Reviewing the WKB approximation for black hole perturbation theory

Chun-Hung Chen

The Institute for Fundamental Study, Naresuan University, Thailand.

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Introduction and Summary

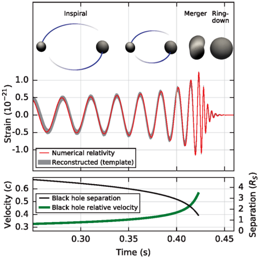
The WKB (or JWKB, Jeffrey(1924), Wentzel–Kramers–Brillouin(1926)) approximation is a powerful tool for obtaining a global approximation to the solution of a linear differential equation, which assume the solution has an exponential form and expand its exponent in an asymptotic series.

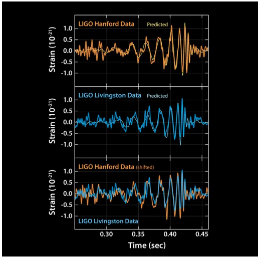
In black-hole perturbation theory, the linearized field equations can often be separated into ordinary differential equations governing the radial and angular perturbations. These equations typically take the form of second-order linear differential equations, providing a natural starting point for the WKB approximation.

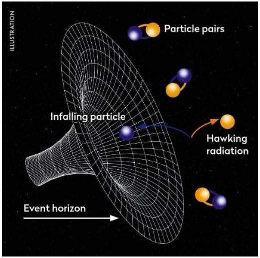
Linear black-hole perturbation theory provides a unified framework for understanding a variety of physical phenomena, including quasinormal modes, quasi-bound states, superradiance, greybody factors, gravitational-wave emission, and late-time tails....

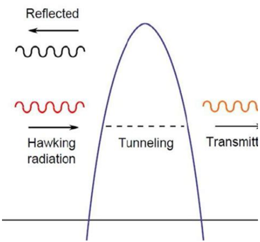
In this talk, we focus on the WKB approximation as a powerful tool for studying the phenomena of QNMs and GBF.

Introduction and Summary









Credits: B.P. Abbott et al. (LIGO Scientific Collaboration and Virgo Collaboration) Phys. Rev. Lett. 116, 061102, https://www.ligo.caltech.edu/, AI collects from Getty Images, and Class. Quantum Grav. 36 185005.

Introduction and Summary

The WKB approximation provides a convenient and powerful framework for studying black-hole perturbations, particularly their quasinormal modes and transmission properties.

Iyer, S. and Will, C. M. (1987), Black-hole normal modes: A WKB approach. I. Foundations and application of a higher-order WKB analysis of potential-barrier scattering, Physical Review D 35, 3621. (1370)

Konoplya, R. A. (2003), Quasinormal behavior of the D-dimensional Schwarzschild black hole and the higher order WKB approach, Physical Review D 68, 024018. (1154)

J. Matyjasek and M. Opala (2017), Quasinormal modes of black holes. The improved semianalytic approach, Physical Review D 96, 024011. (467)

Konoplya, R. A. (2018), Higher order WKB formula for quasinormal modes and grey-body factors: recipes for quick and accurate calculations, Class. Quantum Grav. 36 155002. (533)

Introduction and Summary

Highlight from the Iyer and Will (1987), the third order WKB works, the Schrödinger-like form of the linear equation from the black hole perturbation is given by

$$\left(\frac{d^2}{dr_\star^2} + Q\left(r_\star\right)\right)\Psi\left(r_\star\right) = 0,$$

where $r_\star$ is the tortoise coordinate, the relation between $r_\star$ and the radial position r defined differently by various of linear perturbations and spacetimes.The WKB parameters are given by

$$Q\left(r_\star\right) = \omega^2 - V_{eff} \quad ; \quad z_0^2 = \frac{-2Q\left(r_{max}\right)}{\partial^2_{r_\star} Q\left(r_\star\right)|_{r\to r_{max}}};$$

$$k = \frac{1}{2}\partial^2_{r_\star} Q\left(r_\star\right)|_{r\to r_{max}} \quad ; \quad b_n = \frac{1}{n!k}\partial^n_{r_\star} Q\left(r_\star\right)|_{r\to r_{max}},$$

where r_{max} represented the radial position of the potential peak. The greybody factor (transmission probability) is given by the following formula for the 3rd order approximation

