k-Essence and Galileons

From effective geometry to derivative scalar theories

$$S_k = \int d^4x\,\sqrt{-g}\,\left[-K(X,\phi)\right], \qquad G^{\mu\nu} = 2K_X\,g^{\mu\nu} + 4K_{XX}\,\phi^\mu\phi^\nu$$

- Nonlinear kinetic function *K(X,φ)* — no new higher-derivative mode, but a modified cone & Hamiltonian.
- **Galileons** — second derivatives in the action, second-order field equations.

From a canonical scalar to k-essence

The kinetic term is promoted to an arbitrary function of X and φ — still first order.

Definitions

Kinetic scalar (mostly-plus signature)

$$X \equiv g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi$$

Homogeneous time-dependent field

$$X = -\dot{\phi}^2 < 0$$

Canonical action

$$S_{\rm can} = \int d^4x\,\sqrt{-g}\left[-\tfrac{1}{2}X - V(\phi)\right]$$

k-essence action

$$S_k = \int d^4x\,\sqrt{-g}\,\mathcal{L}\,, \qquad \mathcal{L} = -K(X,\phi)$$

KEY POINT

The nonlinear kinetic function adds **no higher-derivative degree of freedom**. Its effect is to reshape the kinetic matrix, the Hamiltonian, and the characteristic cone of scalar perturbations.

Perfect-fluid interpretation

For a timelike gradient the stress tensor takes perfect-fluid form.

Fluid variables (X < 0)

Four-velocity

$$U_\mu = \frac{\partial_\mu \phi}{\sqrt{-X}}\,, \qquad U_\mu U^\mu = -1$$

Perfect-fluid stress tensor

$$T_{\mu\nu} = (\varepsilon + p)\, U_\mu U_\nu + p\, g_{\mu\nu}$$

Energy density & pressure

$$\varepsilon = -2X K_X + K\,, \qquad p = -K$$

Sound speed & energy

Isentropic sound speed

$$c_s^2 = \frac{\partial p}{\partial \varepsilon} = \frac{p_X}{\varepsilon_X} = \frac{K_X}{K_X + 2X K_{XX}}$$

Homogeneous configuration

$$\boxed{\varepsilon = -2X K_X + K = 2K_X \dot{\phi}^2 + K = \mathcal{H}}$$

- The fluid picture needs X < 0; the effective-metric description is more general and does not require a timelike gradient.

Scalar equation & effective metric — variation

Vary L = −K(X,φ) and integrate by parts to obtain a second-order equation.

From the variation of the action

Variation of the kinetic scalar

$$\delta X = 2\,\phi^{\mu}\nabla_{\mu}\delta\phi$$

Variation of the Lagrangian

$$\delta\mathcal{L} = -K_X\,\delta X - K_\phi\,\delta\phi = -2K_X\phi^{\mu}\nabla_{\mu}\delta\phi - K_\phi\,\delta\phi$$

Integrate the first term by parts

$$\delta S_k = \int d^4x\,\sqrt{-g}\,\left[2\nabla_\mu(K_X\phi^\mu) - K_\phi\right]\delta\phi$$

Scalar field equation

$$\boxed{2\nabla_\mu\left(K_X\nabla^\mu\phi\right) - K_\phi = 0}$$

The equation stays second order for any K — the Ostrogradsky construction of Lecture 1 does not apply.

Scalar equation & effective metric — the cone

Expose the principal part; its coefficient is the inverse effective metric.

Expand the principal part

$$2\nabla_\mu(K_X\nabla^\mu\phi) = 2K_X\Box\phi + 2\big(K_{XX}\nabla_\mu X + K_{X\phi}\phi_\mu\big)\phi^\mu$$

Using

$$\nabla_\mu X = 2\,\phi^\nu\nabla_\mu\nabla_\nu\phi$$

one collects the highest-derivative terms:

$$\big(2K_X g^{\mu\nu} + 4K_{XX}\phi^\mu\phi^\nu\big)\nabla_\mu\nabla_\nu\phi + 2K_{X\phi}X - K_\phi = 0$$

Effective (inverse) metric

$$\boxed{G^{\mu\nu} = 2K_X\, g^{\mu\nu} + 4K_{XX}\,\phi^\mu\phi^\nu}$$

Characteristic hypersurfaces

$$G^{\mu\nu}\xi_\mu\xi_\nu = 0$$

Canonical massive scalar (K_XX = 0)

$$K = \tfrac{1}{2}X + \tfrac{1}{2}m^2\phi^2, \quad K_{XX} = 0 \implies \Box\phi - m^2\phi = 0$$

BACKGROUND DEPENDENCE

K defines the theory, but *G^{μν}* is evaluated on a chosen background. The same theory can have different sound cones — even different stability — on different backgrounds.

