Lecture 1

Dynamical Instabilities, Ghosts, and Ostrogradsky's Theorem

Theoretical consistency of theories

Tachyonic instability

unstable long wavelengths — a background, not a theory, defect

Gradient instability

wrong-sign gradients — short wavelengths blow up fastest

Ghosts

wrong-sign kinetic term — the Hamiltonian is unbounded below

Ostrogradsky

nondegenerate higher time derivatives carry a hidden ghost

Four questions about consistency

A logically prior test than phenomenology: is the theory dynamically sound?

Linear stability	Tachyon (long-λ) and gradient (short-λ) instabilities of a scalar background
Ghosts	Sign of the Hamiltonian; wrong-sign kinetic term; vacuum decay
Ostrogradsky	Nondegenerate higher time derivatives ⇒ a hidden ghost; why GR and f(R) evade it

A canonical massive scalar

A healthy reference point before we start breaking it

Plane-wave analysis

Massive scalar, mostly-plus signature

$$(\Box - m^2)\,\phi = 0$$

$$-\ddot{\phi} + \Delta\phi - m^2\phi = 0, \qquad \Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$$

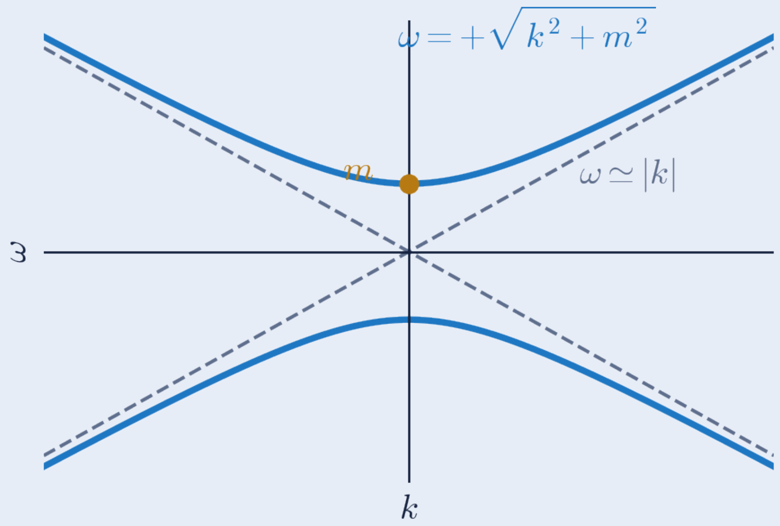
Plane-wave ansatz

$$\phi(t, \mathbf{x}) = \phi_0\, e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}$$

gives the dispersion relation

$$\boxed{\omega^2 = k^2 + m^2, \qquad \omega = \pm\sqrt{k^2 + m^2}}$$

Dispersion relation



- Every Fourier mode has a real frequency: the amplitude oscillates rather than grows, so the background is linearly stable.

Tachyonic instability

flip the sign of the mass squared

Negative mass squared

Reverse the sign of the mass term

$$m^2 = -\mu^2, \qquad \mu^2 > 0$$

The mode equation becomes

$$\omega^2 = k^2 - \mu^2$$

For long wavelengths k < μ the frequency is imaginary

$$\omega = \pm i\,\Omega_k, \qquad \Omega_k = \sqrt{\mu^2 - k^2}$$

A slow, growing mode

The perturbation contains a growing mode

$$\boxed{\phi_k \propto e^{\Omega_k t}}$$

with characteristic time scale

$$\tau_{\mathrm{tach}} \sim \Omega_k^{-1} \gtrsim \mu^{-1}$$

- Only sufficiently small wave numbers are unstable. If μ is tiny the instability is slow — physically relevant, not instantly catastrophic.

