STABILITY, HORNDESKI, AND BEYOND HORNDESKI

Characteristics, Hyperbolicity, and Causality

The second half of the stability block: propagation speeds, characteristic cones, hyperbolicity, and the relation between Hamiltonian positivity and the choice of time.

Velocities

Characteristics

Hyperbolicity

Effective cones

LECTURE 2

Phase, group, and front velocities

Three velocities

$$v_{\mathrm{ph}} = \frac{\omega}{k}, \qquad v_g = \frac{\partial \omega}{\partial k}, \qquad v_f = \lim_{k\to\infty} \frac{\omega(k)}{k}$$

- Phase velocity follows a surface of constant phase.
- Group velocity tracks the peak of a narrow packet.
- Front velocity is the speed of the first sharp disturbance — the quantity relevant for signal propagation.

Canonical massive scalar

Dispersion relation

$$\omega^2 = k^2 + m^2$$

The three velocities

$$v_{\mathrm{ph}} > 1, \qquad v_g < 1, \qquad v_f = 1$$

- A superluminal phase velocity, yet the front — and the signal — travels at v = 1.

LECTURE 2

The tachyonic branch and the signal speed

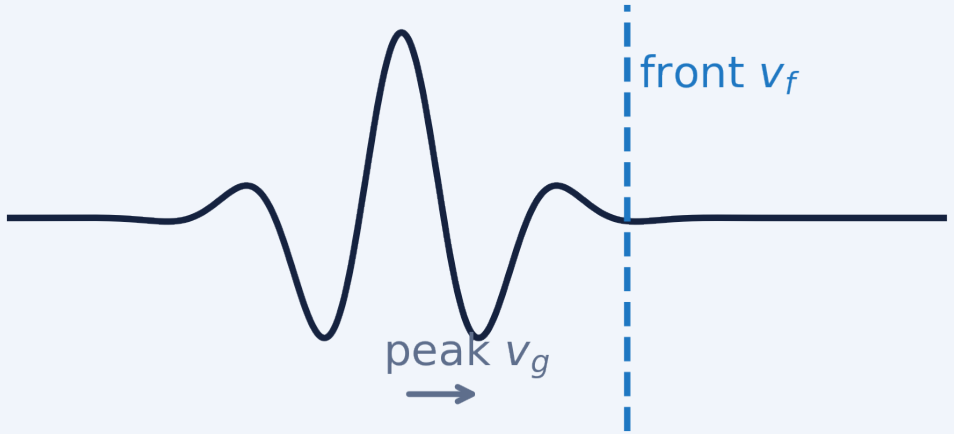
Oscillatory tachyonic branch

Real-frequency modes require $k > \mu$

$$\omega^2 = k^2 - \mu^2, \qquad k > \mu$$

Now the velocities invert

$$v_{\rm ph} < 1, \qquad v_g > 1, \qquad v_f = 1$$



COMMON MISTAKE

The maximum of a dispersive packet need not move with its first front. The signal speed is set by the front — the characteristic cone of the equation.

$$v_f = \lim_{k\to\infty} \frac{\omega(k)}{k}$$

$$\iff \text{high-frequency characteristics}$$

It needs arbitrarily short wavelengths, so it is controlled by the large-k limit — tying the signal velocity to the principal part. Neither a superluminal phase nor group velocity transmits new information.

Principal part and characteristic hypersurfaces

Quasilinear equation

Second-order, quasilinear scalar equation

$$G^{\mu\nu}(\phi, \partial\phi, x)\,\partial_\mu\partial_\nu\phi + F(\phi, \partial\phi, x) = 0$$

Only the principal part — the highest-derivative term — determines the local propagation cone. The symmetric principal tensor plays the role of an inverse effective metric for perturbations.

Characteristic surface

Level surface and its normal covector

$$\Phi(x) = \text{const}, \qquad n_\mu = \partial_\mu\Phi$$

Characteristic when

$$\boxed{G^{\mu\nu} n_\mu n_\nu = 0}$$

- Characteristics are null hypersurfaces of the effective metric — which need not coincide with those of the spacetime metric.