Hyperbolicity & stability — the no-ghost condition

A fixed-X, large-velocity limit isolates the sign of the time-kinetic term.

Local-frame Hamiltonian

Local inertial frame

$$X = -\dot{\phi}^2 + (\nabla\phi)^2$$

Canonical momentum

$$p = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = 2K_X \dot{\phi}$$

Hamiltonian density

$$\mathcal{H} = p\dot{\phi} - \mathcal{L} = 2K_X \dot{\phi}^2 + K$$

Fix X, send $\dot{\phi} \to \infty$

With X held fixed, $(\nabla\varphi)^2$ grows with $\dot{\phi}^2$, while K and K_X stay fixed. The leading term is:

$$\mathcal{H} \simeq 2K_X \dot{\phi}^2$$

If K_X < 0 the energy is unbounded below. Hence a necessary condition:

$$\boxed{K_X > 0}$$

Hyperbolicity & stability — Lorentzian signature

The (t,x) principal tensor must have negative determinant for real characteristics.

Principal tensor in (t, x)

General local background ϕ(t,x)

$$G^{\mu\nu} \sim \begin{pmatrix} -K_X + 2K_{XX}\dot{\phi}^2 & -2K_{XX}\dot{\phi}\,\phi' \\ -2K_{XX}\dot{\phi}\,\phi' & K_X + 2K_{XX}\phi'^2 \end{pmatrix}$$

Hyperbolicity condition

Lorentzian signature in 1+1 needs

$$\det G^{\mu\nu} < 0$$

Combined with K_X > 0, the determinant below yields the second condition.

Determinant of the principal tensor

$$\det G^{\mu\nu} \propto \left(-K_X + 2K_{XX}\dot{\phi}^2\right)\left(K_X + 2K_{XX}\phi'^2\right) - 4K_{XX}^2\dot{\phi}^2\phi'^2 = -K_X\left(K_X + 2XK_{XX}\right)$$

$$\boxed{K_X > 0\,, \qquad K_X + 2XK_{XX} > 0}$$

General background & a spacelike gradient

The determinant used a fully general ϕ(t,x): the conditions do not assume homogeneity.

Locally spacelike gradient

Take

$$\bar\phi = \bar\phi(x)\,, \qquad X = \bar\phi'^{\,2} > 0$$

Perturbations along the gradient propagate with the speed opposite.

Parallel speed

$$\boxed{c_\parallel^2 = \frac{K_X + 2X\,K_{XX}}{K_X}}$$

Transverse directions stay luminal in the local metric frame.

- Same two conditions — **K_X > 0** and **K_X + 2X K_XX > 0** — control hyperbolicity for both timelike and spacelike scalar gradients.

Perturbations and the sound speed

Expand to quadratic order on a homogeneous timelike background.

Quadratic action → wave equation

Split the field

$$\phi = \bar{\phi} + \pi$$

Principal quadratic action

$$S^{(2)}_{\rm kin} = \int d^4x\,\sqrt{-g}\,\Big[-K_X(\nabla\pi)^2 - 2K_{XX}\,\bar{\phi}^{\mu}\bar{\phi}^{\nu}\nabla_\mu\pi\nabla_\nu\pi\Big]$$

Homogeneous timelike background

$$\mathcal{L}^{(2)} = (K_X + 2X\,K_{XX})\,\dot{\pi}^2 - K_X(\nabla\pi)^2$$

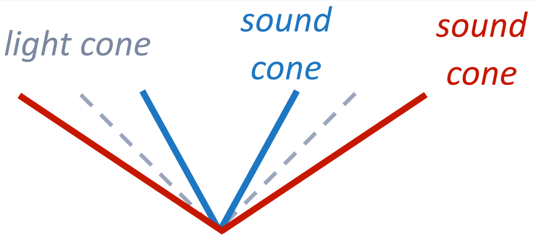
Short-wavelength perturbation

$$\ddot{\pi} - c_s^2\Delta\pi = 0$$

Sound speed

$$\boxed{c_s^2 = \frac{K_X}{K_X + 2X\,K_{XX}} = \frac{1}{1 + 2X\,K_{XX}/K_X}}$$

Conditions (27) ⇒ $\mathbf{c_s^2 > 0}$; the cone may be sub- or super-luminal.