Tachyons and spontaneous symmetry breaking

a negative mass squared need not be fatal

Finite-temperature potential

A standard thermal effective potential

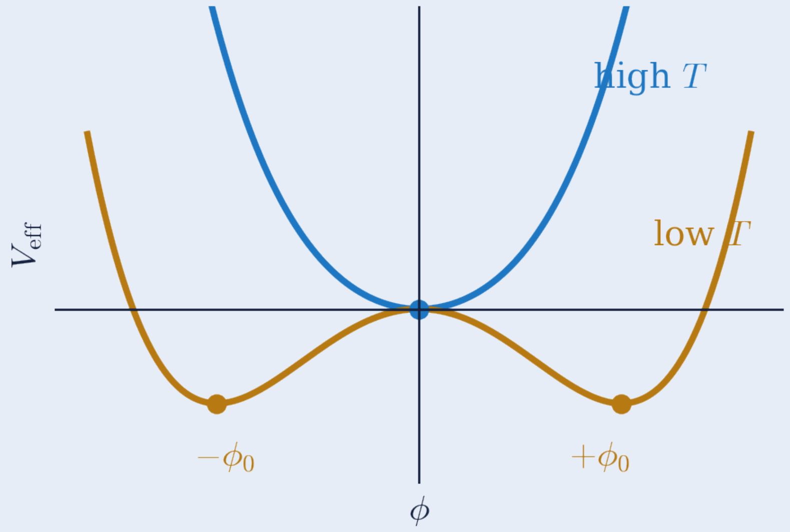
$$V_{\text{eff}}(\phi, T) = -\frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4 + \frac{cT^2}{2}\phi^2$$

Curvature at the origin

$$\boxed{m_{\text{eff}}^2(T) = cT^2 - \mu^2}$$

- High T: m²_eff > 0, origin stable, symmetry restored. Low T: m²_eff < 0, origin tachyonic, field rolls to a broken minimum.

Rolling to a healthy vacuum



KEY POINT

A tachyon is an instability of a particular background. If stable minima exist elsewhere, the system evolves to a healthy vacuum — the logic of spontaneous symmetry breaking.

Gradient instability

wrong sign on the spatial-gradient term

Short wavelengths blow up

Gradient term flips sign, time kinetic term canonical

$$-\ddot{\phi} - \Delta\phi - m^2\phi = 0$$

which gives

$$\omega^2 = m^2 - k^2$$

Now the short modes k > m are unstable, growing as

$$\Omega_k = \sqrt{k^2 - m^2} \simeq k \quad (k \gg m)$$

No shortest instability time

With an EFT cutoff, the fastest mode has

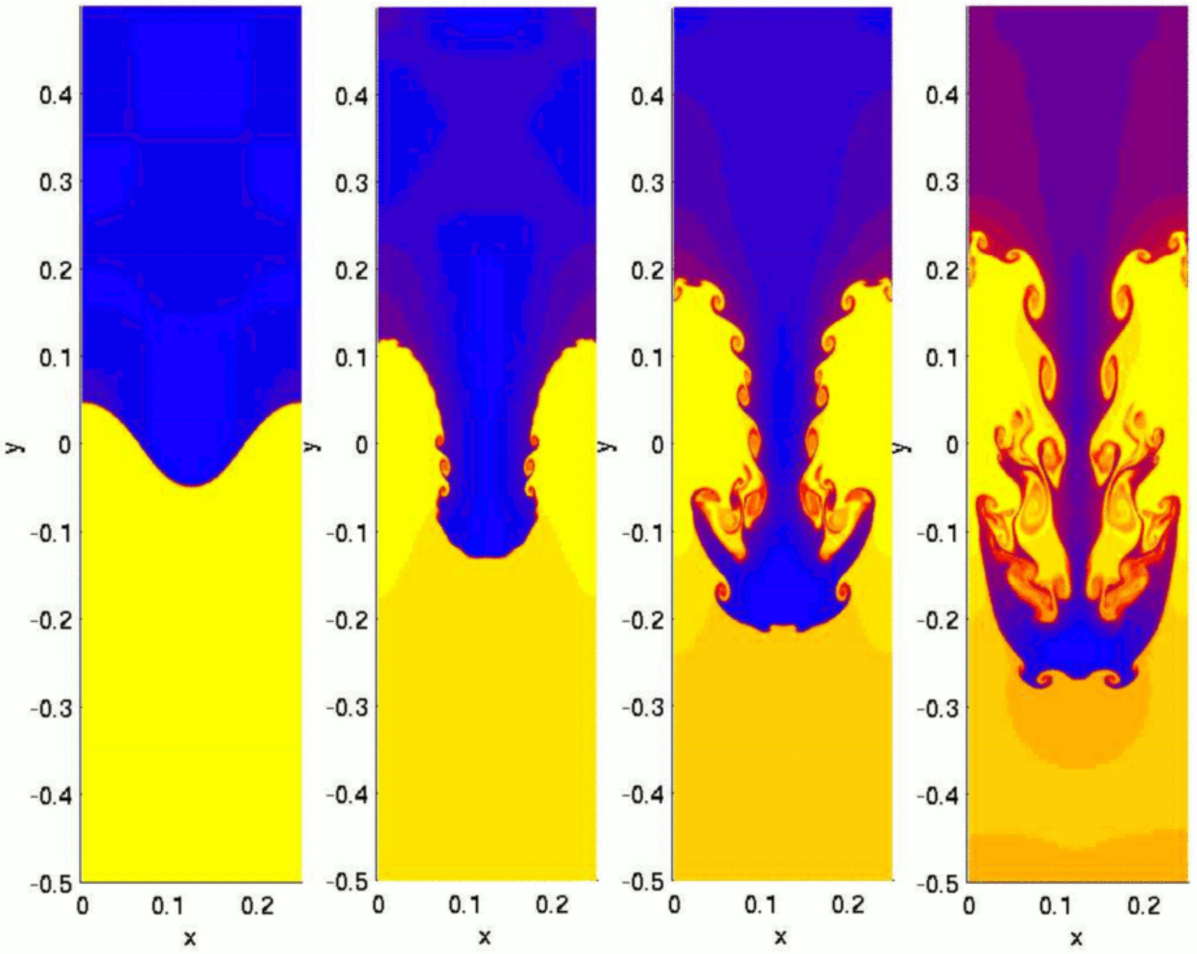
$$\tau_{\mathrm{grad}} \sim k_{\mathrm{cut}}^{-1}$$

