Cosmological Perturbation Theory

From a Smooth Background Space to the Origin of Cosmic Structure

A perfectly smooth universe based on the Cosmological Principle cannot explain the formation of stars and galaxies.

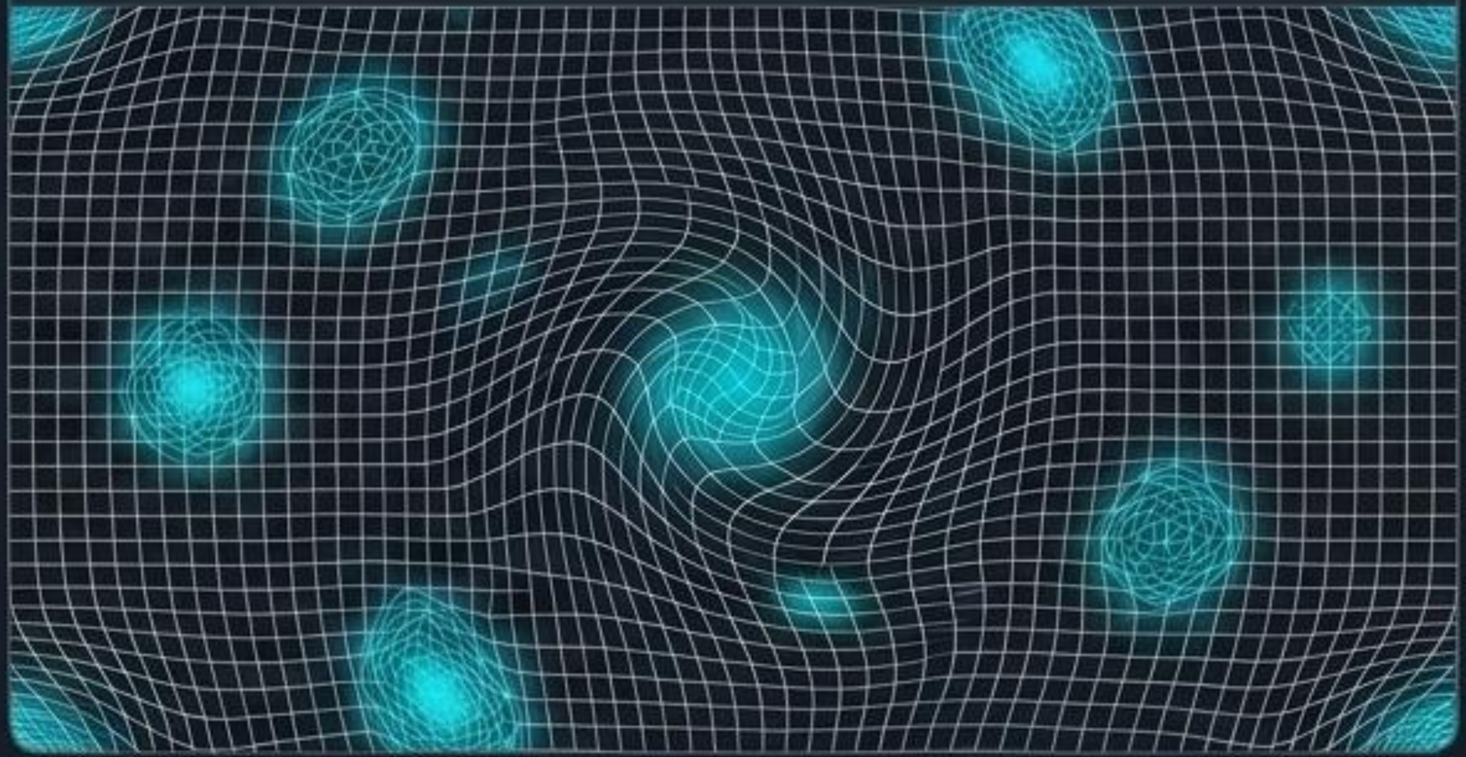
Key Insight: Understanding tiny fluctuations through a linear mathematical framework.

