Introduction to Modified Gravity Theories

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Motivation of alternative theories of gravity

- Motivation of alternative theories of gravity
- How to construct modified theories of gravity
- Gravity theories with extra fields beyond metric tensor
- Gravity theories with Higher Derivative of metric tensor
- Minimally modified gravity theory
- Teleparallel gravity theories

- A possible explonation for the accelerated expansion of the univverse.
- General Relativity (GR) predicts singularities under quite general conditions.
- GR cannot describe quantum nature of gravity.

see for example: T. Clifton, et al., Phys. Rept. **513**, (2012) 1-189.

How to construct modified theories of gravity

- The Weak Equivalence Principle(WEP) states that the inertial mass and gravitational mass of any object are equal.
  $\Rightarrow$ In sufficiently small regions of spacetime, the laws governing freely falling particles in a gravitational field are equivalent to those in a uniformly accelerated frame.

- Einstein Equivalence Principle (EEP): In sufficiently small regions of spacetime, the effects of gravity can be neglected, and the laws of physics reduce locally to those of special relativity.
  see for example: S. M. Carroll, [arXiv:gr-qc/9712019

How to construct modified theories of gravity

- Vectors on a curved manifold can be compared by performing the parallel transport which is defined using affine connection.

- The length of curves and vectors are measured by introducing the metric tensor. see for example: T. P. Sotiriou, [arXiv:0710.4438].

How to construct modified theories of gravity

GR is constructed under the following assumptions: [1]

- The affine connection ${\Gamma^{\lambda}}_{\mu\nu}$ is the Levi–Civita connection given by

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} \left(-\partial_{\rho} g_{\mu\nu} + \partial_{\mu} g_{\nu\rho} + \partial_{\nu} g_{\rho\mu} \right) , \tag{1}$$

  accompanying with the requirement that $\nabla_{\alpha} g_{\mu\nu} = 0$ and $\Gamma^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\nu\mu}$.
- A Riemannian geometry consists of the manifold and metric tensor.
- Gravitational interaction is mediate only by metric tensor.

[1] T. P. Sotiriou, [arXiv:0710.4438]

How to construct modified theories of gravity

- Lovelock's theorem states that, in four-dimensional Riemannian spacetime, if we construct a gravitational theory from an action involving only the metric tensor and its derivatives, the only field equations that contain derivatives of the metric up to second order are Einstein's equations, possibly with a cosmological constant [2].
- The modified theories of gravity can be constructed by the following procedures.
  - × Add extra fields beyond metric to describe gravity.
  - × Allow the field equations to be higher than second order.
  - × Work in a spacetime with a dimension different from four.
  - × Relax the requirement that the field equations be described solely by a rank-((2,0)) tensor.
  - × Relax the symmetry of the field equations under the exchange of their indices.

[2] T. Clifton, et al., Phys. Rept. **513**, (2012) 1-189

Gravity theories with extra fields

- Dirac's varying-(G) conjecture was revisited in the 1960s by Brans and Dicke, leading to the development of scalar-tensor theories of gravity [3].
- The Lagrangian for Brans-Dicke theory can be written as

$$\mathcal{L}_{\mathrm{BD}} = \sqrt{-g}\left(\frac{1}{2}\xi\phi^2 R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right) + L_{\mathrm{matter}} \ , \quad (2)$$

  where we have set $1/\sqrt{8\pi G} = 1$.
- effective gravitational constant:

$$\frac{1}{8\pi G_{\mathrm{eff}}} = \xi\phi^2 \, . \quad (3)$$

[3] T. Clifton, et al., Phys. Rept. **513**, (2012) 1-189

Gravity theories with extra fields

- The term $\xi\phi^2 R$ describes a non-minimal coupling between scalar field and curvature.
- Due to the non-minimal coupling, a scalar field becomes a part of degree of freedom for gravitational interaction.
- This theory belong to a class of scalar-tensor theoriey of gravity.

