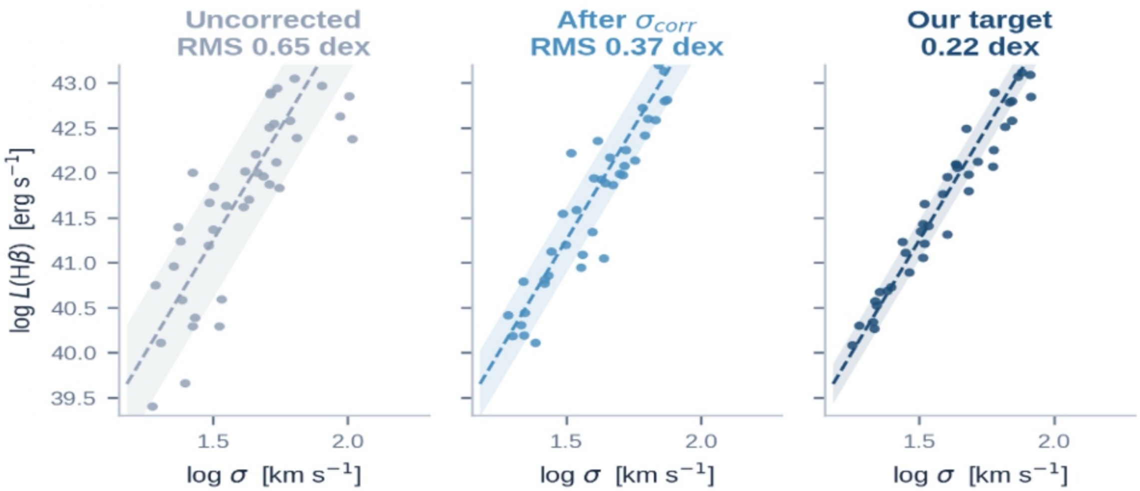
The relation that turns these galaxies into standardizable candles

$z \approx 1.5 - 7$

Where it stays useful, well past the reach of Type Ia supernovae

0.37 dex

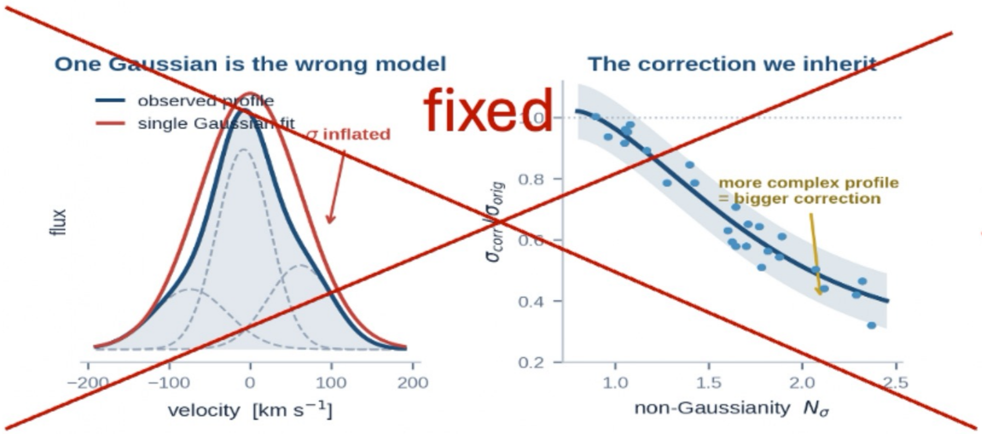
The scatter we still need to explain, and shrink toward 0.22



Today we have an RMS scatter of 0.37 dex, which corresponds to a 43% distance error per galaxy. so we need approximately 1800 galaxies to reach usable precision.

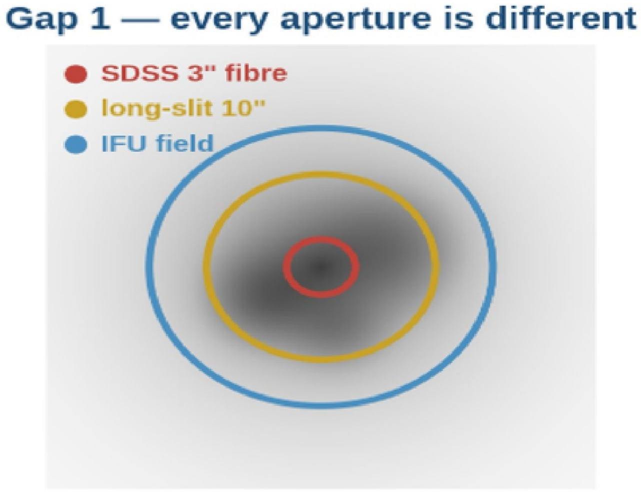
WHILE Type Ia supernovae become less effective in the **redshift desert, z~1.5–7**.