Obstacle 1: The Universe is Not Perfectly Smooth

FLRW Hypothesis

If considered on a large scale, the universe is homogeneous and isotropic according to the FLRW metric

Observed Universe (CMB Anisotropies ~ 10^{-5})



Solution: Linear Perturbation Framework

$$g_{\mu\nu} = \underbrace{\bar{g}_{\mu\nu}}_{\text{Smooth Background (Background)}} + \underbrace{h_{\mu\nu}}_{\text{Tiny Perturbation (Tiny Perturbation)}}$$

Linearized Equations: Linking Geometry and Matter

Spacetime Perturbations

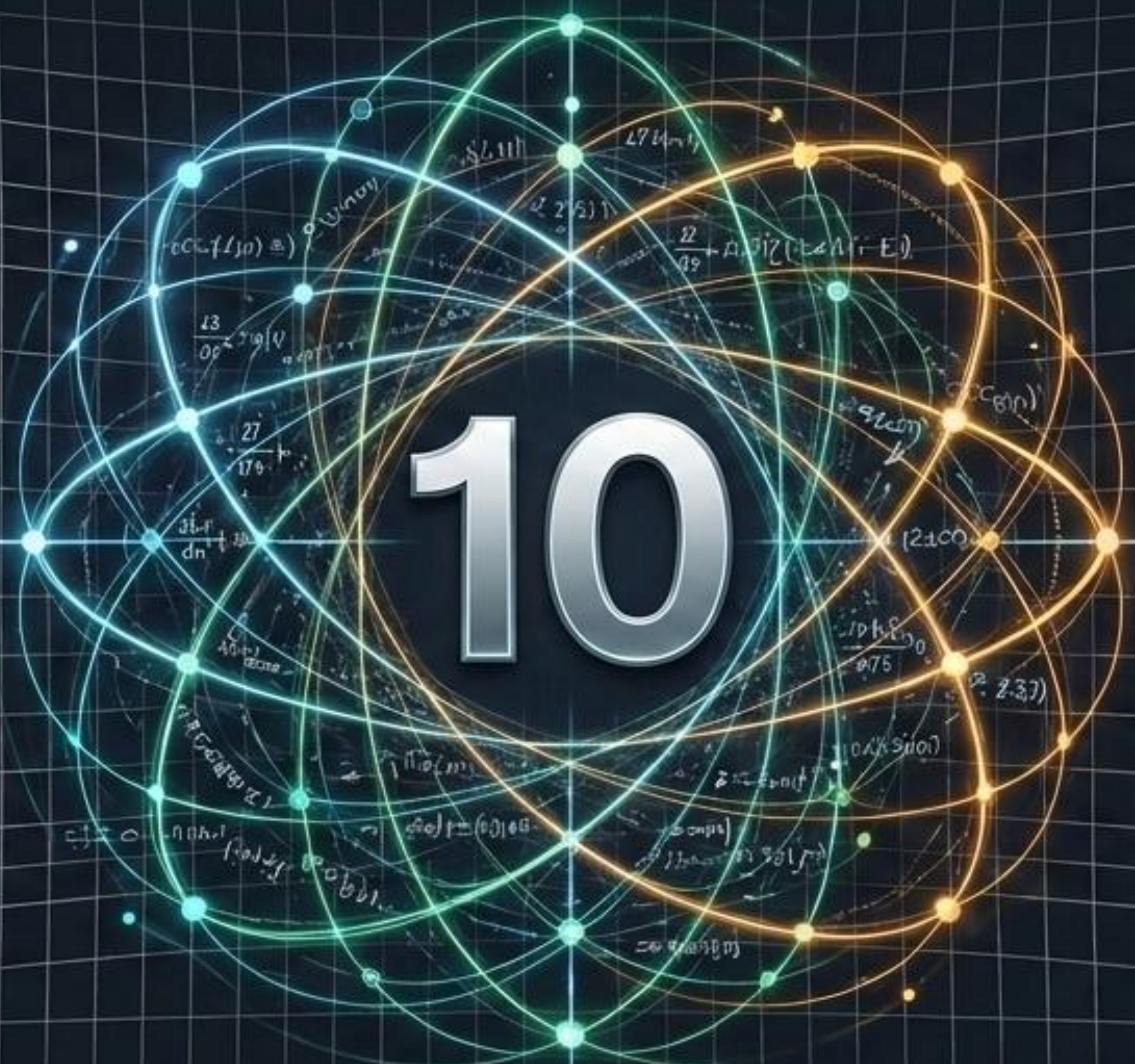
Matter Perturbations

Linear Approximation

- Since the amplitude of perturbations is very small in the early epoch, we consider only first-order terms.
- The scale factor function $a(\tau)$ from the FLRW metric remains the main variable controlling expansion.