$$\mathcal{T} = \frac{1}{1+e^{2S(\omega)}},$$

where

$$\begin{aligned} S(\omega) \quad = \quad & \pi k^{1/2}\left[\frac{1}{2}z_0^2 + \left(\frac{15}{64}b_3^2 - \frac{3}{16}b_4\right)z_0^4 + \left(\frac{1155}{2048}b_3^4 - \frac{315}{256}b_3^2 b_4 + \frac{35}{128}b_4^2 + \frac{35}{64}b_3 b_5 - \frac{5}{32}b_6\right)z_0^6\right] \\ & + \pi k^{-1/2}\left[\frac{3}{16}b_4 - \frac{7}{64}b_3^2 - \left(\frac{1365}{2048}b_3^4 - \frac{525}{256}b_3^2 b_4 + \frac{85}{128}b_4^2 + \frac{95}{64}b_3 b_5 - \frac{25}{32}b_6\right)z_0^2\right] \\ & + \mathcal{O}\left(k^{1/2}z_0^8,\ k^{-1/2}z_0^4,\ k^{-3/2}z_0^0\right). \end{aligned}$$

Introduction and Summary

Furthermore, for the quasinormal modes

$$\omega^2 = \left[V_0 + \left(-2V_0''\right)^{1/2} \frac{\Lambda}{i}\right] - i\left(n + \frac{1}{2}\right)\left(-2V_0''\right)^{1/2}\left(1 + \frac{\Omega}{n + \frac{1}{2}}\right).$$

Here, $V_0^{(m)}$ denotes the of $\partial_{r_\star}^{(m)} V_{eff}|_{r \to r_{max}}$, and the prime now denotes the derivative with respective to the tortoise coordinate. The functions Λ and Ω can be expressed as

$$\begin{aligned}
\Lambda &= \frac{i}{(-2V_0'')^{1/2}}\left[\frac{1}{8}\left(\frac{V_0^{(4)}}{V_0''}\right)\left(\frac{1}{4} + \alpha^2\right) - \frac{1}{288}\left(\frac{V_0'''}{V_0''}\right)^2\left(7 + 60\alpha^2\right)\right], \\
\Omega &= \frac{(n + \frac{1}{2})}{(-2V_0'')}\left[\frac{5}{6912}\left(\frac{V_0'''}{V_0''}\right)^4\left(77 + 188\alpha^2\right) - \frac{1}{384}\left(\frac{V_0'''^2 V_0^{(4)}}{V_0''^3}\right)\left(50 + 100\alpha^2\right)\right. \\
&\quad \left. + \frac{1}{2304}\left(\frac{V_0^{(4)}}{V_0''}\right)^2\left(67 + 68\alpha^2\right) + \frac{1}{288}\left(\frac{V_0''' V_0^{(5)}}{V_0'''^2}\right)\left(19 + 28\alpha^2\right) - \frac{1}{288}\left(\frac{V_0^{(6)}}{V_0''}\right)\left(5 + 4\alpha^2\right)\right],
\end{aligned}$$

where $\alpha = n + 1/2$, $n = 0, 1, 2, \ldots$.

Introduction and Summary

So, basically, whenever you obtain a new second-order ordinary differential equation from black-hole perturbation theory—whether it comes from a GR black hole or a modified-gravity black hole, and whatever type of perturbation you are considering—if you can cast it into a Schrödinger-like form, identify the effective potential barrier, and determine its maximum, then you are already in a good position to obtain semi-analytical results for greybody factors and quasinormal modes. And then… cheers! You're one step closer to your diploma—or maybe even your next publication.

Thank you for your attention!!

However... let us start again,
and take a closer look at the whole procedure, following the third-order WKB formalism of Iyer and Will (1987).