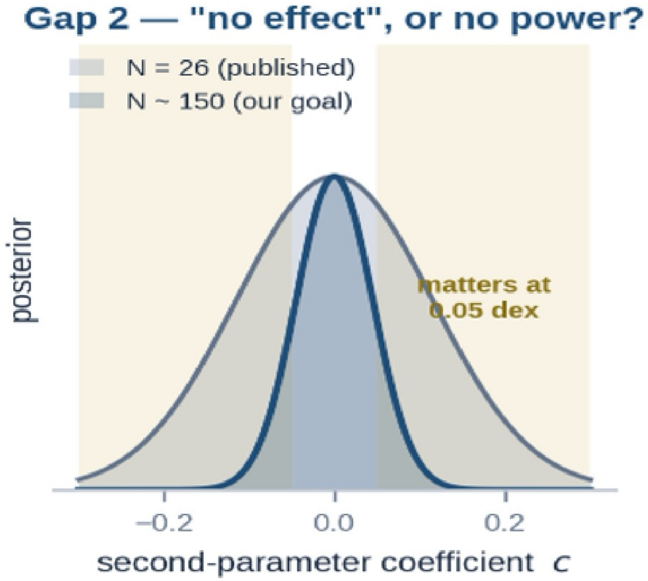
! H II galaxies can provide an independent distance indicator across this range.



However, several major problems have already been fixed, reducing the RMS scatter from 0.67 dex to 0.35 dex today.

Two Problems I think I Can Kill





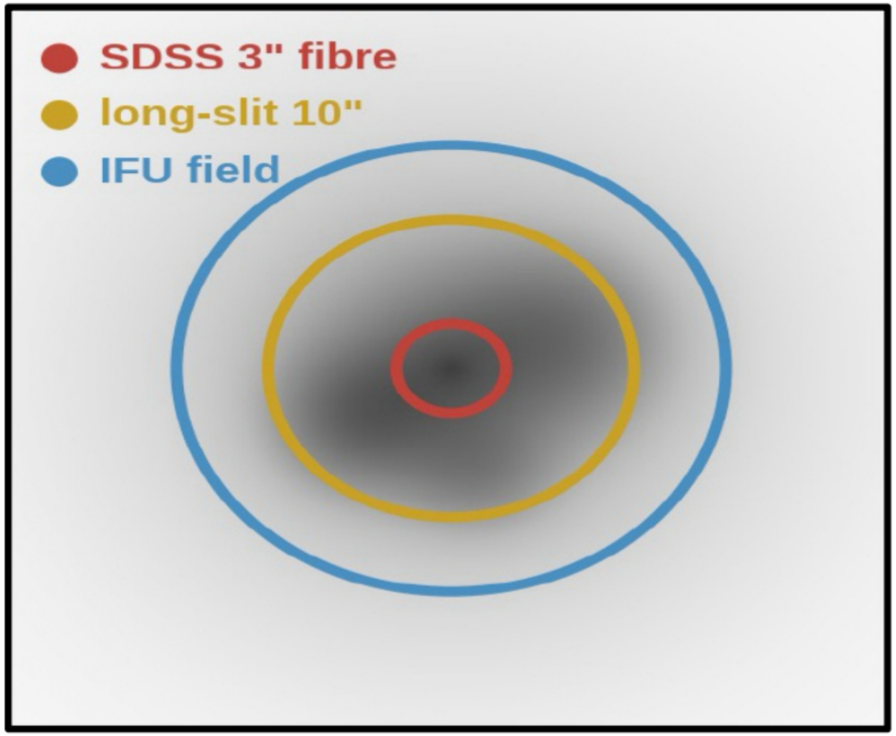
1 Aperture Mismatch

The calibration is stitched from mismatched apertures — a 3″ SDSS fibre, a 10″ long-slit, wide IFU fields. SDSS fluxes are already known to run low, but no one has calibrated that loss against galaxy structure, or asked what it does to σ.