The geometric-optics limit

High-frequency ansatz

$$\phi = A(x)\, e^{i\omega\Phi(x)}, \qquad \omega \to \infty$$

Derivatives at leading order in ω

$$\partial_\mu \phi = e^{i\omega\Phi}\left(\partial_\mu A + i\omega A\, \partial_\mu \Phi\right)$$
$$\partial_\mu \partial_\nu \phi = e^{i\omega\Phi}\left[-\omega^2 A\, \partial_\mu\Phi\, \partial_\nu\Phi + O(\omega)\right]$$

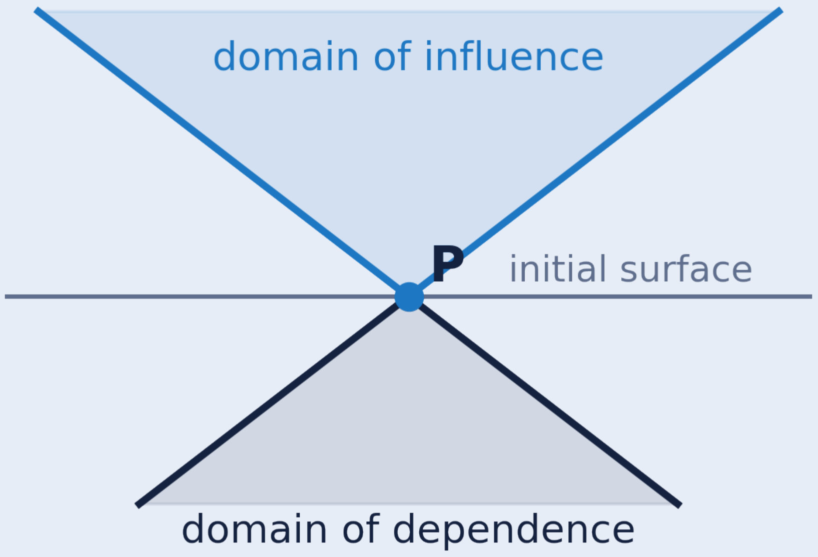
Insert into the principal part

$$-\omega^2 A\, e^{i\omega\Phi}\, G^{\mu\nu} \partial_\mu\Phi\, \partial_\nu\Phi + O(\omega) = 0$$

Divide and take ω → ∞

$$\boxed{G^{\mu\nu}\, \partial_\mu\Phi\, \partial_\nu\Phi = 0}$$

Domains of influence and dependence



- The same cone sets the local domains of dependence and influence.

Tangent and normal descriptions

Tangent from the normal

Raise the normal with the effective metric

$$N^\mu \equiv G^{\mu\nu} n_\nu = G^{\mu\nu}\partial_\nu \Phi$$

It is then tangent to the surface

$$N^\mu \partial_\mu \Phi = G^{\mu\nu}\partial_\mu \Phi \, \partial_\nu \Phi = 0$$

- The raised normal lies in the characteristic surface it defines.

Null for both descriptions

Contract with the covariant metric

$$\boxed{(G^{-1})_{\mu\nu} N^\mu N^\nu = G^{\alpha\beta} n_\alpha n_\beta = 0}$$

- One characteristic, two encodings: a null normal covector of the effective metric, or a null tangent vector of its inverse.

From the effective cone to the characteristic slope

Two-dimensional effective metric

$$G^{\mu\nu} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \qquad (G^{-1})_{\mu\nu} = \frac{1}{\det G}\begin{pmatrix} c & -b \\ -b & a \end{pmatrix}$$

Tangent components along the characteristic

$$N^\mu = \left(\frac{dt}{d\lambda}, \frac{dx}{d\lambda}\right)$$

Null condition of the effective metric

$$c\,(N^0)^2 - 2b\,N^0 N^1 + a\,(N^1)^2 = 0$$

The characteristic slope

Define the slope

$$\xi \equiv \frac{dx}{dt} = \frac{N^1}{N^0}$$

Divide the null condition by $(N^0)^2$

$$\boxed{a\,\xi^2 - 2b\,\xi + c = 0}$$

- The characteristic speeds are read directly from the null cone of the effective metric.

Hyperbolicity and the initial-value problem

Solving for the slopes

Principal part of the 2D equation

$$a\,\phi_{tt} + 2b\,\phi_{tx} + c\,\phi_{xx} = 0$$

Two characteristic slopes

$$\boxed{\xi_\pm = \frac{b \pm \sqrt{b^2 - ac}}{a}}$$

Type of the equation

Hyperbolic condition

$$\boxed{b^2 - ac > 0 \iff \det G^{\mu\nu} < 0}$$

Otherwise

$$\det G = 0 : \ \text{parabolic}, \qquad \det G > 0 : \ \text{elliptic}$$

- Real, distinct slopes ⇒ Lorentzian effective metric.