Outline

General solution

As we mentioned, the WKB approximation defined the asymptotic solution as an exponential series, let us start with writing the master equation and the approximation solution as

$$\epsilon^2 \frac{d^2}{dx^2}\Psi(x) + Q(x)\Psi(x) = 0,$$

and

$$\Psi(x) \sim \exp\left[\epsilon^{n-1} S_n(x)\right],$$

where ϵ is the order parameter. Consider the series method for solving ODE, we may rewrite the master equation as

$$\begin{aligned}
S_0'^2(x) \quad &+ \quad \epsilon\left(S_0''(x) + 2S_0'(x)S_1'(x)\right) \\
&+ \quad \epsilon^2\left(S_1''(x) + S_1'^2(x) + 2S_0'(x)S_2'(x)\right) \\
&+ \quad \epsilon^3\left(S_2''(x) + 2S_0'(x)S_3'(x) + 2S_1'(x)S_2'(x)\right) \\
&\quad\; +Q(x) = 0,
\end{aligned}$$

where "prime" denote ∂_x. Solving this expression vanish order by order, the ϵ^0 order is given by

$$S_0'(x) + Q(x) = 0 \quad \Rightarrow \quad S_0 = \pm i\int^x \sqrt{Q(\eta)}d\eta.$$

The ϵ order is given by

$$\begin{aligned}
S_0''(x) + 2S_0'(x)S_1'(x) = 0, \qquad &\Rightarrow \qquad 2S_1'(x)\left(\pm i\sqrt{Q}\right) \pm i\frac{1}{2}\frac{Q'(x)}{\sqrt{Q(x)}} = 0, \\
\Rightarrow S_1' = -\frac{1}{4}\frac{Q'(x)}{Q(x)}, \qquad &\Rightarrow \qquad S_1 = -\frac{1}{4}\ln Q(x).
\end{aligned}$$

Following the same procedure, one may further derive the higher ϵ^n order.

General solution

For the ϵ^2 order

$$
\begin{aligned}
& S_1''(x) + S_1'^2(x) + 2S_0'(x)S_2'(x) = 0, \\
\Rightarrow \qquad & -\frac{1}{4}\frac{Q''Q - Q'^2}{Q^2} + \frac{1}{16}\frac{Q'^2}{Q^2} \pm 2i\sqrt{Q}S_2' = 0 \\
\Rightarrow \qquad & S_2(x) = \mp\frac{i}{8}\int^x \left(\frac{Q''(\eta)}{Q^{3/2}(\eta)} - \frac{5}{4}\frac{Q'^2(\eta)}{Q^{5/2}(\eta)}\right) d\eta.
\end{aligned}
$$

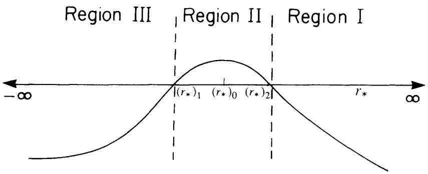
For the ϵ^3 order

$$
\begin{aligned}
& S_2''(x) + 2S_0'(x)S_3'(x) + 2S_1'(x)S_2'(x) = 0, \\
\Rightarrow \qquad & \mp\frac{i}{8}\frac{d}{dx}\left(\frac{Q''}{Q^{3/2}} - \frac{5}{4}\frac{Q'^2}{Q^{5/2}}\right) \pm 2i\sqrt{Q}S_3' \pm \left(\frac{Q'}{Q}\right)\left(\frac{i}{8}\left(\frac{Q''}{Q^{3/2}} - \frac{5}{4}\frac{Q'^2}{Q^{5/2}}\right)\right) = 0, \\
& \Rightarrow S_3(x) = \frac{1}{16}\int^x \frac{d}{d\eta}\left[\left(\frac{1}{\sqrt{Q(\eta)}}\right)\left(\frac{Q''}{Q^{3/2}} - \frac{5}{4}\frac{Q'^2}{Q^{5/2}}\right)\right] d\eta = \frac{1}{16}\left[\frac{Q''(x)}{Q^2(x)} - \frac{5}{4}\frac{Q'^2(x)}{Q^3(x)}\right].
\end{aligned}
$$

These shall be the general solutions for the WKB approximation. Considering the asymptotic behavior of the leading term $S_0(x) \sim \pm i\tilde{\omega}x$ as $x \to \infty$ and $Q(x) \sim \tilde{\omega}^2$, therefore the upper sign of $S_n(x)$ represented the outgoing (to spacial infinity) solution, and the lower sign represented the ingoing solution (left moving). On the other hand, $S_0(x) \sim \pm i\tilde{k}x$ as $x \to -\infty$ and $Q(x) \sim \tilde{k}^2$, therefore, the upper sign of $S_n(x)$ represented the outgoing wave (to the black hole event horizon) and the lower sign represented the ingoing solution (right moving).