2 Statistical Power

“Not significant” on 26 objects is not “no effect.” No one has stated how large a second parameter that sample could have detected — and at the precision we want, a 0.05 dex term would have been invisible.

Aperture and Survey Matching



Different aperture size for each survey

- Chávez used 8–13″ slits
- SDSS a 3″ fibre, MEGARA a 12.5 × 11.3″ field
- MaNGA 12–32″ bundles

It is the instrument problem. Can be fix by calibration

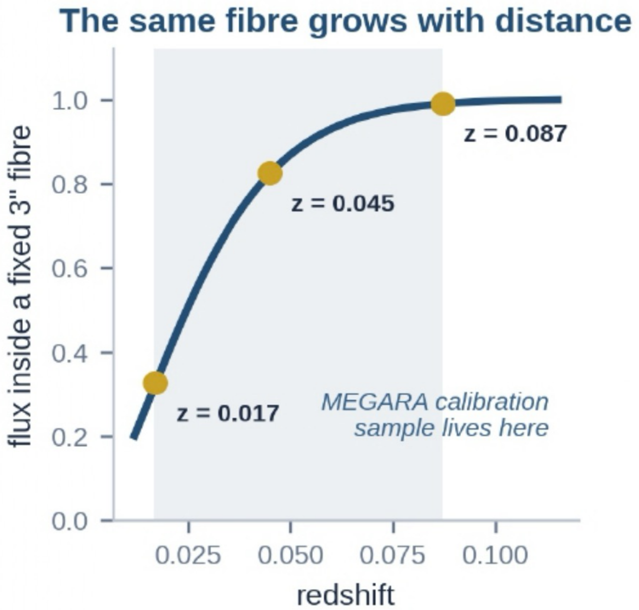
NOTE : **Use data from MaNGA IFU data cube (x,y + wavelength)**
So each spatial pixel have flux, spectrum, velocity, velocity dispersion

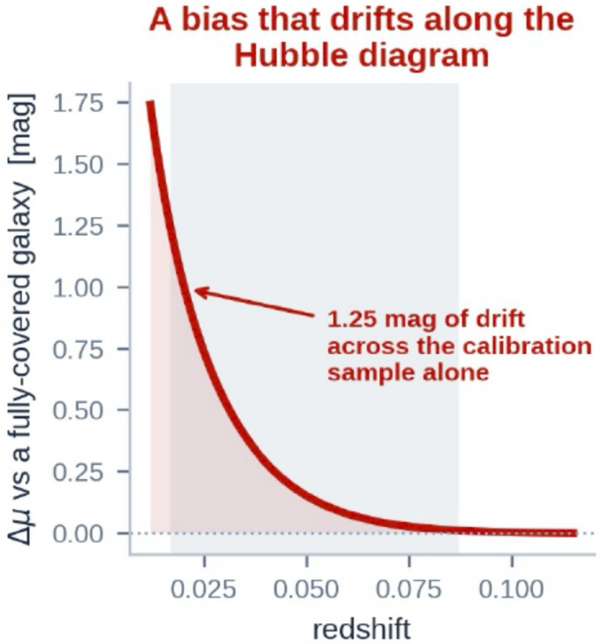
! We could sum up all spatial pixels inside synthetic aperture to measure growth of flux and velocity as function of radius
Which give 2 growth curve. Flux and sigma curve

Map Cube I/O to WCS → Create synthetic aperture → Use Correction to predict structure from image alone

Using effective radius, concentration, sersic index, asymmetry

Aperture Bias (Redshift trap)





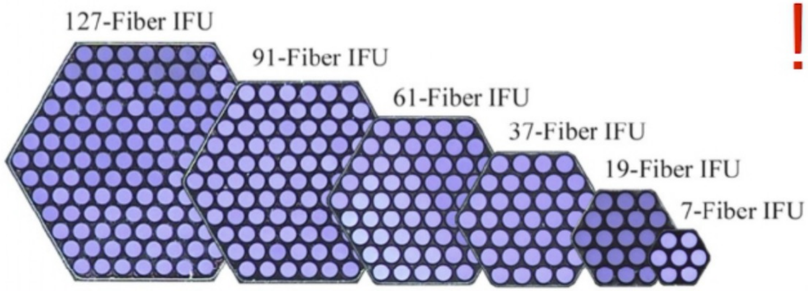
Spectroscopic surveys use a fixed angular fibre size

↓

Nearby galaxies are only partially sampled
More distant galaxies are sampled more completely

↓

the measured Hβ luminosity and velocity dispersion vary systematically with redshift.



