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On the higher-overtone of the Quasinormal-modes

within the Large angular momentum limit

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Outlines

1. Review of the Quasinormal modes
2. The Dolan - Ottewill expansion method
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4. Summary

Review of the Quasinormal modes

Why quasinormal modes are matter ?

- In gravitational waves, the merging of the compact objects like black holes and black holes can be illustrated as shown in figure

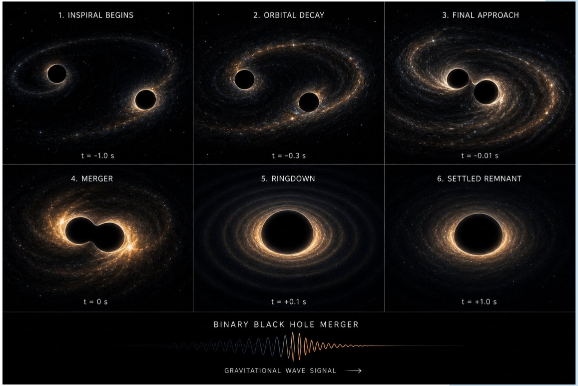

Figure 5: Illustrated the binary black hole merging from [Abbott et al., 2016].

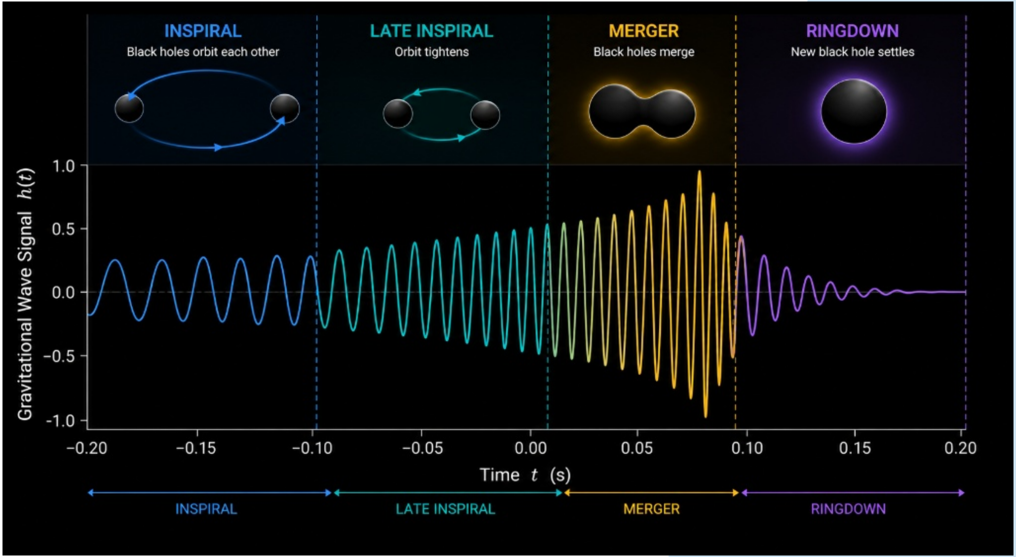

Figure 6: Illustrated the binary black hole merging states from [Abbott et al., 2016]

The method to obtain the quasinormal frequencies

- There are several methods to obtain the quasinormal frequencies ω of the ringdown pahse of the black holes merging.
- One we interest is *the Dolan-Ottewill expansion method*; this method introduces to write the quasinormal frequencies ω in term of the inverse expansion of the angular momentum $L = \ell + \frac{1}{2}$ for any overtone n as

$$\omega_{\ell n} = L\bar{\omega}_{-1}^{(n)} + \bar{\omega}_{0}^{(n)} + L^{-1}\bar{\omega}_{1}^{(n)} + L^{-2}\bar{\omega}_{2}^{(n)} + \cdots . \tag{1}$$

- The Dolan - Ottewill method uses features from *the Regge - Wheeler equation* that describe a perturbation of black holes in the Schwarzschild geometry which can be obtained from the static spherically symmetric spacetime.

The Quasinormal modes boundary conditions

- The quasinormal modes have the boundary conditions for *the ingoing wave* at the horizon ($r_* \to -\infty$) and *the outgoing wave* at the spatial infinity ($r_* \to \infty$) such that [Dolan and Ottewill, 2009]

$$u_{\ell\omega}(r) \sim \begin{cases} e^{-i\omega r_*}, & r_* \to -\infty, \\ A^{(\text{in})}_{\ell\omega} e^{-i\omega r_*} + A^{(\text{out})}_{\ell\omega} e^{i\omega r_*}, & r_* \to \infty, \end{cases} \tag{2}$$

  where the Quasinormal modes correspond to the (complex) frequencies ω at which $A^{(\text{in})}_{\ell\omega} = 0$.
- The radial equation we used in the master equation *must satisfied* this boundary condition. According to the radial function $u_{\ell\omega}(r)$ in Eq. (**??**), this ansatz can be used because it satisfied this boundary condition of the quasinormal modes.
- In the next section, we introduce the Dolan - Ottewill method for the Schwarzschild geometry.