General solution

The general WKB solution is valid everywhere except near the classical turning points. In black-hole perturbation theory, the effective potential typically has the barrier structure shown in the plot. Since ω^2 is generally an unknown quantity that we need to determine, the physical setup typically involves zero to two turning points. However, Region II, the region between the two turning points, requires further analysis. To handle this region, we expand the equation around x_0 (or $(r_\star)_0$ in the plot), the position of the maximum of the effective potential.



A general image for $Q(x) = \omega^2 - V_{eff}(x)$. Iyer and Will (1987) PRD 35, 3621.

Outline

~~Introduction and Summary~~

~~General solution~~

Interior solution

Exterior solution

Matching procedure

Remark

Interior solution

As we mentioned, the WKB fail around the classical turning points, as such we need to do a further analysis around the region between the classical turning points. An further expansion around the position of the potential peak shall be imposed, the $Q(z)$ and the corresponding radial equation can be written as

$$Q(z) = Q_0 + \frac{1}{2} Q_0'' z^2 + \frac{1}{6} Q_0''' z^3 + \frac{1}{24} Q_0^{(4)} z^4 + \frac{1}{120} Q_0^{(5)} z^5 + \frac{1}{720} Q_0^{(6)} z^6,$$

and

$$\epsilon^2 \frac{d^2}{dz^2} \Psi + k \left(z_0^2 + z^2 + bz^3 + cz^4 + dz^5 + fz^6 \right) \Psi = 0,$$

where $z = x - x_0$, and

$$\begin{aligned} & k \equiv \tfrac{1}{2} Q_0'', \ \ z_0{}^2 \equiv -2Q_0/Q_0'' \ , \\ & b \equiv \tfrac{1}{3} Q_0'''/Q_0'', \ \ c = \tfrac{1}{12} Q_0^{(4)}/Q_0'' \ , \\ & d \equiv \tfrac{1}{60} Q_0^{(5)}/Q_0'', \ \ f = \tfrac{1}{360} Q_0^{(6)}/Q_0'' \ . \end{aligned}$$

One notice that the ODE in this region will approach the ODE of parabolic cylinder function $D_\nu(t)$ and $D_{-\nu-1}(it)$ if we do a further transformation as

$$\begin{aligned} t \equiv (4k)^{1/4} e^{-i\pi/4} z/\epsilon^{1/2} \quad , \quad \nu + \frac{1}{2} \equiv -ik^{1/2} z_0^2/2\epsilon, \\ \bar{b} \equiv \frac{1}{4} b (4k)^{-1/4} e^{i\pi/4} \quad , \quad \bar{c} \equiv \frac{1}{4} c (4k)^{-1/2} e^{i\pi/2}, \\ \bar{d} \equiv \frac{1}{4} d (4k)^{-3/4} e^{3i\pi/4} \quad , \quad \bar{f} \equiv \frac{1}{4} f (4k)^{-1} e^{i\pi}, \end{aligned}$$

and the radial equation become

$$\frac{d^2}{dt^2} \Psi + \left[\nu + \frac{1}{2} - \frac{1}{4} t^2 - \epsilon^{1/2} \bar{b} t^3 - \epsilon \bar{c} t^4 - \epsilon^{3/2} \bar{d} t^5 - \epsilon^4 \bar{f} t^6 \right] \Psi = 0.$$

The leading order be exactly the parabolic cylinder function.

Interior solution

To handle the higher ϵ order, we assume the solution $\Psi \sim f(t)D_\nu(g(t))$, putting in the radial equation one have

$$
\begin{aligned}
& f\left[\dot{g}^2\ddot{D}_\nu(g) + (nu + \frac{1}{2} - \frac{1}{4}t^2)D_\nu(g)\right] \\
+ \quad & \left(2\dot{f}\dot{g} + f\ddot{g}\right)\dot{D}_\nu(g) + \left(-f\dot{g}^2(\nu + \frac{1}{2} - \frac{1}{4}g^2) + \ddot{f} + Q(t)f\right)D_\nu(g) = 0.
\end{aligned}
$$