The growth rate increases with k: ultraviolet modes grow arbitrarily fast. The Rayleigh–Taylor instability — heavy fluid over light — is the familiar analogy.

TACHYON vs GRADIENT

Tachyon: long-λ, may just find a new vacuum. Gradient: short-λ, rate grows with k — usually far more dangerous in an effective field theory.

Rayleigh–Taylor instability as a physical phenomenon



Surface of two fluids, the heavy fluid on top of light fluid

$$c_s \propto \frac{i}{\sqrt{k}}$$

Growing mode at high momenta:

$$\phi \sim e^{A\sqrt{k}\,t}$$

Gradient instability can be physical

Gradient instabilities in effective theories

Canonical complex scalar with U(1) potential (2 d.o.f.)

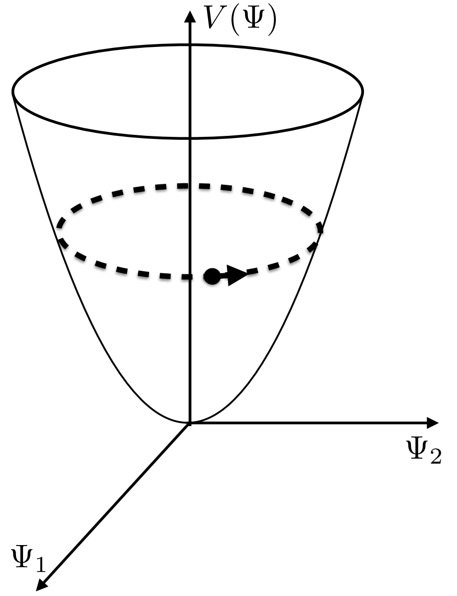
$$\mathcal{L} = -\frac{1}{2}|\partial\Psi|^2 - V(|\Psi|)$$

Effectively k-essence with 1 d.o.f.

$$\omega^2 = m^2 - k^2$$

It is possible to have

$c_s^2 =$ const < 0 for low k



Positive energy for a canonical scalar

the Hamiltonian sign test, in its simplest form

Lagrangian and momentum

Canonical scalar

$$\mathcal{L}_\phi = -\tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2$$
$$= \tfrac{1}{2}\dot{\phi}^2 - \tfrac{1}{2}(\nabla\phi)^2 - \tfrac{1}{2}m^2\phi^2$$

Conjugate momentum

$$p_\phi = \frac{\partial\mathcal{L}_\phi}{\partial\dot{\phi}} = \dot{\phi}$$

Energy bounded below

Legendre transform

$$\mathcal{H}_\phi = p_\phi\dot{\phi} - \mathcal{L}_\phi$$
$$= \tfrac{1}{2}\dot{\phi}^2 + \tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}m^2\phi^2 \;\geq\; 0$$

- Each term is a square: the energy is bounded from below. This is the property a ghost destroys.