Gravity theories with extra fields

- Varying the following BD action with respect to $g^{\mu\nu}$:

$$\mathcal{S}_{\rm BD} = \int d^4x \Big\{ \sqrt{-g}\frac{1}{2}\xi\phi^2 R + \underbrace{\sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right)}_{\equiv L_\phi} + L_{\rm matter} \Big\} . \tag{4}$$

- We get

$$2\varphi G_{\mu\nu} = T_{\mu\nu}^{\rm matter} + T_{\mu\nu}^{\phi} - 2\left(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\right)\varphi \,, \tag{5}$$

where $\Box \equiv \nabla_\mu\nabla^\mu$.

- In the above equation:

$$\varphi = \frac{1}{2}\xi\phi^2 \,. \tag{6}$$

Gravity theories with extra fields

- The energy-momentum tensor of matter and scalar field are defined as

$$\frac{\delta\left(\sqrt{-g}L_{\text{matter}}\right)}{\delta g^{\mu\nu}} = -\frac{1}{2}\sqrt{-g}\,T^{\text{matter}}_{(\mu\nu)}\ , \tag{7}$$

$$\frac{\delta\left(\sqrt{-g}L_{\phi}\right)}{\delta g^{\mu\nu}} = -\frac{1}{2}\sqrt{-g}\,T^{\phi}_{(\mu\nu)}\,. \tag{8}$$

- Application of the covariant derivative to the EOM yields [4]

$$\nabla_\mu T^{\mu\nu}_{\text{matter}} = -\nabla_\mu T^{\mu\nu}_{\phi} + 2\left(\left[\nabla^\nu, \Box\right] + G^{\mu\nu}\partial_\mu\right)\varphi = 0\,, \tag{9}$$

- This covariant conservation law provides a necessary and sufficient condition to assures that the WEP holds.

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[4] Y. Fujii and K. Maeda, "The Scalar-Tensor Theory of Gravitation, Cambridge University Press (2003).

Gravity theories with extra fields

- The BD theory can be generalized to

$$S = \int d^4 \sqrt{-g} f(\phi, X) R + S_\phi + S_{\mathrm{matter}} \ , \qquad (10)$$

  where $f(\phi, X)$ is a generic function of ϕ and its kinetic term $X \equiv -\partial_\mu \phi \partial^\mu \phi / 2$.
- To build general scalar tensor theories of gravity, the ghost and Ostrogradski instabilities must be avoided [5].
- ghost: $X \propto -\dot{\phi}^2/2 \Rightarrow H = X + V < 0$
- Ostrogradski instability: The EOM contain time derivatives higher than second order.

[5] T. Clifton, et al., Phys. Rept. **513**, (2012) 1-189.

Gravity theories with extra fields

- In general, the Lagrangian $L_\phi = L_\phi(\phi, \partial_\mu\phi, \partial_\mu\partial_\nu\phi)$ leads to the EOM:

$$\frac{\partial L_\phi}{\partial \phi} - \partial_\mu \frac{\partial L_\phi}{\partial \phi_\mu} + \partial_\mu \partial_\nu \frac{\partial L_\phi}{\partial \phi_{\mu\nu}} = 0\,, \qquad (11)$$

where $\phi_\mu \equiv \partial_\mu\phi$ and $\phi_{\mu\nu} \equiv \partial_\nu\partial_\mu\phi$.

- Under suitable symmetry of the second-order derivatives of the field in the Lagrangian, EOM can contain up to second-order derivatives of the field.
- Let the Lagrangian is invariant under the transformation

$$\phi_\mu \to \phi_\mu + b_\mu\,, \qquad \phi \to \phi + b_\mu x^\mu + c\,, \qquad (12)$$

where b_μ and c are constants.

- The scalar field theory that is invariant under such transformation is the Galileons theory [6].

[6] C. Deffayet, et al., Phys. Rev. D **84**, (2011) 064039.