Obstacle 2: The Complexity of Coupled Equations

Perturbation Matrix $h_{\mu\nu}$ is a Symmetric Tensor which consists of 10 Degrees of Freedom (DOF)



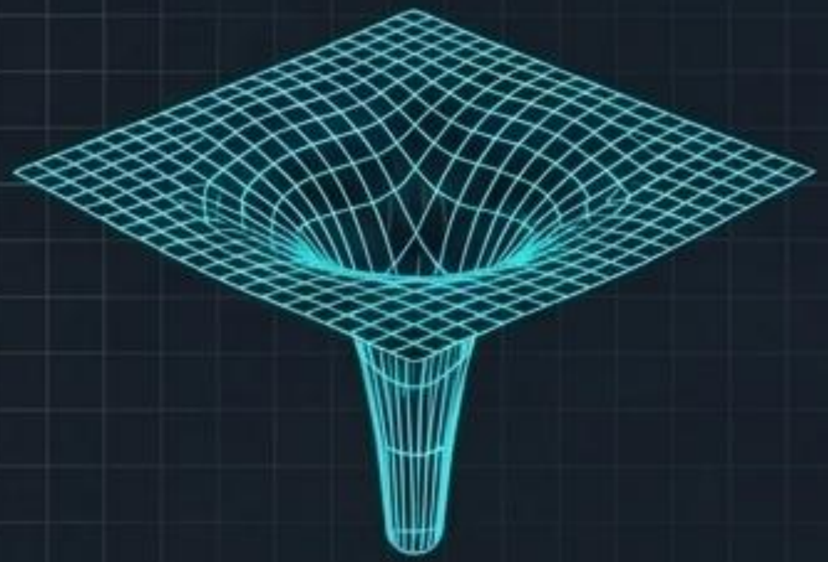
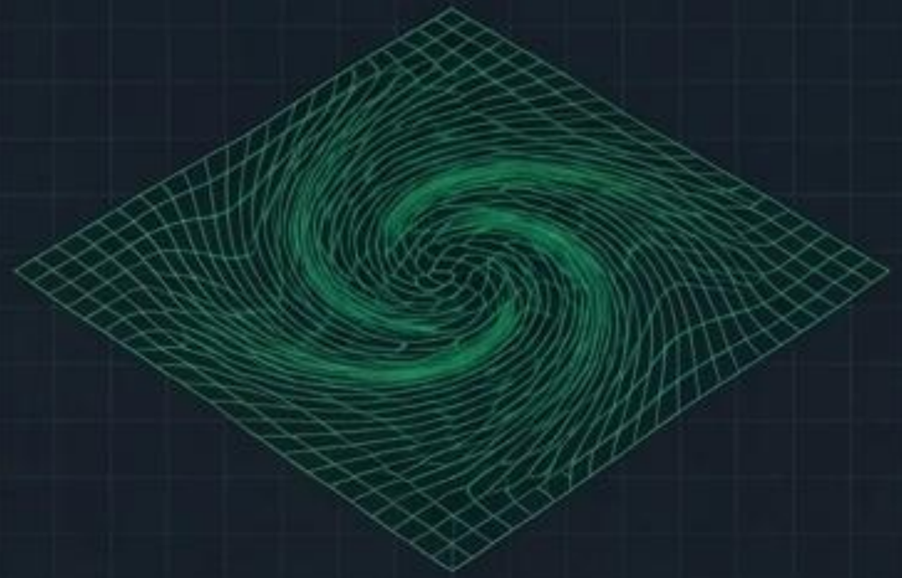
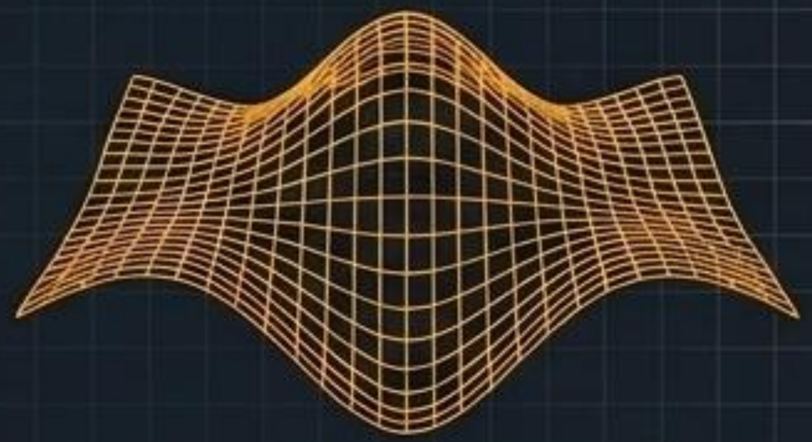
Problem: All 10 variables are entangled leading to a system of complicated partial differential and fully coupled equations (Fully Coupled PDEs)