The first line vanish because of the parabolic cylinder function, the $\dot{D}_\nu(g)$ tern vanish by solving $f = \dot{g}^{-1/2}$, the above equation become an ODE as

$$\dot{g}^2(\nu + \frac{1}{2} - \frac{1}{4}g^2) + \frac{1}{2}\dddot{g}/\dot{g} - \frac{3}{4}\ddot{g}^2/\dot{g}^2 - Q(t) = 0.$$

Setting

$$g(t) = t + \epsilon^{n/2}A_n(t) \quad , \quad A_n(t) = \sum_i a_{2i}^{(n)} t^{n+1-2i},$$

where sum over i for the cases $n + 1 - 2i \geq 0$, and solving $A_n(t)$ by tracking ϵ order by order. One found that we need to rewrite one of the definition as $\nu + \frac{1}{2} \equiv -ik^{1/2}z_0^2/2\epsilon - \epsilon\Lambda - \epsilon^2\Omega$, or the terms proportional to t^0 will not vanish for $A_2(t)$ and $A_4(t)$ respectively.

Interior solution

The interior solution be obtained

$$\Lambda=\tfrac{1}{2}(3\bar{c}-7\bar{b}^{2})+(\nu+\tfrac{1}{2})^{2}(6\bar{c}-30\bar{b}^{2}),$$

$$\Omega=-(\nu+\tfrac{1}{2})(1155\bar{b}^{4}-918\bar{b}^{2}\bar{c}+67\bar{c}^{2}+190\bar{b}\bar{d}-25\bar{f})-(\nu+\tfrac{1}{2})^{3}(2820\bar{b}^{4}-1800\bar{b}^{2}\bar{c}+68\bar{c}^{2}+280\bar{b}\bar{d}-20\bar{f}).$$

$$A_1(t)=\tfrac{16}{3}\bar{b}(\nu+\tfrac{1}{2}+\tfrac{1}{8}t^{2}),$$

$$A_2(t)=(\nu+\tfrac{1}{2})(3\bar{c}-\tfrac{103}{9}\bar{b}^{2})t+\tfrac{1}{2}(\bar{c}-\tfrac{13}{9}\bar{b}^{2})t^{3},$$

$$\begin{aligned}A_3(t)=&\tfrac{1}{5}(\tfrac{2744}{9}\bar{b}^{3}-216\bar{b}\bar{c}+48\bar{d})+\tfrac{1}{5}(\nu+\tfrac{1}{2})^{2}(\tfrac{126\,544}{81}\bar{b}^{3}-784\bar{b}\bar{c}+\tfrac{256}{3}\bar{d})\\&+\tfrac{1}{15}(\nu+\tfrac{1}{2})(\tfrac{11\,558}{27}\bar{b}^{3}-234\bar{b}\bar{c}+32\bar{d})t^{2}+\tfrac{1}{5}(\tfrac{173}{27}\bar{b}^{3}-\tfrac{17}{3}\bar{b}\bar{c}+2\bar{d})t^{4},\end{aligned}$$

$$\begin{aligned}A_4(t)=&[(-\tfrac{10\,619}{36}\bar{b}^{4}+\tfrac{519}{2}\bar{b}^{2}\bar{c}-\tfrac{91}{4}\bar{c}^{2}-60\bar{b}\,\bar{d}+10\bar{f})\\&+(\nu+\tfrac{1}{2})^{2}(-\tfrac{2\,770\,843}{2430}\bar{b}^{4}+\tfrac{3733}{5}\bar{b}^{2}\bar{c}-\tfrac{77}{2}\bar{c}^{2}-\tfrac{5276}{45}\bar{b}\,\bar{d}+10\bar{f})]t\\&+(\nu+\tfrac{1}{2})(-\tfrac{94\,357}{1215}\bar{b}^{4}+\tfrac{2846}{45}\bar{b}^{2}\bar{c}-\tfrac{17}{3}\bar{c}^{2}-\tfrac{538}{45}\bar{b}\,\bar{d}+\tfrac{5}{3}\bar{f})t^{3}+(-\tfrac{9013}{3240}\bar{b}^{4}+\tfrac{187}{60}\bar{b}^{2}\bar{c}-\tfrac{11}{24}\bar{c}^{2}-\tfrac{14}{15}\bar{b}\,\bar{d}+\tfrac{1}{3}\bar{f})t^{5}\end{aligned}$$