Gravity theories with extra fields

- Examples of the Galileons Lagrangian:

$$L_4 = \frac{1}{2}\varepsilon^{\alpha_1\alpha_2\sigma_1\sigma_2}\varepsilon^{\beta_1\beta_2}_{\sigma_1\sigma_2}\phi_\lambda\phi^\lambda\phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2} \tag{13}$$

  where ε is the totally antisymmetric Levi-Civita tensor

- In the curved spacetime, $\partial_\mu \to \nabla_\mu$.

Gravity theories with extra fields

- The Lagrangian for the Horndeski theory is the linear combination of the Lagrangians [7]

$$L_2 \equiv G_2(\phi, X)\,, \tag{14}$$
$$L_3 \equiv G_3(\phi, X)\Box\phi \tag{15}$$
$$L_4 \equiv G_4(\phi, X)R - 2\,G_{4X}(\phi, X)(\Box\phi^2 - \phi^{\mu\nu}\phi_{\mu\nu})\,, \tag{16}$$
$$\begin{aligned} L_5 \equiv & G_5(\phi, X)G_{\mu\nu}\phi^{\mu\nu} + \frac{1}{3}G_{5X}(\phi, X)(\Box\phi^3 - 3\,\Box\phi\,\phi_{\mu\nu}\phi^{\mu\nu} \\ & + 2\,\phi_{\mu\nu}\phi^{\mu\sigma}\phi^{\nu}{}_{\sigma})\,,\,, \end{aligned} \tag{17}$$

where $G_{\mu\nu}$ is the Einstein tensor, subscript $_X$ denotes derivatives with respect to the kinetic term X.

[7] J. Gleyzes, et al., JCAP **08**, (2013) 025.

Gravity theories with extra fields

- Let us consider the Lagrangian

$$L = \dot{q}_1^2/2 + \dot{q}_2 - V(q_1, q_2). \tag{18}$$

- According to the Euler-Lagrange equation:

$$\dot{P}_1 = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} = -\frac{\partial V}{\partial q_1}\,, \quad \dot{P}_2 = \underbrace{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2}}_{=0} = -\frac{\partial V}{\partial q_2}\,, \tag{19}$$

- The equation for q_2 is the constraint equation.

Gravity theories with extra fields

- The Lagrangian is degenerate if

$$\det \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} = 0\,. \tag{20}$$

- The action for degenerate higher order scalar-tensor (DHOST) theories is [8]

$$S = \int d^4x \sqrt{-g} \left\{ f(\phi, X) R + \sum_{i=1}^{5} L_i^{(2)} + \sum_{j=1}^{10} L_j^{(3)} \right\}. \tag{21}$$

- $L_i^{(2)}$ are given by

$$\begin{aligned} L_1^{(2)} &\equiv A_1 \phi_{\mu\nu}\phi^{\mu\nu}\,, \quad L_2^{(2)} \equiv A_2 (\Box\phi)^2\,, \\ L_3^{(2)} &\equiv A_3 (\Box\phi)\phi^\mu \phi_{\mu\nu}\phi^\nu\,, \quad L_4^{(2)} \equiv A_4 \phi^\mu \phi_{\mu\rho}\phi^{\rho\nu}\phi_\nu\,, \\ L_5^{(2)} &\equiv A_5 (\phi^\mu \phi_{\mu\nu}\phi^\nu)^2\,, \end{aligned} \tag{22}$$

