Review of cosmology

8^{th} International School on High-Energy and Astro-Physics (SHEAP2026): Modified gravity

Pitayuth Wongjun

The Institute for Fundamental Study,
Naresuan University, Phitsanulok 65000, Thailand

August 13, 2026

Contents

Chapter 1

FLRW metric

We have learned so far the geometry of the simple astronomical objects by using General Relativity. It is found that some distinguished observational predictions of General Relativity are compatible with observational data. This makes General Relativity become the one of fundamental theory of nature. It is worthwhile to test General Relativity with cosmological objects. This can be evaluated by considering geometry of the universe. The basic concept for applying General Relativity to describe the evolution of the universe is main content of this chapter.

1.1 Cosmological principle

Due to the nonlinearity of the Einstein field equation, it is not easy to obtain general solutions of this equation. So far, we have learned that it is possible to extract some solutions from this equation by imposing reasonable symmetry to the theory. For example, the Schwarzschild solution can be obtained by solving the Einstein field equation with static and spherical symmetry. In cosmology, the symmetries of spacetime are provided by the cosmological principle. The cosmological principle is a principle based on the assumption that "the universe is isotropic and homogeneous in three-space". This principle provides a smooth or uniform universe at large scale. It is important to note that this principle is compatible with observational data, specifically Cosmic Microwave Background (CMB) radiation data. Therefore, this assumption (cosmological principle) provide a reasonable symmetries to study geometry and dynamics of the universe by using General Relativity.

Since the cosmological principle provides us the symmetry only in three-dimensional space. Therefore, one can sprite the spacetime into product space as follows

$$\underbrace{\mathbb{R}}_{\text{real}} \times \underbrace{\Sigma}_{\text{3-space}} \,, \tag{1.1}$$

where $\mathbb{R}$ represents (real) time and Σ represents a homogeneous and isotropic 3-space. Therefore, the spacetime can be treated as a series of non-interacting spacelike hypersurface labeling by universal time parameter as shown in figure 1.1. The the 3-dimensional hypersurface which is homogeneous and isotropic will be obeyed translational and rotational invariants respectively. This corresponds to maximally symmetric space. This kind of space will provide us the simple way to find the general form of the metric. We will discuss the maximally symmetric space in next subsection in detail.

1.2 Maximally symmetric space

By definition, the maximally symmetric space is a space which has number of degrees of freedom of the metric is equal to number of isometry of the metric. Note that isometry is a symmetry of the metric which is not necessary to be a physical symmetry. To understand clearly, let us consider n-dimensional flat Euclidean space, $\mathbb{R}^n$. This space will be described by a range-2 ($n \times n$) symmetric metric $g_{\mu\nu}$. The number of degrees of freedom of this metric is

$$n(n+1)/2. \tag{1.2}$$

Considering the space with has rotational and translational invariants, we have

$$\left.\begin{array}{lll} \text{translation} & \Rightarrow n \text{ degrees of freedom} \\ \text{rotation} & \Rightarrow \dfrac{n}{2}(n-1) \;\text{ degrees of freedom} \end{array}\right\} n + \frac{n}{2}(n-1) = \frac{n}{2}(n+1)\,. \tag{1.3}$$

We can see that the number of degree s of freedom of the metric is equal to the number of the isometry of the metric. This corresponds to the maximally symmetric space. Note that, in n dimensions, we can rotate x axis to one of the other $n-1$ axes and the next axis can be rotated to other $n-2$ axes, which means all of the axes can be rotated

$$(n-1) + (n-2) + (n-3) + \ldots + 1 = \frac{n}{2}(n-1) \quad \text{ways.} \tag{1.4}$$

It makes our sense that the more number of isometry, the fewer functions needed to specify the properties of the spacetime. One of important consequence of the maximally symmetric space is that there is only one number needed to specify. This number is independent of the coordinates and corresponds to the curvature K. Maximally symmetric space will provide the fact that $R_{\rho\sigma\mu\nu}$ is the same everywhere. Since the curvature tensor depends on the metric, by using the symmetry of the indices, one has

$$R_{\rho\sigma\mu\nu} \propto g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu} = K\left(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}\right). \tag{1.5}$$

Consequently, the Ricci tensor and Ricci scalar can be written as

$$\begin{aligned} R_{\sigma\nu} &= g^{\rho\mu}R_{\rho\sigma\mu\nu} = g^{\rho\mu}K\left(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}\right), \\ &= K\left(\delta^{\mu}_{\mu}g_{\sigma\nu} - \delta^{\mu}_{\nu}g_{\sigma\mu}\right) = K(n-1)\,g_{\sigma\nu}, \qquad (1.6) \\ R &= g^{\sigma\nu}R_{\sigma\nu} = g^{\sigma\nu}K(n-1)\,g_{\sigma\nu} = K(n-1)\,\delta^{\nu}_{\nu} = Kn(n-1)\,. \qquad (1.7) \end{aligned}$$

Finally, one find that the number corresponding to the space curvature can be written in terms of Ricci scalar as follows

$$K = \frac{R}{n(n-1)}. \tag{1.8}$$

This is a main result follows from the maximally symmetric space. We will use this result in next subsection in order to find the general form of the metric describing the dynamics of the universe.