We need mathematical tools to decouple these variables.

Solution: SVT Decomposition

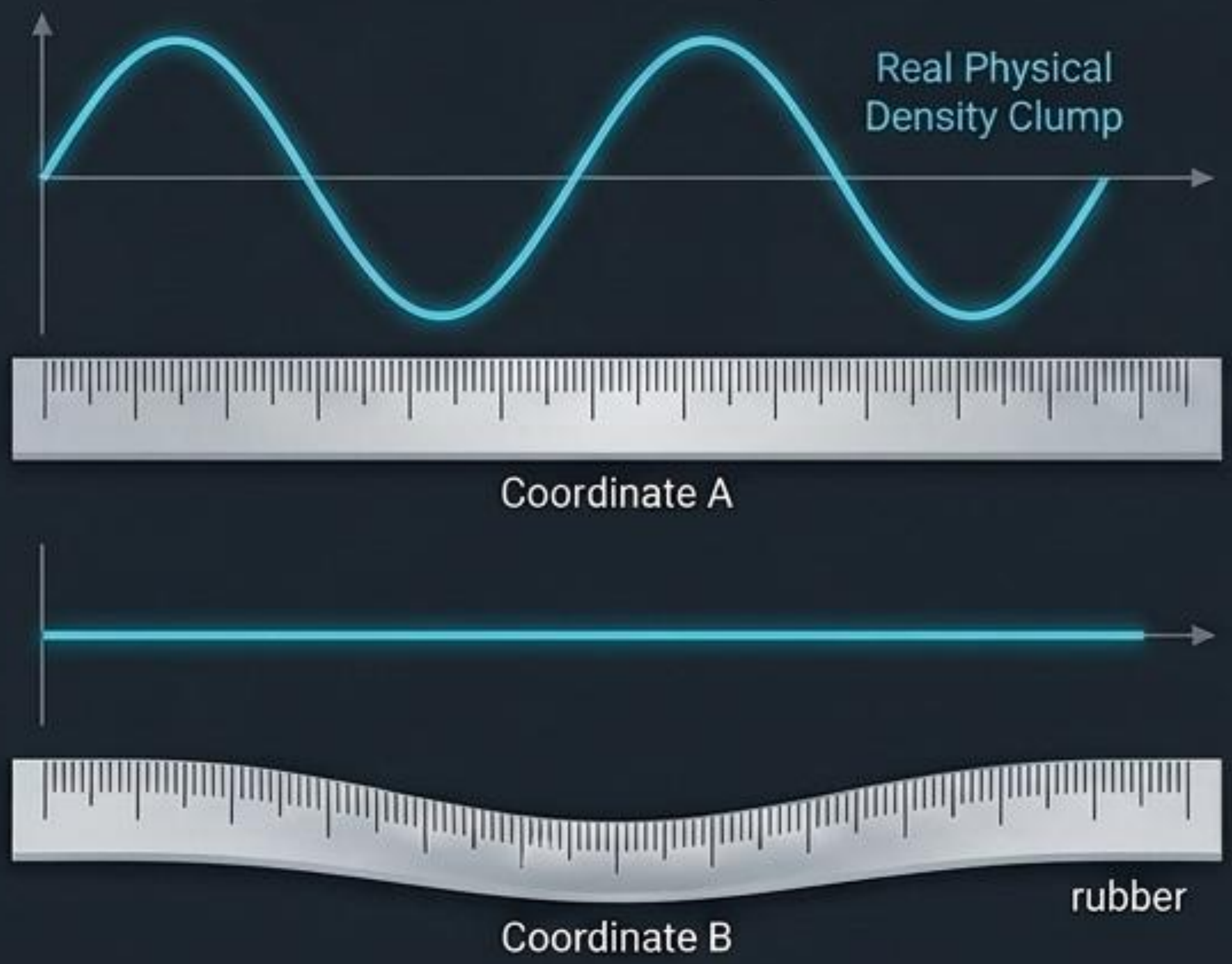
In linear theory, all three mode types are decoupled.