[8] D. Langlois and K. Noui, JCAP **1602** (2016) 034

Gravity theories with extra fields

- $L_i^{(3)}$ are given by

$$
\begin{aligned}
L_1^{(3)} &\equiv B_1(\Box\phi)^3\,, \quad L_2^{(3)} \equiv B_2(\Box\phi)\,\phi_{\mu\nu}\phi^{\mu\nu}\,,\\
L_3^{(3)} &\equiv B_3\phi_{\mu\nu}\phi^{\nu\rho}\phi^{\mu}_{\rho}\,, \quad L_4^{(3)} \equiv B_4\,(\Box\phi)^2\,\phi_\mu\phi^{\mu\nu}\phi_\nu\,,\\
L_5^{(3)} &\equiv B_5\Box\phi\,\phi_\mu\phi^{\mu\nu}\phi_{\nu\rho}\phi^\rho\,, \quad L_6^{(3)} \equiv B_6\phi_{\mu\nu}\phi^{\mu\nu}\phi_\rho\phi^{\rho\sigma}\phi_\sigma\,,\\
L_7^{(3)} &\equiv B_7\phi_\mu\phi^{\mu\nu}\phi_{\nu\rho}\phi^{\rho\sigma}\phi_\sigma\,, \quad L_8^{(3)} \equiv B_8\phi_\mu\phi^{\mu\nu}\phi_{\nu\rho}\phi^\rho\,\phi_\sigma\phi^{\sigma\lambda}\phi_\lambda\,,\\
L_9^{(3)} &\equiv B_9\Box\phi\,(\phi_\mu\phi^{\mu\nu}\phi_\nu)^2\,, \quad L_{10}^{(3)} \equiv B_{10}\,(\phi_\mu\phi^{\mu\nu}\phi_\nu)^3\,,
\end{aligned}
$$

Gravity theories with Higher Derivative

- Other possibility to extend GR is to allow the field equations to be higher than second order.
- This class of theories may encounter the Ostrogradski and ghost instabilities [9].
- Let us consider the $f(R)$ gravity theory.

[9] T. Clifton, et al., Phys. Rept. **513**, (2012) 1-189.

Gravity theories with Higher Derivative

- The Lagrangian for the $f(R)$ gravity is [10]

$$\mathcal{L} = \sqrt{-g} f(R) + L_{\text{matter}} \ , \tag{23}$$

- Variation of the action for $f(R)$ gravity with respect to $g^{\mu\nu}$ yields

$$f_R R_{\mu\nu} - \frac{1}{2} f g_{\mu\nu} - f_{R;\mu\nu} + g_{\mu\nu} \Box f_R = T^{\text{matter}}_{\mu\nu} \ , \tag{24}$$

where $_{;\mu} \equiv \nabla_\mu$, and $f_R \equiv df/dR$.

- The $f(R)$ gravity is free from Ostrogradski and ghost instabilities.

[10] T. Clifton, et al., Phys. Rept. **513**, (2012) 1-189.

Gravity theories with Higher Derivative

- Let us write the action in the equivalent form using field χ as
$$\mathcal{L} = \sqrt{-g}\,[f(\chi) + f_R(\chi)(R-\chi)]\,, \tag{25}$$
- Variation with respect to χ gives
$$f_{RR}(\chi)(R-\chi) = 0\,, \quad \Rightarrow \quad R = \chi \text{ if } f_{RR} \neq 0\,. \tag{26}$$
- defining
$$\phi \equiv f_R(\chi)\,. \tag{27}$$

Gravity theories with Higher Derivative

- In terms of the new field ϕ:

$$\mathcal{L} = \sqrt{-g}\,[\phi R - 2\Lambda(\phi)]\ . \tag{28}$$

- The potential term is defined as

$$\Lambda(\phi) \equiv \frac{1}{2}\,[\chi(\phi)\phi - f(\chi(\phi))]\ . \tag{29}$$

Minimally modified gravity theory

- How to construct modified theories of gravity that have two propagating degrees of freedom for gravity?
- The ADM metric:

$$ds^2 = -N^2 dt^2 + h_{ij}(N^i dt + dx^i)(N^j dt + dx^j)\,, \qquad (30)$$

  where N, N^i, and h_{ij} are the lapse function, shift vector, and induced metric on the three-dimensional hypersurfaces.
- The metric h_{ij} is the projection of the metric $g_{\mu\nu}$ on three-dimensional hypersurfaces:

$$h_{\alpha\beta} = \Pi^{\mu}_{\alpha}\Pi^{\nu}\beta g_{\mu\nu}\,, \quad \Pi^{\mu}_{\alpha} \equiv \delta^{\mu}_{\alpha} + n^{\mu}n_{\alpha}\,. \qquad (31)$$