KEY POINT

The same principal tensor fixes both the propagation cone and the PDE type. Only the hyperbolic case admits the standard local initial-value problem, with data placed on a hypersurface spacelike with respect to the effective metric.

Hamiltonians and time direction

Quadratic Lagrangian

$$L = -\tfrac{1}{2}\, G^{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi$$

Separate time and space derivatives

$$L = -\tfrac{1}{2}G^{00}\dot\phi^2 - G^{0i}\dot\phi\,\partial_i\phi - \tfrac{1}{2}G^{ij}\partial_i\phi\,\partial_j\phi$$

Momentum, then invert for the velocity

$$p = -G^{00}\dot\phi - G^{0i}\partial_i\phi$$

$$\dot\phi = -\frac{p + G^{0i}\partial_i\phi}{G^{00}}$$

Legendre transform

$$\mathcal{H} = p\dot\phi - L$$

gives the Hamiltonian density

$$\boxed{\mathcal{H} = -\frac{1}{2G^{00}}\left(p + G^{0i}\partial_i\phi\right)^2 + \tfrac{1}{2}G^{ij}\partial_i\phi\,\partial_j\phi}$$

Its sign depends on the chosen time coordinate and slicing, so stability must be formulated in a coordinate-independent way — the goal of the next sections.

A scalar with sound speed c_s — the boost

In the preferred frame

Lagrangian with sound speed

$$L = \tfrac{1}{2}\dot{\phi}^2 - \frac{c_s^2}{2}(\phi')^2, \qquad c_s^2 > 0$$

Manifestly positive Hamiltonian

$$\boxed{\mathcal{H} = \tfrac{1}{2}p^2 + \frac{c_s^2}{2}(\phi')^2 \; > \; 0}$$

- In its own rest frame the mode is healthy: energy bounded below.

Boost of velocity v

Coordinates (relative to the metric)

$$\tilde{t} = \frac{t + vx}{\sqrt{1 - v^2}}, \qquad \tilde{x} = \frac{x + vt}{\sqrt{1 - v^2}}$$

Derivatives transform as

$$\partial_t = \tfrac{1}{\sqrt{1-v^2}}\left(\partial_{\tilde{t}} + v\,\partial_{\tilde{x}}\right)$$
$$\partial_x = \tfrac{1}{\sqrt{1-v^2}}\left(v\,\partial_{\tilde{t}} + \partial_{\tilde{x}}\right)$$

- Substituting into L gives the boosted Lagrangian next.

Boosted Lagrangian, effective metric, and its inverse

Boosted Lagrangian

Substitute the boost, then collect $\dot\phi$ and ϕ' terms

$$L = \frac{1}{1-v^2}\Big[\tfrac{1}{2}(1-c_s^2v^2)\,\dot\phi^2 + (1-c_s^2)\,v\,\dot\phi\,\phi' - \tfrac{1}{2}(c_s^2-v^2)(\phi')^2\Big]$$

$$L = \frac{1}{2(1-v^2)}\Big[(\phi_{,\tilde t} + v\phi_{,\tilde x})^2 - c_s^2(v\phi_{,\tilde t} + \phi_{,\tilde x})^2\Big]$$

Effective metric and its inverse

Inverse effective metric G'^{ab} (principal part)

$$G'^{ab} = \frac{1}{1-v^2}\begin{pmatrix} -(1-c_s^2v^2) & -(1-c_s^2)v \\ -(1-c_s^2)v & c_s^2-v^2 \end{pmatrix}$$

Covariant effective metric $(G'^{-1})_{\alpha\beta}$

$$(G'^{-1})_{ab} = \frac{1}{c_s^2(1-v^2)}\begin{pmatrix} v^2-c_s^2 & -(1-c_s^2)v \\ -(1-c_s^2)v & 1-c_s^2v^2 \end{pmatrix}$$

Boosted Hamiltonian: when a coefficient flips sign

$$\mathcal{H} = \frac{1-v^2}{2(1-c_s^2 v^2)}\left[p - \frac{(1-c_s^2)v}{1-v^2}\phi'\right]^2 + \frac{c_s^2 - v^2}{2(1-v^2)}(\phi')^2$$

Subluminal mode ($c_s < 1$)

$$c_s < 1: \quad |v| > c_s \;\Rightarrow\; G'^{11} < 0$$

The gradient coefficient $c_s^2 - v^2$ turns negative once $|v| > c_s$, so the Hamiltonian is unbounded below.