The Dolan - Ottewill expansion method

The Dolan - Ottewill expansion method (Continue)

- First, let us introduce *the master equation* for static spherical symmetric spacetime,

$$\left[\frac{d^2}{dr_*^2} + \omega^2 - V_{eff}(r)\right] R(r) = 0, \tag{3}$$

where r_* is *the tortoise coordinate* such that,

$$\frac{d}{dr_*} = f(r)\frac{d}{dr}, \tag{4}$$

$R(r)$ is an arbitrary radial equation, $V_{eff}(r)$ is an effective potential and $f(r) = 1 - \frac{2}{r}$ in *the Schwarzschild spacetime* because we fixed $M = 1$.

The Dolan - Ottewill expansion method (Continue)

- For the black hole perturbation, Regge and Wheeler had obtained the effective potential as

$$V_{eff}(r) = f(r)\left(\frac{\ell(\ell+1)}{r^2} + \frac{2(1-s^2)}{r^3}\right), \tag{5}$$

where ℓ is the angular momentum parameter and s is a spin parameter.

- Consider the quasinormal frequencies as the inverse summation of angular momentum $L = \ell + \frac{1}{2}$

$$\omega_{\ell n} = L\bar{\omega}_{-1}^{(n)} + \bar{\omega}_0^{(n)} + L^{-1}\bar{\omega}_1^{(n)} + L^{-2}\bar{\omega}_2^{(n)} + \cdots. \tag{6}$$

- Let us introduce the ansatz

$$R(r) = u_{\ell\omega}(r) = \exp\left(i\omega\int^{r_*}\left(1+\frac{6}{r}\right)^{\frac{1}{2}}\left(1-\frac{3}{r}\right)dr\right)v_{\ell\omega}(r). \tag{7}$$

The Dolan - Ottewill expansion method (Continue)

- Substituting Eq. (7) into Eq. (3), we obtain

$$\frac{d}{dr}\left(f(r)\frac{d}{dr}v_{l\omega}(r)\right)+2i\omega\left(1+\frac{6}{r}\right)^{\frac{1}{2}}\left(1-\frac{3}{r}\right)\frac{d}{dr}v_{l\omega}(r)$$
$$+\left[\frac{27\omega^2-L^2}{r^2}+\frac{27i\omega}{r^3}(1+\frac{6}{r})^{-\frac{1}{2}}+\frac{1}{4r^2}+\frac{2(1-s^2)}{r^3}\right]v_{l\omega}(r)=0, \tag{8}$$

- Let us first compute *the fundamental mode* $(n=0)$. The quasinormal frequencies for $(n=0)$ is obtained as

$$\omega_{\ell(n=0)}=L\bar{\omega}_{-1}^{(0)}+\bar{\omega}_0^{(0)}+L^{-1}\bar{\omega}_1^{(0)}+L^{-2}\bar{\omega}_2^{(0)}+\cdots. \tag{9}$$

- A radial equation $v_{\ell\omega}(r)$ for $(n=0)$ is

$$v_{\ell\omega}(r)=\exp\left(S_0^{(0)}(r)+L^{-1}S_1^{(0)}(r)+L^{-2}S_2^{(0)}(r)+L^{-3}S_3^{(0)}(r)+\cdots\right), \tag{10}$$

where $S_k(r)$ is a radial function.

The Dolan - Ottewill expansion method (Continue)

- Substituting Eq. (9) and (10) and *collecting* together like power of L, we obtain a set of independent equations which may be used to determine the coefficients $\bar{\omega}_k$ and the radial functions $S_k(r)$. We will get the equations for each order of L such that

$$L^2: \quad 27\bar{\omega}_{-1}^2 - 1 = 0 \rightarrow \bar{\omega}_{-1} = \pm\frac{1}{\sqrt{27}}, \tag{11}$$

$$L^1: \quad 2i\bar{\omega}_{-1}(1+\frac{6}{r})^{\frac{1}{2}}(1-\frac{3}{r})S_0' + \frac{54\bar{\omega}_{-1}\bar{\omega}_0}{r^2} + \frac{27i\bar{\omega}_{-1}}{r^3}(1+\frac{6}{r})^{-\frac{1}{2}} = 0, \tag{12}$$

$$\begin{aligned} L^0: \quad & 2i(1+\frac{6}{r})^{\frac{1}{2}}(1-\frac{3}{r})(\bar{\omega}_{-1}S_1' + \bar{\omega}_0 S_0') + f(r)(S_0'' + (S_0')^2) + f'(r)S_0' \\ & + \frac{27(2\bar{\omega}_{-1}\bar{\omega}_1 + \bar{\omega}_0^2)}{r^2} + \frac{27i\bar{\omega}_0}{r^3}(1+\frac{6}{r})^{-\frac{1}{2}} + \frac{1}{4r^2} - \frac{2(1-s^2)}{r^3} = 0, \end{aligned} \tag{13}$$

$$L^{-1}: \quad \ldots.$$

The Dolan - Ottewill expansion method (Continue)