! If this effect is not corrected, it introduces a redshift-dependent error into the L(Hβ)–σ relation that can be mistaken for a cosmological signal.

testing this directly by resampling a low-redshift cube to higher redshift and measuring the induced shift in distance modulus.

How Big a Second Parameter Could Hide?

01 **Reproduce the Regression**

Early on, rebuild the regression as $\log L = a + b \log \sigma + \Sigma\, c_i X_i$, with an explicit intrinsic-scatter term built in from the start.

02 **Set the Upper Bound**

Rather than asking only whether a variable is significant, report the full posterior and quote a 95% upper limit for every coefficient c_i.

03 **Injection–Recovery**

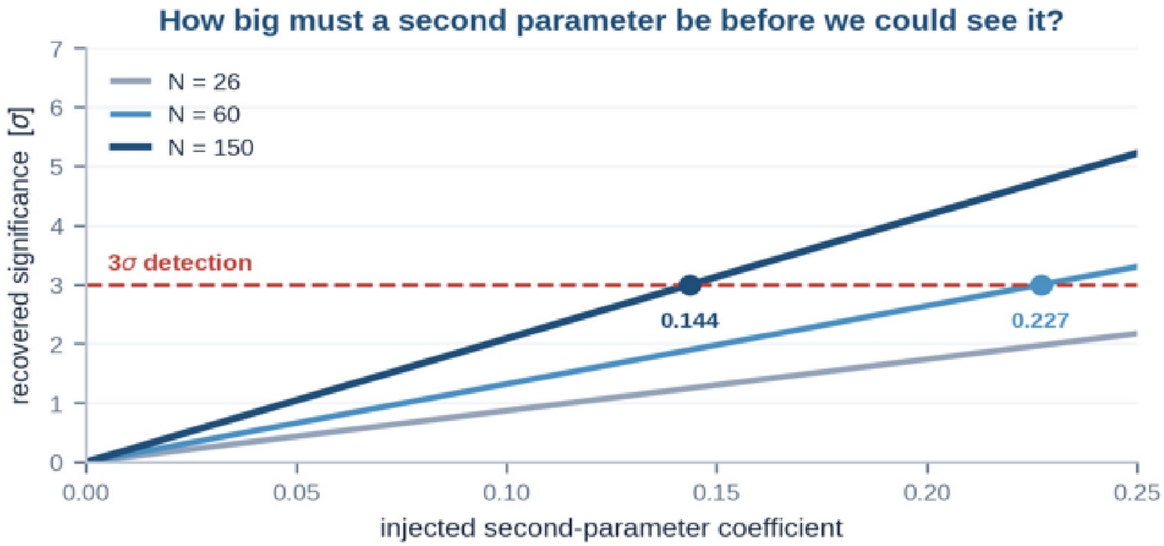
Inject a simulated second parameter of known amplitude, then ask how large it has to be before it is recovered at *3σ* confidence — shown at right.

04 **Revisit the Old Variables**

Test parameters the paper measured but never regressed: rotation (f_{rot}), dust (A_V), electron density (n_e), star-formation rate (Σ_{SFR}), and the A/B/C morpho-kinematic class.