The same check for linearized gravity

constraints do not obscure the sign test

Fierz–Pauli action (with zero mass)

Einstein–Hilbert to quadratic order around ημν + hμν

$$S^{(2)} = \frac{M_P^2}{2}\int d^4x\,\Big(-\tfrac{1}{4}h_{\mu\nu,\sigma}h^{\mu\nu,\sigma} + \tfrac{1}{2}h_{\mu\nu,\sigma}h^{\mu\sigma,\nu} + \tfrac{1}{4}h_{,\sigma}h^{,\sigma} - \tfrac{1}{2}h_{,\sigma}h^{\mu\sigma}{}_{,\mu}\Big)$$

Physical transverse–traceless components

$$\partial_i h^{TT}_{ij} = 0, \qquad h^{TT}_{ii} = 0$$

Reduced TT action and energy

$$S^{(2)}_{TT} = \frac{1}{16\pi G}\int d^4x\,\Big(-\tfrac{1}{4}\partial_\alpha h^{TT}_{ij}\,\partial^\alpha h^{TT}_{ij}\Big) = \frac{1}{64\pi G}\int d^4x\,\big[(\dot h^{TT}_{ij})^2 - (\partial_k h^{TT}_{ij})^2\big]$$

Hamiltonian density of the two polarizations

$$\mathcal{H}_{TT} = \frac{1}{64\pi G}\big[(\dot h^{TT}_{ij})^2 + (\partial_k h^{TT}_{ij})^2\big] \;\ge\; 0$$

- After reduction to physical modes the graviton energy is manifestly positive — the presence of constraints does not hide the sign.

A wrong-sign kinetic term

the definition of a ghost

Flip the overall sign

A scalar with opposite overall sign

$$\mathcal{L}_{\text{ghost}} = +\tfrac{1}{2}\partial_\mu\chi\,\partial^\mu\chi + \tfrac{1}{2}m^2\chi^2$$
$$= -\tfrac{1}{2}\dot{\chi}^2 + \tfrac{1}{2}(\nabla\chi)^2 + \tfrac{1}{2}m^2\chi^2$$

Hamiltonian density

$$\mathcal{H}_{\text{ghost}} = -\tfrac{1}{2}\dot{\chi}^2 - \tfrac{1}{2}(\nabla\chi)^2 - \tfrac{1}{2}m^2\chi^2$$
$$\leq \ 0$$

Ghost vs tachyon

The energy can be made arbitrarily negative. Such a degree of freedom is a ghost.

Tachyon — standard kinetic term, unstable potential near the chosen background.

Ghost — wrong sign in the kinetic sector itself, so there is no ground state even if the potential looks harmless.

Vacuum decay in the presence of a ghost

why a ghost is so severe

Coupling to a healthy field

Let an ordinary field φ interact with a ghost χ

$$S = -\int \frac{1}{2}(\partial\phi)^2 + \int \frac{1}{2}(\partial\chi)^2 + S_{int}[\phi,\chi]$$

$$\mathcal{L}_{\text{int}} = g\,\phi^2\chi^2$$

The vacuum can decay while conserving four-momentum

$$0 \;\longrightarrow\; \phi(k_1) + \phi(k_2) + \chi(p_1) + \chi(p_2)$$

Positive- and negative-energy quanta balance at arbitrarily large momentum, so the available phase space is unbounded.

An unbounded decay rate

Decay rate per unit time and volume

$$\boxed{\Gamma \sim g^2 \int^{k_{\text{cut}}} d^4k \;\sim\; g^2 k_{\text{cut}}^4}$$

KEY POINT

A ghost makes the Hamiltonian unbounded below. Once it couples to healthy fields, nothing prevents production of arbitrarily large positive and negative energy — one of the most severe pathologies a modified-gravity theory can have.

Ghost instability can be physical

Gradient instabilities in effective theories

Canonical complex scalar with U(1) potential (2 d.o.f.)