$$\psi\sim\dot{g}^{-1/2}[AD_{\nu}(g(t))+BD_{-\nu-1}(ig(t))].$$

Outline

Exterior solution

The "Exterior" here means the overlap region with region II in the region I and III. Where outside the classical turning points that WKB general solutions valid. However it closed to the classical turning, we still need to consider $Q(z)$, but rewrite it around the classical turning point as

$$Q(z)=k[-\bar{z}^2+z^2+b(z^3-\bar{z}^3)+c(z^4-\bar{z}^4)+d(z^5-\bar{z}^5)+f(z^6-\bar{z}^6)]\,.$$

$$\bar{z}=\pm z_0-\tfrac{1}{2}bz_0{}^2\pm\tfrac{1}{8}(5b^2-4c)z_0{}^3-\tfrac{1}{2}(2b^3-3bc+d)z_0{}^4\pm\tfrac{1}{128}(231b^4-504b^2c+224bd+112c^2-64f)z_0{}^5\,,$$

together $Q(z)$ with the WKB general solutions, one may obtained

$$\begin{aligned}
S_0\sim \pm ik^{1/2}\{&\tfrac{1}{2}z^2+\tfrac{1}{6}bz^3+\tfrac{1}{32}(4c-b^2)z^4+\tfrac{1}{80}(b^3-4bc+8d)z^5\\
&-\tfrac{1}{768}(5b^4-24b^2c+32bd+16c^2-64f)z^6+\cdots\\
&+\bar{z}^2[-\tfrac{1}{4}+\tfrac{1}{4}bz-\tfrac{1}{32}(3b^2-4c)z^2+\tfrac{1}{96}(5b^3-12bc+8d)z^3+\cdots]\\
&+\bar{z}^3[\tfrac{1}{4}b^2z-\tfrac{1}{32}(3b^3-4bc)z^2+\cdots]\\
&+\bar{z}^4[\tfrac{1}{16}z^{-2}-\tfrac{3}{16}bz^{-1}+\tfrac{1}{256}(79b^2-28c)+\tfrac{1}{128}(35b^3-28bc+24d)z+\cdots]\\
&+\bar{z}^5(\tfrac{1}{8}bz^{-2}-\tfrac{3}{8}b^2z^{-1}+\cdots)+\bar{z}^6[\tfrac{1}{64}z^{-4}-\tfrac{5}{96}bz^{-3}+\tfrac{3}{256}(17b^2+4c)z^{-2}+\cdots]\\
&-[\tfrac{1}{2}\bar{z}^2+\tfrac{1}{2}b\bar{z}^3+\tfrac{5}{64}(3b^2+4c)\bar{z}^4+\tfrac{1}{32}(15b^3-12bc+16d)\bar{z}^5+\cdots]\ln(2z/\bar{z})\}\,,\\
e^{S_1}\sim Q^{-1/4}\sim k^{-1/4}z^{-1/2}\{&1-\tfrac{1}{4}bz+\tfrac{1}{32}(5b^2-8c)z^2-\tfrac{1}{128}(15b^3-4bc+32d)z^3+\cdots\\
&+\bar{z}^2[\tfrac{1}{4}z^{-2}-\tfrac{5}{16}bz^{-1}+\tfrac{5}{128}(9b^2-8c)+\cdots]+\bar{z}^3(\tfrac{1}{4}bz^{-2}-\tfrac{5}{16}b^2z^{-1}+\cdots)\\
&+\bar{z}^4[\tfrac{5}{32}z^{-4}-\tfrac{45}{128}bz^{-3}+\tfrac{13}{1024}(45b^2-8c)z^{-2}+\cdots]\}\,,\\
S_2\sim \mp\tfrac{1}{8}ik^{-1/2}\{&\tfrac{3}{2}z^{-2}-\tfrac{1}{2}bz^{-1}+\tfrac{1}{12}(b^2-2c)+\tfrac{1}{16}(35b^3-92bc+72d)z-\tfrac{1}{3}\bar{z}^{-2}+\tfrac{1}{3}b\bar{z}^{-1}\\
&+\bar{z}^2[\tfrac{19}{8}z^{-4}-\tfrac{31}{12}bz^{-3}+\tfrac{1}{32}(75b^2-68c)z^{-2}+\cdots]\\
&-[\tfrac{1}{8}(7b^2-12c)+\cdots]\ln(2z/\bar{z})\}\,,\\
S_3\sim k^{-1}[&-\tfrac{3}{16}z^{-4}+\tfrac{1}{8}bz^{-3}-\tfrac{3}{64}(3b^2-4c)z^{-2}+\cdots]\,.
\end{aligned}$$