Even more nonlinear: From quasilinear k-essence to the cubic Galileon

Second derivatives may enter the Lagrangian yet cancel in the equation of motion.

Monge–Ampère structure

General 2D second-order form

$$A\left(u_{xx}u_{yy} - u_{xy}^2\right) + B\,u_{xx} + C\,u_{xy} + D\,u_{yy} + E = 0$$

A = 0 is quasilinear; *A ≠ 0* is nonlinear in the highest derivatives.

Simplest 1+1 example

$$\phi_{tt}\phi_{xx} - \phi_{tx}^2 = 0$$

Relativistic realization

$$\boxed{S_3 = \int d^4x\,(\partial\phi)^2\Box\phi}$$

Equation of motion

$$(\Box\phi)^2 - (\partial_\mu\partial_\nu\phi)(\partial^\mu\partial^\nu\phi) = 0$$

The Lagrangian has second derivatives; the Euler–Lagrange equation does not exceed second order.

This cancellation is the seed of Horndeski theory.

Explicit cancellation of the third derivatives

Two integrations by parts; the third-derivative pieces cancel exactly.

Variation of $S_3 = \int d^4x \; X \, \Box\phi$

$$\delta S_3 = \int d^4x \left[2\partial_\mu\phi \, \partial^\mu\delta\phi \, \Box\phi + X \, \Box\delta\phi \right]$$

First term — integrate by parts twice

$$2\int d^4x \, \partial_\mu\phi \, \partial^\mu\delta\phi \, \Box\phi = -2\int d^4x \, \delta\phi \left[(\Box\phi)^2 + \partial_\mu\phi \, \partial^\mu\Box\phi \right]$$

Second term — integrate by parts twice

$$\int d^4x \, X \, \Box\delta\phi = \int d^4x \, \delta\phi \, \Box X$$

Using the identity

$$\Box X = 2\,\partial_\mu\partial_\nu\phi \, \partial^\mu\partial^\nu\phi + 2\,\partial_\mu\phi \, \partial^\mu\Box\phi$$

the two $\partial\varphi\cdot\partial\Box\varphi$ terms cancel. Hence:

$$\boxed{\delta S_3 = 2\int d^4x \, \delta\phi \left[\partial_\mu\partial_\nu\phi \, \partial^\mu\partial^\nu\phi - (\Box\phi)^2 \right]}$$

Covariant cubic Galileon — variation

Curve spacetime; vary and integrate by parts before commuting derivatives.

Scalar variation of S^{cov}_3

Covariant action and shorthand

$$S_3^{\mathrm{cov}} = \frac{1}{2}\int d^4x\,\sqrt{-g}\,(\nabla\phi)^2\Box\phi \qquad \phi_\mu \equiv \nabla_\mu\phi\,, \qquad \phi_{\mu\nu} \equiv \nabla_\mu\nabla_\nu\phi$$

Scalar variation

$$\delta S_3^{\mathrm{cov}} = \int d^4x\,\sqrt{-g}\,\Big[\Box\phi\,\phi^\nu\nabla_\nu\delta\phi + \tfrac{1}{2}\phi_\mu\phi^\mu\,\nabla_\nu\nabla^\nu\delta\phi\Big]$$

Integrate by parts

$$\delta S_3^{\mathrm{cov}} = \int d^4x\,\sqrt{-g}\,\Big[-\nabla_\nu\big(\Box\phi\,\phi^\nu\big) + \tfrac{1}{2}\nabla_\nu\nabla^\nu\big(\phi_\mu\phi^\mu\big)\Big]\delta\phi$$

Expand the derivatives

$$0 = -\phi^\nu\nabla_\nu\Box\phi - (\Box\phi)^2 + \phi_{\mu\nu}\phi^{\mu\nu} + \phi_\mu\nabla_\nu\phi^{\mu\nu}$$

The last term still contains ∇ν φ^{μν} — evaluated by commuting covariant derivatives next.