SVT Diagnostic Matrix

Scalar Modes (4 DOFs)	Vector Modes (4 DOFs → 2 Physical)	Tensor Modes (2 DOFs)
		
Meaning: Density and Clumping	**Meaning:** Fluid Vorticity	**Meaning:** Gravitational Waves
Result: Main driver for forming galaxies and large-scale structure	**Result:** Decays with the expansion of the universe	**Result:** Travels freely and leaves imprints on structure

Obstacle 3: The Gauge Problem

The Ruler Metaphor



Squished marks counteract the sine wave, resulting in an apparent flat line.

Coordinate transformations can create fake physical results.

$$x^\mu \to \tilde{x}^\mu = x^\mu + \xi^\mu$$

Example: Density change from coordinate transformation

$$\tilde{\delta\rho} = \delta\rho - \bar{\rho}'\xi^0$$

Problem: Is what we observe a real matter density perturbation, or just a distortion of the observer's coordinates? Leading to Fictitious Modes lurking within the 10 DOFs.

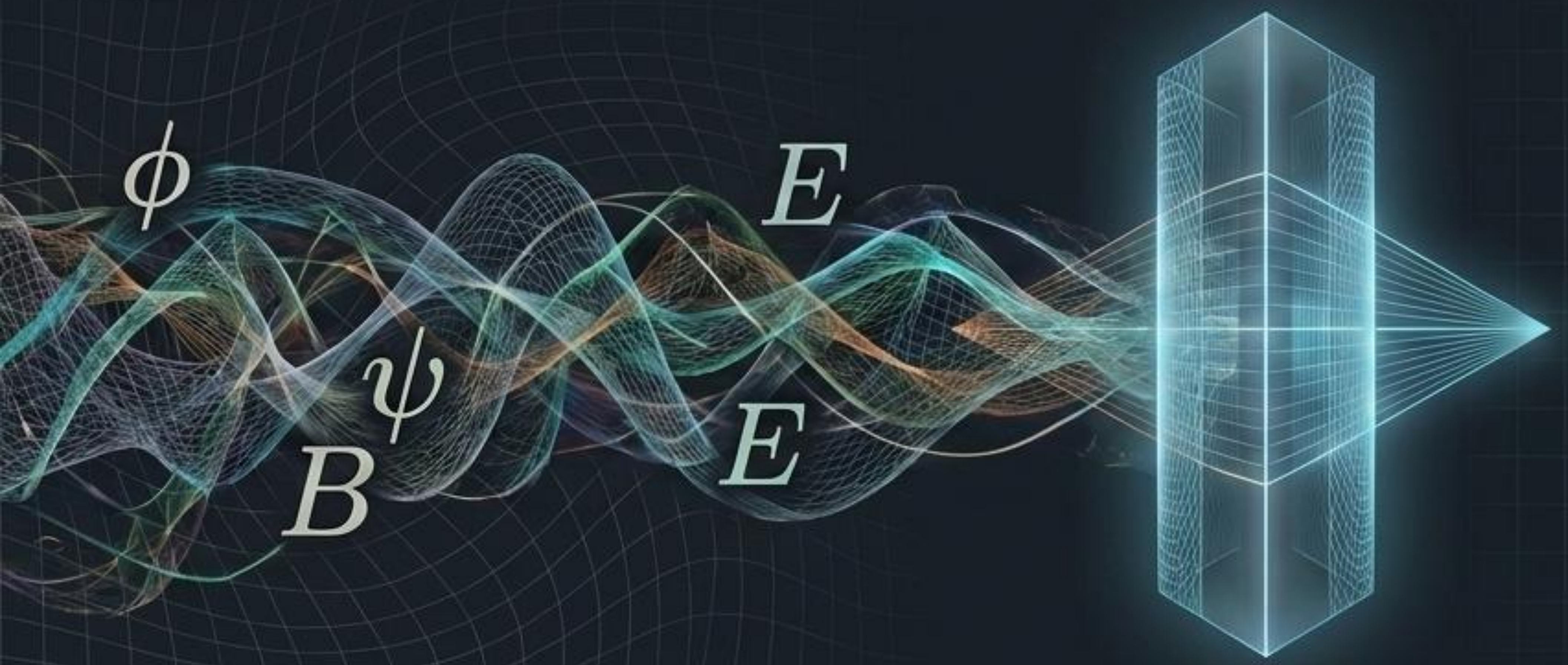
Solution 3A: Gauge Invariants