Exterior solution

Therefore, the exterior solution can be written as

$$\Psi(x) \sim \exp\left[\epsilon^{n-1} S_n(x)\right].$$

Some tricks in working on the equations in previous page:

The integration of S_2 divergence around the classical turning point, therefor a two times integration by parts will let the integration numerically workable, which means

$$\begin{aligned} S_2(x) &= \mp\frac{i}{8}\int_{\bar{z}}^{z}\left(\frac{Q''}{Q^{3/2}} - \frac{5}{4}\frac{Q'^2}{Q^{5/2}}\right)dz'. \\ &\sim \mp\frac{i}{8}\left(\frac{5Q'}{6Q^{3/2}}\Big|_{\bar{z}}^{z} - \frac{Q''}{Q'Q^{1/2}}\Big|_{\bar{z}}^{z} + \frac{1}{2}\int_{\bar{z}}^{z}\left(\frac{Q''}{Q'}\right)' Q^{-1/2}dz'\right). \end{aligned}$$

The coefficients before $\ln(2z/\bar{z})$ need to converge to $\nu + 1/2$ until the corresponding ϵ order, or

the leading order will change.

Outline

Matching procedure

Therefore, leading order terms that contribute by the third order approximation, the exterior and interior solutions can be written as

$$\psi \sim [Z_{\rm in}^{\rm I} e^{-ik^{1/2}z^2/2\epsilon}(z/\epsilon^{1/2})^{-(\nu+1)}(4k)^{-(\nu+1)/4} e^{i\pi(\nu+1)/4} R$$
$$+ Z_{\rm out}^{\rm I} e^{ik^{1/2}z^2/2\epsilon}(z/\epsilon^{1/2})^{\nu}(4k)^{\nu/4} e^{-i\pi\nu/4} R^{-1}](4/\epsilon^2 k e^{i\pi})^{1/8}, \quad z \gtrsim z_0 ,$$
$$\psi \sim [Z_{\rm in}^{\rm III} e^{-ik^{1/2}z^2/2\epsilon}(-z/\epsilon^{1/2})^{-(\nu+1)}(4k)^{-(\nu+1)/4} e^{i\pi(\nu+1)/4} R$$
$$+ Z_{\rm out}^{\rm III} e^{ik^{1/2}z^2/2\epsilon}(-z/\epsilon^{1/2})^{\nu}(4k)^{\nu/4} e^{-i\pi\nu/4} R^{-1}](4/\epsilon^2 k e^{i\pi})^{1/8}, \quad z \lesssim -z_0 ,$$

$$\psi \sim A e^{ik^{1/2}z^2/2\epsilon}(z/\epsilon^{1/2})^{\nu}(4k)^{\nu/4} e^{-i\pi\nu/4}$$
$$+ B e^{-ik^{1/2}z^2/2\epsilon}(z/\epsilon^{1/2})^{-(\nu+1)}(4k)^{-(\nu+1)/4} e^{-i\pi(\nu+1)/4}, \quad z \gtrsim z_0 ,$$
$$\psi \sim \left[A + B\frac{(2\pi)^{1/2} e^{-i\pi\nu/2}}{\Gamma(\nu+1)} \right] e^{ik^{1/2}z^2/2\epsilon}(-z/\epsilon^{1/2})^{\nu}(4k)^{\nu/4} e^{3i\pi\nu/4}$$
$$+ \left[B e^{3i\pi(\nu+1)/2} - A\frac{(2\pi)^{1/2} e^{i\pi\nu}}{\Gamma(-\nu)} \right] e^{-ik^{1/2}z^2/2\epsilon}(-z/\epsilon^{1/2})^{-(\nu+1)}(4k)^{-(\nu+1)/4} e^{-3i\pi(\nu+1)/4}, \quad z \lesssim -z_0 .$$