- In terms of the ADM metric:

$$\mathcal{L}_E = \sqrt{-g}R = N\sqrt{h}L(N, N^i, \dot{h}_{ij}, h_{ij}, D_i)\,. \qquad (32)$$

Minimally modified gravity theory

- From the definition of the conjugate momentum:

$$\pi_N = \frac{\delta}{\delta \dot{N}} \int d^3x \mathcal{L}_E = 0\,, \quad \pi^i = \frac{\delta}{\delta \dot{N}_i} \int d^3x \mathcal{L}_E = 0\,,$$
$$\pi^{ij} = \frac{\delta}{\delta \dot{h}_{ij}} \int d^3x \mathcal{L}_E\,. \tag{33}$$

- The total Hamiltonian can be written as

$$H_T = \int d^3x \Big(N H_0(h_{ij}, \pi^{ij}, D_i) + N^i H_i(h_{ij}, \pi^{ij}, D_i) + \lambda_N \pi_N + \lambda_i \pi^i \Big)\,, \tag{34}$$

where H_0 and H_i are Hamiltonian and momentum constraints, while λ and λ_i are Lagrange multipliers.

Minimally modified gravity theory

- The constancy of the constraints π_N and π^i yields

$$\dot{\pi}_N = \{\pi_N, H_T\} = \frac{\partial \pi_N}{\partial N}\frac{\partial H_0}{\partial \pi_N} - \frac{\partial \pi_N}{\partial \pi_N}\frac{\partial H_0}{\partial \pi_N} = -H_0 \approx 0\,, \quad (35)$$

$$\dot{\pi}^i = \{\pi^i, H_T\} = \frac{\partial \pi^i}{\partial N^i}\frac{\partial H_0}{\partial \pi^i_N} - \frac{\partial \pi^i}{\partial \pi^i}\frac{\partial H_0}{\partial \pi^i} = -H_i \approx 0 \quad (36)$$

  where ≈ 0 denotes vanishes on the constraint hypersurface.
- All constraints π_N, π^i, H_0 and H_i are first class [11].
- There are 10 - 1 - 3 - 1 - 3 = 2 degrees of freedom for gravity in GR.

[11] J. Gleyzes, et al., JCAP **08**, (2013) 025.

Minimally modified gravity theory

- For scalar-tensor theories H_0 also depends on N [12].

$$\dot{\pi}_N = -\frac{\partial}{\partial N}(NH_0) \approx 0\,. \tag{37}$$

- Hence, $H_0 \to \tilde{H}_0 \equiv H_0 + N\partial H_0/\partial N$
- Nevertheless

$$\{\pi_N, \tilde{H}_0\} = -\frac{\partial \tilde{H}_0}{\partial N} = -2\frac{\partial H_0}{\partial N} - \frac{\partial^2 H_0}{\partial N^2} \neq 0\,. \tag{38}$$

- The π_N and $\tilde{H}_0$ are second class.
- There are 10 - 1/2 - 3 - 1/2 - 3 = 3 degrees of freedom for gravity in scalar-tensor theories.
- In Minimally Modified Gravity (MMG) theory, the total Hamiltonian must be linear function of N [13]

[12] J. Gleyzes, et al., JCAP **08**, (2013) 025.

[13] S. Mukohyama and K. Noui, JCAP **07** (2019) 049.

Minimally modified gravity theory

- The Hamiltonian for MMG theory can be written as

$$H_T = \int d^3x \big(N\mathcal{H}_0(h_{ij}, \pi^{ij}, D_i) + N^i H_i(h_{ij}, \pi^{ij}, D_i) \\ + \lambda_N \pi_N + \lambda_i \pi^i \big) \, , \qquad (39)$$

- The constraint $\mathcal{H}_0$ shouldbe be first class and not a Hamiltonian constraint in GR.
- Example is [14]

$$\mathcal{H}_0 = f(H_0) \, , \qquad (40)$$

  where $f(H_0)$ is generic function of the Hamiltonian constraint H_0.
- The constraint $\mathcal{H}_0$ becomes second class when matter field is introduced in the theory.