Constructing observer-independent variables to find physical truth

Noisy Raw Data

Coordinate Independence Filter

Bardeen potentials



$$\Phi = \phi - H\sigma - \sigma'$$

$$\Psi = \psi + H\sigma$$

(Where $\sigma = E' - B$ is the Shear of Spacetime)

These variables represent true gravity, used to link directly to observable values without worrying about coordinate choice.

Solution 3B: Gauge-Fixing Toolkit

Instead of using Gauge Invariants, we can 'gauge-fix' to directly eliminate Fictitious DOFs.

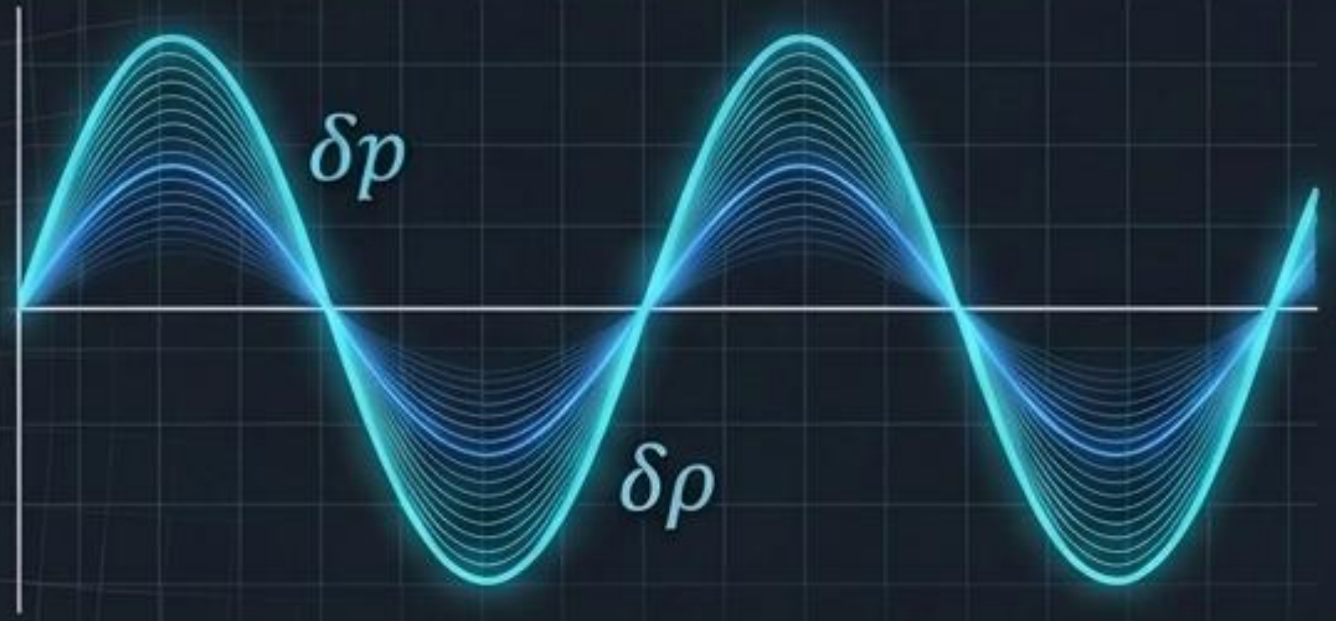
The Gauge Selection Matrix

Gauge Name (Gauge)	Conditions set to zero	Appropriate Usage
Conformal Newtonian	$\sigma = 0, E = 0$	Analyze small-scale physics (similar to Newtonian potentials Φ, Ψ)
Synchronous	$\phi = 0, B = 0$	Often used with Dark Matter particles (Dark Matter rest frame)
Comoving Orthogonal	$\tilde{v} = 0, \tilde{v} + B = 0$	Particle Inflation calculations (Inflation)
Uniform Density	$\tilde{\delta}\rho = 0$	Consider perturbations relative to constant density surfaces