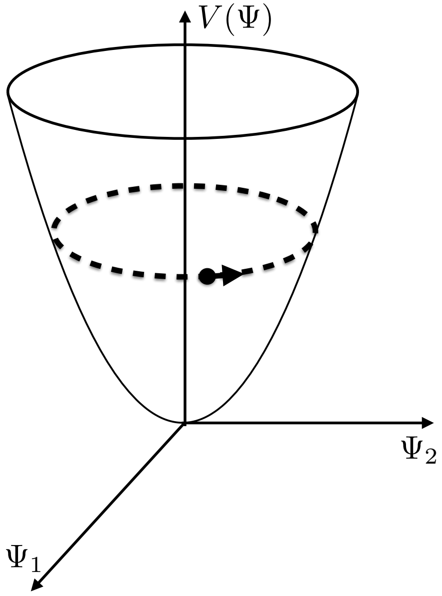
$$\mathcal{L} = -\frac{1}{2}|\partial\Psi|^2 - V(|\Psi|)$$

Effectively k-essence with 1 d.o.f.

$$\omega^2 = m^2 - k^2$$

It is possible to have

$c_s^2 =$ const < 0 for low k and effectively dof is a ghost



A higher-derivative mechanical example

first-order form and the primary constraint

Fourth-order equation, two dof

The simplest higher-derivative Lagrangian

$$\mathcal{L} = \tfrac{1}{2}\ddot{q}^2$$

Generalized Euler–Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{q}}\Big) + \frac{d^2}{dt^2}\Big(\frac{\partial \mathcal{L}}{\partial \ddot{q}}\Big) = 0$$

$$q^{(4)} = 0$$

Four initial data → two mechanical degrees of freedom, not one.

Auxiliary variable & Hamiltonian

Auxiliary s, multiplier χ, then integrate by parts

$$\mathcal{L} = \tfrac{1}{2}s^2 + \chi(\ddot{q} - s)$$

$$\mathcal{L} = \tfrac{1}{2}s^2 - \dot{\chi}\,\dot{q} - \chi s$$

Equations of motion and momenta

$$\ddot{\chi} = 0, \qquad s - \chi = 0, \qquad \ddot{q} - s = 0$$

$$p_q = -\dot{\chi}, \qquad p_s = 0, \qquad p_\chi = -\dot{q}$$

Primary constraint and canonical Hamiltonian

$$\Phi_1 \equiv p_s \approx 0$$

$$H_T = -p_q p_\chi - \tfrac{1}{2}s^2 + \chi s + v\,p_s$$

Dirac chain and the unbounded Hamiltonian

the reduced Hamiltonian is linear in a momentum

Constraint analysis

Preserve the primary constraint

$$\dot{\Phi}_1 = \{p_s, H_T\} = -\frac{\partial H_T}{\partial s} = s - \chi \approx 0$$

→ a secondary constraint

$$\Phi_2 \equiv s - \chi \approx 0$$

Its preservation fixes the multiplier, no new constraint

$$\dot{\Phi}_2 = \{s - \chi, H_T\} = v + p_q \approx 0 \;\Rightarrow\; v = -p_q$$

Reduced Hamiltonian (linear in p_a)

$$\boxed{H = -p_q p_\chi + \tfrac{1}{2}\chi^2}$$

It reproduces the equation

Hamilton's equations

$$\dot{q} = \frac{\partial H}{\partial p_q} = -p_\chi, \qquad \dot{p}_q = -\frac{\partial H}{\partial q} = 0$$

$$\dot{\chi} = \frac{\partial H}{\partial p_\chi} = -p_q, \qquad \dot{p}_\chi = -\frac{\partial H}{\partial \chi} = -\chi$$

hence, exactly the fourth-order equation

$$\ddot{q} = \chi, \qquad q^{(3)} = -p_q, \qquad q^{(4)} = -\dot{p}_q = 0$$

- H is linear in p_q — unbounded above and below. The higher-derivative theory carries this sickness intrinsically.

The hidden ghost

diagonalize to expose the wrong-sign mode

Eliminate the auxiliary field

Removing s leaves

$$\mathcal{L} = -\dot{\chi}\,\dot{q} - \tfrac{1}{2}\chi^2$$

Rotate to normal coordinates

$$\chi = \frac{\varphi + \psi}{\sqrt{2}}, \qquad q = \frac{\varphi - \psi}{\sqrt{2}}$$

One wrong-sign kinetic term

The Lagrangian becomes

$$\boxed{\mathcal{L} = -\tfrac{1}{2}\dot{\varphi}^2 + \tfrac{1}{2}\dot{\psi}^2 - \tfrac{1}{4}(\varphi + \psi)^2}$$

- ϕ has a healthy kinetic term, ψ the wrong sign. The Ostrogradsky instability is the higher-derivative manifestation of a ghost.