Superluminal mode ($c_s > 1$)

$$c_s > 1: \quad |v| > \frac{1}{c_s} \;\Rightarrow\; G'^{00} > 0$$

The coefficient of the momentum square turns negative once $|v| > 1/c_s$, again making it unbounded below.

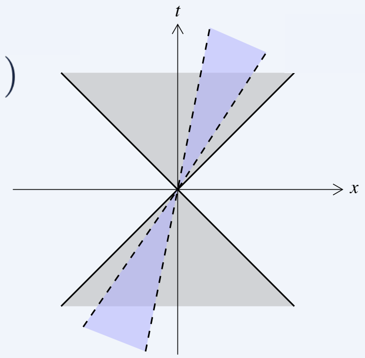
In both cases the coordinate Hamiltonian is unbounded below, although the preferred-frame Hamiltonian stays positive. The resolution is geometric — next slide.

Bounded below? The geometry of the time direction

Subluminal: bad evolution vector

Norm of ∂_t' in the covariant metric

$$(G'^{-1})_{ab}T^aT^b = (G'^{-1})_{00} = \frac{v^2 - c_s^2}{c_s^2(1 - v^2)}$$

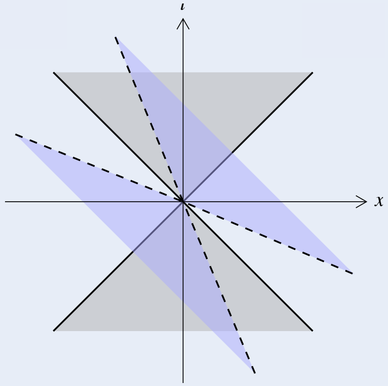


For $|v| > c_s$ this is positive: ∂_t' is spacelike and leaves the effective cone.

Superluminal: bad initial surface

Norm of the normal dt′ in the inverse metric

$$G'^{ab}n_an_b = G'^{00} = -\frac{1 - c_s^2v^2}{1 - v^2}$$



For $|v| > 1/c_s$ this turns positive: dt′ is no longer timelike, so t′ = const is not spacelike.

KEY POINT

Bounded below w.r.t. an admissible time direction ⇒ the vacuum is stable. The converse is false: an unbounded coordinate Hamiltonian may only reflect a time direction not adapted to the effective causal structure.

Several cones and an invariant stability criterion

A common time orientation

Covector timelike for every inverse metric

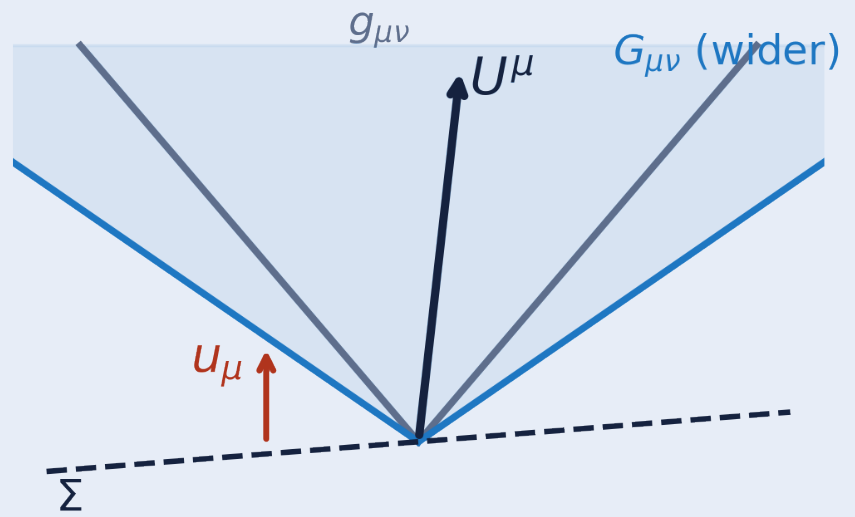
$$g^{\mu\nu}u_\mu u_\nu < 0, \qquad G^{\mu\nu}u_\mu u_\nu < 0, \ \ldots$$

Vector timelike for every covariant metric

$$g_{\mu\nu}U^\mu U^\nu < 0, \qquad (G^{-1})_{\mu\nu}U^\mu U^\nu < 0, \ \ldots$$

- Their level surfaces can serve as common spacelike initial hypersurfaces.

Two nested cones



$$\boxed{T^\nu{}_\mu U^\mu u_\nu \ \geq \ 0}$$

A wider cone — superluminal propagation relative to matter — is not by itself a ghost or a vacuum instability. What matters is a common time orientation, a common spacelike initial surface, and a positive physical energy with respect to those choices.