Fluid Dynamics: Adiabatic vs. Entropy Perturbations

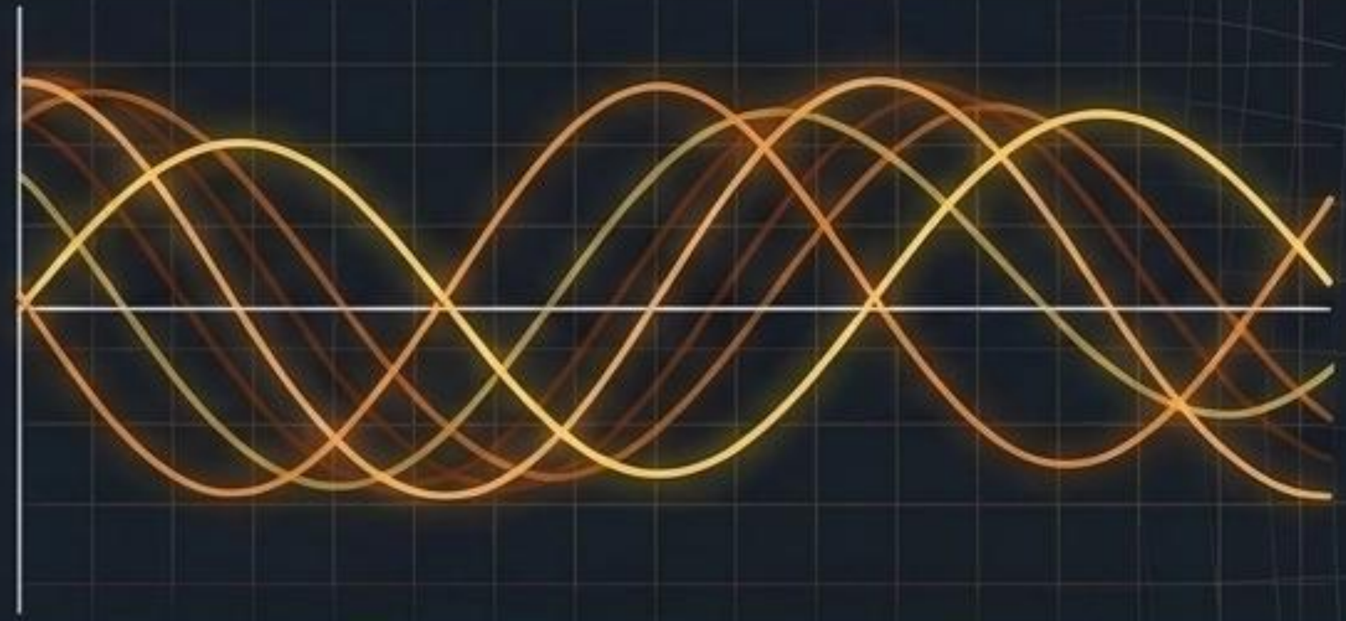
Equation of State: $\delta p = c_a^2 \delta \rho + \delta p_{non}$