A fourth-order scalar field

the field-theory analogue, reduced to second order

Higher-derivative scalar

A four-derivative Lagrangian

$$\mathcal{L} = -\tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi + \frac{\alpha}{2}(\Box\phi)^2$$

gives a fourth-order equation

$$\Box\phi + \alpha\Box^2\phi = 0$$

Auxiliary B, multiplier χ, integrate by parts

$$\widetilde{\mathcal{L}} = -\tfrac{1}{2}(\partial\phi)^2 + \frac{\alpha}{2}B^2 + \chi(\Box\phi - B)$$

$$\widetilde{\mathcal{L}} = -\tfrac{1}{2}(\partial\phi)^2 - \partial_\mu\chi\,\partial^\mu\phi + \frac{\alpha}{2}B^2 - \chi B$$

Two second-order fields

Algebraic B = χ/α removes B

$$B = \chi/\alpha$$

$$\widetilde{\mathcal{L}} = -\tfrac{1}{2}(\partial\phi)^2 - \partial_\mu\chi\,\partial^\mu\phi - \frac{\chi^2}{2\alpha}$$

Field redefinition

$$\phi = \varphi - \chi$$

diagonalizes the kinetic terms

$$\boxed{\widetilde{\mathcal{L}} = -\tfrac{1}{2}(\partial\varphi)^2 + \tfrac{1}{2}(\partial\chi)^2 - \frac{\chi^2}{2\alpha}}$$

The ghost pole in the propagator

the same conclusion, read off the poles

Propagator decomposition

The higher-derivative propagator splits into two simple poles:

$$\boxed{\frac{1}{p^2 + \alpha p^4} = \frac{1}{p^2} - \frac{1}{p^2 + \alpha^{-1}}}$$

DERIVATIVE COUNTING IS NOT ENOUGH

A fourth-order equation generically carries an extra mode, but the physical question is the sign of that mode after reduction to independent second-order fields. The relative minus sign between the two poles is exactly the wrong-sign residue — a ghost. In this nondegenerate example the extra mode is unavoidably a ghost.

General nondegenerate systems

the theorem in its general form

Setup and nondegeneracy

A higher-derivative mechanical system

$$S = \int dt\, L(q_a, \dot{q}_a, \ddot{q}_a)$$

Nondegenerate higher-derivative sector

$$\det\left(\frac{\partial^2 L}{\partial \ddot{q}_a\, \partial \ddot{q}_b}\right) \neq 0$$

Ostrogradsky variables and momenta

$$Q_a = \dot{q}_a$$

$$P_a = \frac{\partial L}{\partial \dot{q}_a} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}_a}\right)$$

$$\Pi_a = \frac{\partial L}{\partial \ddot{q}_a}$$

A Hamiltonian linear in P_a

$$H = P_a \dot{q}^a + \Pi_a \dot{Q}^a - L.$$

$$\Downarrow$$

$$H = P_a Q^a + \Pi_a \ddot{q}^a - L(q, Q, \ddot{q})$$

Nondegeneracy solves for the acceleration

$$\ddot{q}_a = A_a(q, Q, \Pi)$$

The Hamiltonian is then

$$H = P_a Q_a + \Pi_a A_a - L\big(q, Q, A(q, Q, \Pi)\big)$$

- H is linear in the momenta P_a, hence unbounded above and below. This is the general Ostrogradsky result.

Nondegeneracy is essential

where the loophole for healthy theories lives

A degenerate Hessian

If instead

$$\det\left(\frac{\partial^2 L}{\partial \ddot{q}_a\,\partial \ddot{q}_b}\right) = 0$$

constraints may remove the extra mode. This possibility underlies the construction of higher-derivative scalar–tensor theories without an Ostrogradsky ghost — Horndeski and beyond.

Linear in $\ddot{q}$ is harmless

A term linear in the highest derivative: if

$$f(q,\dot{q}) = \frac{\partial F(q,\dot{q})}{\partial \dot{q}}$$

then it is a total derivative up to lower orders

$$f(q,\dot{q})\,\ddot{q} = \frac{dF}{dt} - \frac{\partial F}{\partial q}\,\dot{q}$$

- It differs from a first-derivative expression only by a total derivative, so it adds no genuine higher-derivative dynamics.