Covariant cubic Galileon — the curvature term

Commuting covariant derivatives generates a Ricci coupling; equations stay second order.

Commutator on the vector ϕ^μ

$$\nabla_\nu \phi^{\mu\nu} = \nabla^\mu \Box\phi + R^\mu{}_\nu \phi^\nu$$

The *φ^μ ∇_μ□φ* terms cancel, leaving a curvature coupling.

Scalar equation of motion

$$\boxed{-(\Box\phi)^2 + \phi_{\mu\nu}\phi^{\mu\nu} + R_{\mu\nu}\phi^\mu\phi^\nu = 0}$$

Second order in both the scalar and the metric — the Ricci term is required by the commutator.

Metric variation — also at most second order

$$-\frac{2}{\sqrt{-g}}\frac{\delta S_3^{\rm cov}}{\delta g^{\mu\nu}} = -\Box\phi\,\phi_\mu\phi_\nu + 2\phi^\alpha\phi_{(\mu}\nabla_{\nu)}\phi_\alpha - g_{\mu\nu}\phi^\alpha\phi^\beta\phi_{\alpha\beta}$$

Galilean symmetry & the flat-space Hamiltonian

The shift-by-a-linear-function symmetry names the theory; begin the Hamiltonian analysis.

Galilean symmetry

Flat-space action

$$\mathcal{L} = -\tfrac{1}{2}(\partial\phi)^2 + \tfrac{\alpha}{2}(\partial\phi)^2\Box\phi$$

Invariant up to a total derivative under

$$\boxed{\phi(x) \longrightarrow \phi(x) + c + b_\mu x^\mu}$$

Expand the cubic term

$$\tfrac{\alpha}{2}\Box\phi\,(\partial\phi)^2 = \tfrac{\alpha}{2}\{\ddot\phi\dot\phi^2 - \ddot\phi(\nabla\phi)^2 - \Delta\phi\,\dot\phi^2 + \Delta\phi(\nabla\phi)^2\}$$

First term is a total time derivative

$$\ddot\phi\dot\phi^2 = \frac{1}{3}\frac{d}{dt}\dot\phi^3$$

- The remaining terms are reorganized by integration by parts (time, then space) on the next slide.

Hamiltonian on unrestricted field space

Two integrations by parts give a compact Lagrangian; the Hamiltonian is not globally bounded.

Integrations by parts

In time

$$-\tfrac{\alpha}{2}\ddot{\phi}(\nabla\phi)^2 \;\doteq\; \alpha\dot{\phi}\,\nabla\dot{\phi}\cdot\nabla\phi$$

Then in space

$$\alpha\dot{\phi}\,\nabla\dot{\phi}\cdot\nabla\phi = \tfrac{\alpha}{2}\nabla(\dot{\phi}^2)\cdot\nabla\phi \;\doteq\; -\tfrac{\alpha}{2}\dot{\phi}^2\Delta\phi$$

Resulting Lagrangian

$$\mathcal{L} = \tfrac{1}{2}\big(1-2\alpha\Delta\phi\big)\dot{\phi}^2 - \tfrac{1}{2}\big(1-\alpha\Delta\phi\big)(\nabla\phi)^2$$

Momentum & Hamiltonian

Momentum

$$p = \big(1-2\alpha\Delta\phi\big)\dot{\phi}$$

Hamiltonian

$$\boxed{\mathcal{H} = \frac{1}{2}\frac{p^2}{1-2\alpha\Delta\phi} + \frac{1}{2}\big(1-\alpha\Delta\phi\big)(\nabla\phi)^2}$$

For any $\alpha \neq 0$, large $\Delta\varphi$ flips a coefficient — H not globally bounded.

LOCAL VS GLOBAL STABILITY

H unbounded over all configurations does not mean every solution is unstable — perturbations around a background have their own kinetic matrix.

A useful generalization — a Horndeski sector

$$S_3 = \int d^4x\,\sqrt{-g}\,G_3(X,\phi)\,\Box\phi$$