[14] K. Aoki, et al., Eur.Phys.J.C **80** (2020) 708.

Minimally modified gravity theory

- To solve the matter coupling problem, the gauge fixing terms must be introduced such that this gauge term and $\mathcal{H}_0$ are second class in vacuum. [15].
- Example is [16]

$$\int d^3x\sqrt{h}\mu^i D_i \frac{\pi^i_i}{\sqrt{h}}\,. \tag{41}$$

[15] K. Aoki, et al., Phys. Rev. D **98** (2018) 044022.
[16] K. Aoki, et al., Eur.Phys.J.C **80** (2020) 708.

Teleparallel gravity theories

- In this class of modified gravity theories, metric tensor and affine connection are treted as independent dynamical variables – a metric-affine geometry.
- The assumptions $\nabla_\alpha g_{\mu\nu} = 0$ and $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$ are relaxed.
- Non-metricity tensor: $Q_{\alpha\mu\nu} \equiv \nabla_\alpha g_{\mu\nu}$
  Torsion tensor: $T^\alpha_{\mu\nu} \equiv \Gamma^\alpha_{\mu\nu} - \Gamma^\alpha_{\nu\mu}$
- The commond assumption for this class of theories is the vanishing of the Riemann tensor: [17]

$$R^\alpha_{\mu\nu\rho} = \partial_\nu \Gamma^\alpha_{\rho\mu} - \partial_\rho \Gamma^\alpha_{\nu\mu} + \Gamma^\alpha_{\nu\lambda}\Gamma^\lambda_{\rho\mu} - \Gamma^\alpha_{\rho\lambda}\Gamma^\lambda_{\nu\mu} = 0\,. \qquad (42)$$

[17] L. Heisenberg, [arXiv:2309.15958]

Teleparallel gravity theories

- Teleparallel Equivalent of General Relativity:

$$R^{\alpha}_{\mu\nu\rho} = 0\,, \quad Q_{\alpha\mu\nu} = 0\,, \quad T^{\alpha}_{\mu\nu} \neq 0\,. \tag{43}$$

- Using the relation

$$R = 0 = R(g) + T + \text{divergent term}\,. \tag{44}$$

- Here,

$$T \equiv -\frac{1}{4} T_{\alpha\mu\nu} T^{\alpha\mu\nu} - \frac{1}{2} T_{\alpha\mu\nu} T^{\mu\alpha\nu} + T_{\alpha} T^{\alpha}\,, \tag{45}$$

where $T_{\mu} = T^{\alpha}{}_{\mu\alpha}$.

- The action is

$$S = -\frac{1}{2} \int d^4x \sqrt{-g}\, T\,. \tag{46}$$

Teleparallel gravity theories

- Symmetric Teleparallel Equivalent of GR:

$$R^{\alpha}_{\mu\nu\rho} = 0\,, \quad T^{\alpha}_{\mu\nu} = 0\,, \quad Q_{\alpha\mu\nu} \neq 0\,. \tag{47}$$

- Using the relation

$$R = 0 = R(g) + Q + \text{divergent term}\,. \tag{48}$$

- Here,

$$Q = \frac{1}{4}Q_{\alpha\beta\gamma}Q^{\alpha\beta\gamma} - \frac{1}{2}Q_{\alpha\beta\gamma}Q^{\beta\alpha\gamma} - \frac{1}{4}Q_{\alpha}Q^{\alpha} + \frac{1}{2}Q_{\alpha}\tilde{Q}^{\alpha}\,, \tag{49}$$

where $Q_\alpha = Q_{\alpha\lambda}{}^{\lambda}$ and $\tilde{Q}_\alpha = Q^{\lambda}{}_{\lambda\alpha}$.

Teleparallel gravity theories

- The action is

$$S = -\frac{1}{2}\int d^4x\sqrt{-g}Q\,. \tag{50}$$