Why Einstein–Hilbert is not Ostrogradsky

second derivatives, but only linearly

The action and the Ricci tensor

Einstein–Hilbert action

$$S_{EH} = \frac{1}{16\pi G}\int d^4x\,\sqrt{-g}\,R$$

Ricci tensor

$$R_{\mu\nu} = \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\mu \Gamma^\alpha_{\alpha\nu} + \Gamma^\alpha_{\mu\nu}\Gamma^\beta_{\alpha\beta} - \Gamma^\beta_{\mu\alpha}\Gamma^\alpha_{\nu\beta}$$

Christoffel symbols

Built from first derivatives of the metric

$$\Gamma^\alpha_{\mu\nu} = \tfrac{1}{2} g^{\alpha\beta}(\partial_\mu g_{\nu\beta} + \partial_\nu g_{\mu\beta} - \partial_\beta g_{\mu\nu})$$

- R contains second derivatives of the metric — through the two ∂Γ terms — but they occur only linearly. That linearity is the escape.

The ΓΓ bulk action

integrate the second derivatives away

Integrate by parts

The two linear ∂Γ terms integrate by parts:

$$\sqrt{-g}\, g^{\mu\nu} \partial_\alpha \Gamma^\alpha_{\mu\nu} = -\Gamma^\alpha_{\mu\nu} \partial_\alpha (\sqrt{-g}\, g^{\mu\nu}) + \text{t.d.}$$

$$-\sqrt{-g}\, g^{\mu\nu} \partial_\mu \Gamma^\alpha_{\alpha\nu} = \Gamma^\alpha_{\alpha\nu} \partial_\mu (\sqrt{-g}\, g^{\mu\nu}) + \text{t.d.}$$

The ΓΓ form is not manifestly covariant, but it gives the same bulk equations once the boundary term is treated properly.

First-derivative bulk Lagrangian

$$S_{EH} = \frac{1}{16\pi G} \int d^4x\, \sqrt{-g}\, g^{\mu\nu} \left(\Gamma^\beta_{\mu\alpha} \Gamma^\alpha_{\nu\beta} - \Gamma^\alpha_{\mu\nu} \Gamma^\beta_{\alpha\beta}\right)$$
$$+ \text{ boundary term}$$

KEY POINT

The bulk Lagrangian contains only first derivatives of $g_{\mu\nu}$. Einstein–Hilbert does not satisfy the nondegeneracy premise of the Ostrogradsky theorem — so it carries no Ostrogradsky ghost.

f(R): fourth order without a ghost

higher order, but the extra mode is a scalar

The field equations carry fourth metric derivatives

$$f'(R)R_{\mu\nu} - \tfrac{1}{2}f(R)g_{\mu\nu} - (\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)f'(R) = 8\pi G\,T_{\mu\nu}$$

The term $\nabla_\mu\nabla_\nu f'(R)$ contains derivatives of R, hence up to fourth derivatives of the metric.

The extra mode is the scalar f′(R)

The additional degree of freedom is isolated as the scalar f′(R), and metric f(R) gravity is rewritten as a scalar–tensor theory.

Consistency set by the scalar

Its mass scale

$$m_s^2 \propto \frac{1}{f''(R)}$$

Consistency is fixed by the sign of the scalar kinetic term and the stability of its mass — not by the order of the metric equation.

$$f'(R) > 0, \quad f''(R) > 0.$$

Higher order is not the theorem

the nondegeneracy premise is what matters

The general lesson

Ostrogradsky's theorem applies to a nondegenerate dependence on higher time derivatives. GR evades the premise; metric f(R) has fourth-order equations, yet its higher-derivative structure is equivalent to one extra scalar and need not be a ghost.

$$\boxed{\text{higher-order equations} \;\not\Rightarrow\; \text{Ostrogradsky ghost}}$$

The role of constraints

First-order construction with a multiplier

$$\widetilde{\mathcal{L}} = L(q, Q, \dot{Q}) - \lambda\,(Q - \dot{q})$$

Whether the resulting constraints eliminate the extra mode is the whole question. Generic nondegenerate theories: they do not. Degenerate ones: they may — the mechanism made central again in Lecture 4.

What we can now diagnose

Tachyon

Wrong sign of m^2. Long wavelengths grow; may only signal the wrong background.

Gradient

Wrong sign of gradients. Short wavelengths grow fastest; dangerous in EFT.

Ghost

Wrong sign of the kinetic term. Hamiltonian unbounded below; vacuum decays.

Ostrogradsky

Nondegenerate higher time derivatives carry a hidden ghost — unless constraints intervene.

NEXT → Lecture 2

Passing these tests is necessary, not sufficient: the equations must also have a well-posed initial-value problem, and their characteristic surfaces fix how signals propagate. Next: effective metrics, hyperbolicity, and causal cones.