Adiabatic Perturbations



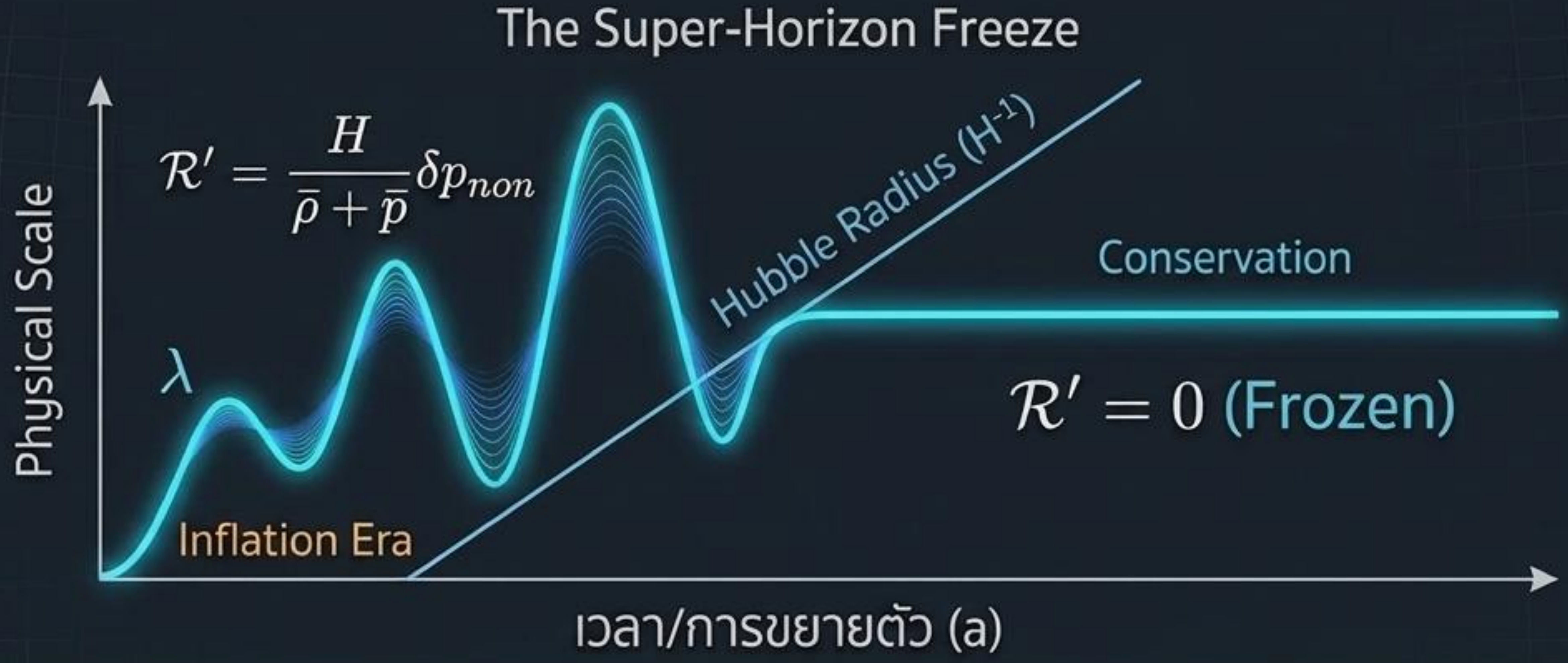
- Pressure variations depend solely on density ($\delta p_{non} = 0$)
- All types of perturbations in the universe move together

Entropy (Isocurvature) Perturbations



- There are latent pressure fluctuations ($\delta p_{non} \neq 0$)
- Caused by interactions between multiple species (Multi-fluid systems)

บทสรุปสูงสุด: การอนุรักษ์ความโค้งในสเกลใหญ่



- เป้าหมายทั้งหมดของการแยก SVT และการตรึงเกจ คือการค้นพบตัวแปรที่ถูกอนุรักษ์ (Conserved Quantity)
- The 'Aha!' Moment: หากพิจารณาการรบกวนแบบ Adiabatic นอกขอบเขต Hubble Radius ค่า $\mathcal{R}$ จะถูกแช่แข็งและไม่เปลี่ยนแปลง ทำให้เราสามารถโยงความผันผวนควอนตัมในยุคเงินเฟ้อ เข้ากับโครงสร้าง CMB ในปัจจุบันได้โดยตรง!

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{2}R + P(\phi, X)\right) \qquad X \equiv -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$$

$$h_{ij} = a^2(1+2\zeta)\delta_{ij} = a^2 e^{2\zeta}, \qquad N = 1+\alpha, \qquad N_i = \partial_i\beta$$

$$S^{(2)} = \frac{1}{2}\int dt d^3x \left(2a^3\frac{X\rho_X}{H^2}\dot{\zeta}^2 - 2a\frac{XP_{,X}}{H^2}(\partial_i\zeta)^2\right)$$

No Ghost condition $X\rho_X > 0$

Laplacian condition $XP_{,X} > 0$