Matching procedure

One may matching them and obtained the result

$$\begin{pmatrix} Z_{out}^{III} \\ Z_{in}^{III} \end{pmatrix} = \begin{pmatrix} e^{i\pi\nu} & \frac{iR^2 e^{i\pi\nu}\sqrt{2\pi}}{\Gamma(\nu+1)} \\ \frac{R^{-2}\sqrt{2\pi}}{\Gamma(-\nu)} & -e^{i\pi\nu} \end{pmatrix} \begin{pmatrix} Z_{out}^{I} \\ Z_{in}^{I} \end{pmatrix},$$

where

$$R = \left(\nu + \frac{1}{2}\right)^{(\nu+\frac{1}{2})/2} \exp\left[-\frac{1}{2}\left(\nu + \frac{1}{2}\right) - \frac{1}{48}\left(\nu + \frac{1}{2}\right)^{-1}\right].$$

Outline

Remark

The quasinormal modes solution be defined by the following equations

$$\begin{pmatrix} Z_{out}^{III} \\ Z_{in}^{III} \end{pmatrix} = \begin{pmatrix} e^{i\pi\nu} & \frac{iR^2 e^{i\pi\nu}\sqrt{2\pi}}{\Gamma(\nu+1)} \\ \frac{R^{-2}\sqrt{2\pi}}{\Gamma(-\nu)} & -e^{i\pi\nu} \end{pmatrix} \begin{pmatrix} Z_{out}^{I} \\ Z_{in}^{I} \end{pmatrix},$$

and

$$\nu + \frac{1}{2} \equiv -ik^{1/2} z_0^2/2\epsilon - \epsilon\Lambda - \epsilon^2\Omega.$$

Where the QNMs boundary condition is purely outgoing wave to the spacial infinity and purely outgoing (incoming) wave to the event horizon. (Note that we follow the description of Iyer and Will (1987) work.) Therefore, if we set $Z_{in}^{III} = 0$, and ν be positive integer n, then $Z_{in}^{I} = 0$ because $\Gamma(-n) \to \infty$. Setting $\epsilon = 1$ and recalling

$$z_0^2 = 2\frac{\omega^2 - V_0}{-V_0''},$$

one can solve the QNMs from the first few formula of this presentation.

Remark

For the greybody factor, we can take $Z_{in}^{III} = 0$ also, but not consider ν as integer, and we have

$$
\begin{aligned}
Z_{out}^{III} &= e^{i\pi\nu} Z_{out}^{I} + \frac{iR^2 e^{i\pi\nu}\sqrt{2\pi}}{\Gamma(\nu+1)} Z_{in}^{I}, \\
0 &= \frac{R^{-2}\sqrt{2\pi}}{\Gamma(-\nu)} Z_{out}^{I} - e^{i\pi\nu} Z_{in}^{I}.
\end{aligned}
$$

Then we have

$$Z_{out}^{I} = \frac{e^{i\pi\nu}\Gamma(-\nu)}{R^{-2}\sqrt{2\pi}} Z_{in}^{I} \quad \Rightarrow \quad \frac{Z_{out}^{III}}{Z_{in}^{I}} = \frac{\Gamma(-\nu)}{R^{-2}\sqrt{2\pi}}.$$

Note that we use the formula $\Gamma(-\nu)\Gamma(\nu+1) = -\frac{\pi}{\sin \pi\nu}$ in the process of simplification. The greybody factor $T \equiv \left| Z_{out}^{III}/Z_{in}^{I} \right|^2$, and $\nu + \frac{1}{2}$ be purely imaginary number for real ω, one may observe from

$$\nu + \frac{1}{2} \equiv -ik^{1/2} z_0^2/2\epsilon - \epsilon\Lambda - \epsilon^2\Omega.$$

With the purely imaginary $\nu + \frac{1}{2}$, we may check $R^{\star} = e^{-i\pi(\nu+1/2)/2} R^{-1}$, and $|\Gamma(-nu)|^2 = -\frac{\pi}{\sin \pi\nu}$. Together all, we have

$$T = (1 + e^{2i\pi(\nu+1/2)})^{-1} \quad , \quad S(\omega) \equiv i\pi(\nu + \frac{1}{2}).$$

Therefore, one obtain the general formula for the greybody factor through WKB method as in the beginning of this presentation.

Thank you again for your attention!!

However…