- Some results for $(n = 0)$

	L^{1}	L^{0}	L^{-1}	L^{-2}	L^{-3}	L^{-4}
scalar,	1	$-\frac{i}{2}$	$\frac{7}{216}$	$-\frac{137}{7776}i$	$\frac{2615}{1259712}$	$\frac{590983}{362797056}i$
EM,	1	$-\frac{i}{2}$	$-\frac{65}{216}$	$\frac{295}{7776}i$	$-\frac{35617}{1259712}$	$\frac{3374791}{362797056}i$
Grav.,	1	$-\frac{i}{2}$	$-\frac{281}{216}$	$\frac{1591}{7776}i$	$-\frac{710785}{1259712}$	$\frac{92347783}{362797056}i$

Table: Series expantion coefficient $\omega_k = \sqrt{27}\bar{\omega}_k^{(0)}$ for the fundamental $(n = 0)$ Schwarzschild QNMs, where $\sqrt{27}\omega_{l,n=0} = \sum_{k=-1}^{\infty} \omega_k L^{-k}$, for scalar $(s = 0)$, electromagnetic $(s = 1)$ and gravitational $(s = 2)$ modes.

- As we shown, we can determine the quasinormal frequencies ω for the fundamental mode using this step.

The Dolan - Ottewill expansion method (Continue)

- For higher overtone ($n > 0$), Dolan and Ottewill present the form of $v_{\ell\omega}(r)$

$$v_{\ell\omega}^{(n)}(r) = \left[(1 - \frac{3}{r})^{n} + \sum_{j=1}^{n}\sum_{k=1}^{\infty} a_{jk}^{(n)} L^{-k} \left(1 - \frac{3}{r}\right)^{n-j}\right] \exp\left(S_0^{(n)}(r) + L^{-1} S_1^{(n)}(r) + \ldots\right), \tag{14}$$

  where $a_{jk}^{(n)}$ is the coefficient.

- Let us illustrate for the overtone $n = 1$, $n = 2$ and $n = 3$, for example,

$$v_{\ell\omega}^{(1)}(r) = \left[1 - \frac{3}{r} + \sum_{k=1}^{\infty} a_{1k}^{(1)} L^{-k}\right] \exp\left(S_0^{(1)}(r) + L^{-1} S_1^{(1)}(r) + \ldots\right), \tag{15}$$

The Dolan - Ottewill expansion method (Continue)

- For $n = 2$ and $n = 3$,

$$v^{(2)}_{\ell\omega}(r) = \left[\left(1 - \frac{3}{r}\right)^2 + \sum_{k=1}^{\infty} a^{(2)}_{1k} L^{-k} \left(1 - \frac{3}{r}\right) + \sum_{k=1}^{\infty} a^{(2)}_{2k} L^{-k}\right]$$

$$\cdot \exp\left(S^{(2)}_0(r) + L^{-1} S^{(2)}_1(r) + \ldots\right), \qquad (16)$$

$$v^{(3)}_{\ell\omega}(r) = \left[\left(1 - \frac{3}{r}\right)^3 + \sum_{k=1}^{\infty} a^{(3)}_{1k} L^{-k} \left(1 - \frac{3}{r}\right)^2 + \sum_{k=1}^{\infty} a^{(3)}_{2k} L^{-k} \left(1 - \frac{3}{r}\right) + \sum_{k=1}^{\infty} a^{(3)}_{3k} L^{-k}\right]$$

$$\cdot \exp\left(S^{(3)}_0(r) + L^{-1} S^{(3)}_1(r) + \ldots\right), \qquad (17)$$

- According to our objectives, our purpose is to determine the general form of $a^{(n)}_{jk}$ within the perturbaation $s = 0, 1, 2$. Hence, we will use the Dolan - Ottewill method to determine the coefficient $a^{(n)}_{jk}$ for the bosonic field and to look a trend that what factors it depends on.

Expected results

expected results

- Obtaining the extension (generalization) of the coefficient $a_{ij}^{(n)}$ for higher - overtone quasinormal modes for the bosonic quasinormal modes via Dolan - Ottewill method.
- The satisfied results for fermionic fields quasinormal modes using Dolan - Ottewill method in Schwarzschild background.

Summary

Summary

- In this work, we investigate the quasinormal modes (QNMs) arising from perturbations of black holes. These modes are formulated as an eigenvalue problem associated with the damped oscillations characterized by complex frequecies. Various analytical and numerical techniques have been developed to determine and approximate the QNM spectra and corresponding solutions.
- In particular, we examine the large angular momentum expansion of QNMs using the Dolan-Ottewill method. This approach provides an inverse series expansion for the quasinormal frequencies ω applicable to arbitrary overtone number. Special attention is given to the fundamental mode ($n = 0$), for which explicit results are presented and discussed.
- Now, we investigate the general form of the coefficient $a_{jk}^{(n)}$ in bosonic field and try to describe the factors that it depends on.

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Summary (Continue)

- In the future work, we will investigate in fermionic field and furthermore on the applications of the Dolan - Ottewill method for determining the excitation factor of black hole quasinormal modes, $B_{\ell\omega}$.

References

Abbott, B. P., et al. (LIGO Scientific Collaboration, and Collaboration), V. (2016).
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Physical Review Letters.

Dolan, S. R. and Ottewill, A. C. (2009).
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