05 **Scale Up, Check the Instrument**

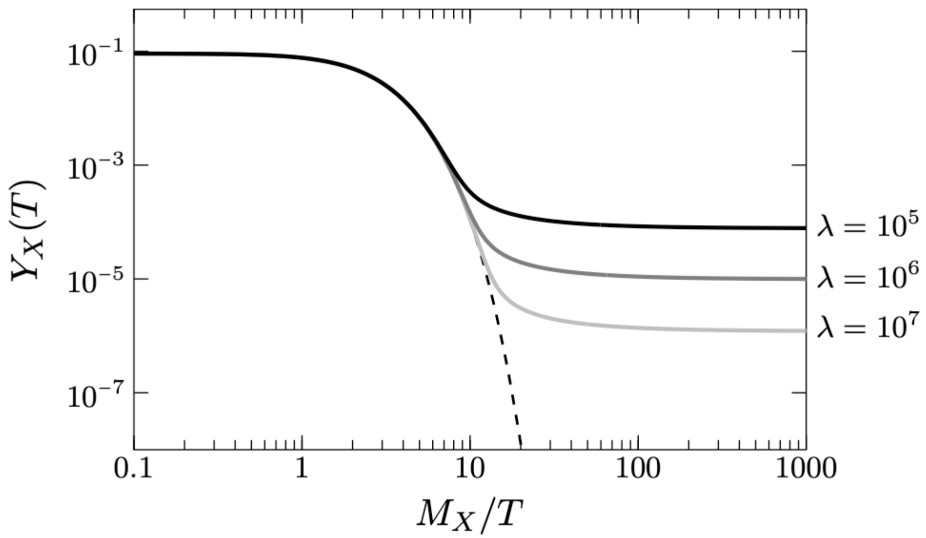
Grow the sample to ~150 galaxies and re-derive $f(N_\sigma)$ independently — but only where resolution allows: SAMI's red arm qualifies, MaNGA does not, since its σ_{inst} is coarser than the galaxies' own σ.



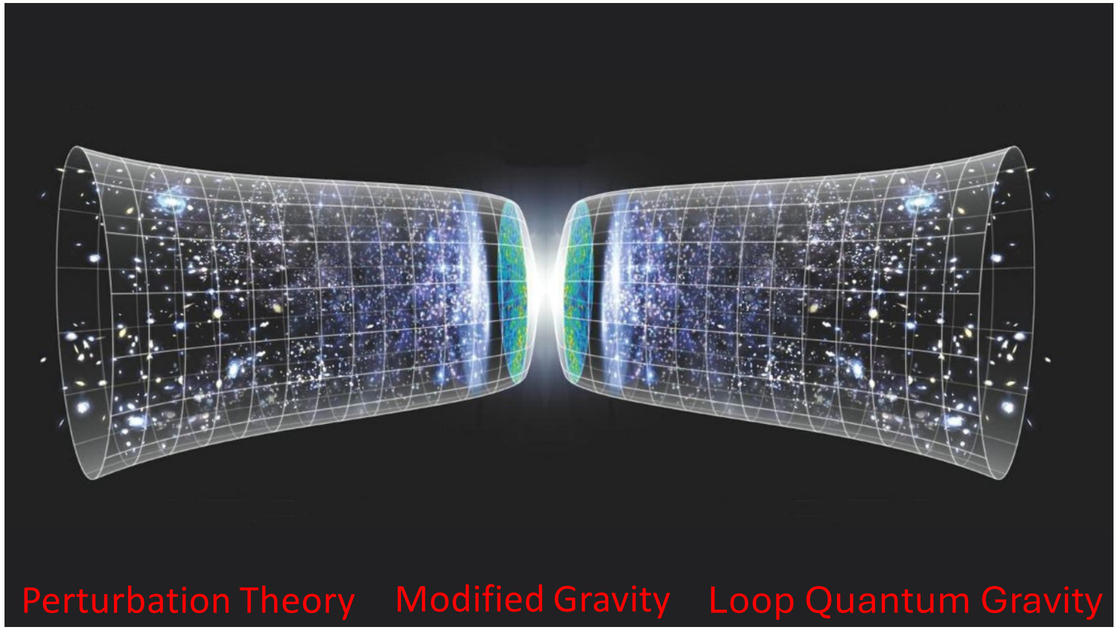
What i'm currently in?

$$\frac{\mathrm{d}Y_X}{\mathrm{d}x} = -\frac{\lambda}{x^2}\left[Y_X^2 - (Y_X^{\mathrm{eq}})^2\right],$$

Dark matter freeze out



I don't know if this is possible or not, but can we discuss it? 🫵🫱



What would it be for big bounce model ?

Contracting > High density > Bounce > Expansion

fundamentally changes the freeze-out problem

Bounce mechanism may produce or redistribute entropy

Under what conditions can a Big Bounce cosmology produce the observed dark-matter relic abundance while remaining consistent with cosmological constraints?