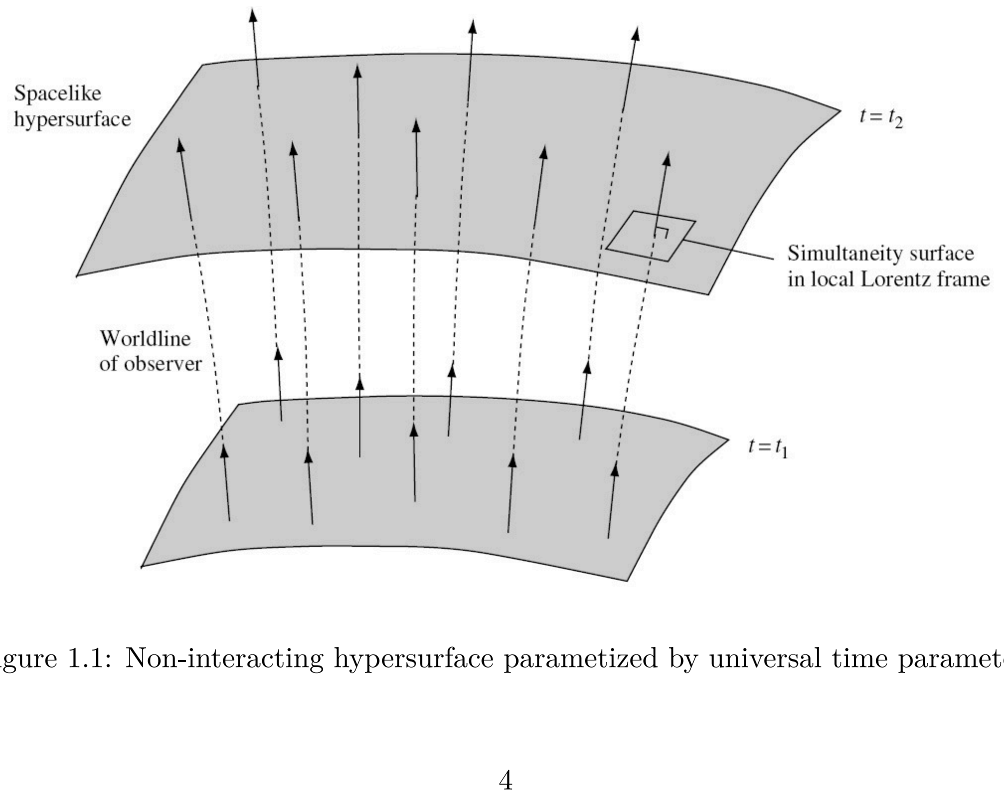


Figure 1.1: Non-interacting hypersurface parametized by universal time parameter.

1.3 Friedmann-Lemaitre-Robertson-Walker metric

Considering the homogeneous and isotropic hypersurface, the general form of the line element can be written as

$$d\sigma^2 = g_{ij}dx^i dx^j. \tag{1.9}$$

In order to illustrate the form of the metric, one may consider a triangle of three points in this space. For isotropy, the triangle must be the same for all time. From this argument, one finds that each hypersurface will be different only by an overall factor $d\sigma_2^2 = S^2(x^k,t)d\sigma_1^2 = S^2(x^k,t)g_{ij}dx^i dx^j$. For homogeneity, it will be satisfied only if the overall factor is independent of the coordinates x^k. Then the most general form of the homogeneous and isotropic hypersurface can be written as

$$d\sigma^2 = S^2(t)\gamma_{ij}dx^i dx^j, \tag{1.10}$$

where h_{ij} is a component of the metric in the original hypersurface depending only on spatial coordinates x^k.The coordinates x^k are the comoving coordinates, while the observers in these coordinates will not see any dynamics of the objects, called comoving observers. Therefore, the most general form of the metric in this spacetime can be written as

$$ds^2 = -dt^2 + S^2(t)\gamma_{ij}dx^i dx^j, \tag{1.11}$$

where the term proportional to $dtdx^i$ disappears due to the requirement of a non-interacting hypersurface. Now we are in the position to find the form of the metric h_{ij}. Let us consider the subspace defined as follows

$$d\sigma^2 = \gamma_{ij}dx^i dx^j = e^{2B}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{1.12}$$

where we have used the coordinate transformation to eliminate the other function. Note that for isotropy one finds that there are only two independent functions to specify the metric. However, from the general coordinate invariance, we can use a coordinate transformation to eliminate one of them. By using the properties of the maximally symmetric space as discussed in the previous subsection, one has

$$\begin{aligned} R_{\mu\nu} &= K(n-1)g_{\mu\nu} = -2Kg_{\mu\nu}, \\ R_{rr} &= \frac{2}{r}B' = 2Ke^{2B}, \qquad (1.13) \\ R_{\theta\theta} &= -e^{-2B}\left(1 - rB'\right) - 1 = 2Kr^2. \qquad (1.14) \end{aligned}$$

We leave detail calculation for Γ^i_{jk} and R_{ij} in the Exercise. From the Eq. (1.13) one has

$$\begin{aligned} B' &= Kre^{2B}, \\ \int e^{-2B}dB &= K\int rdr, \\ e^{-2B} &= -Kr^2 + C, \end{aligned} \tag{1.15}$$

where C is an integration constant. Substitute the Eq. (1.15) into the Eq. (1.14) we have

$$\begin{aligned} e^{-2B}\left(1 - rB'\right) - 1 &= -2Kr^2, \\ e^{-2B}\left(1 - r^2Ke^{2B}\right) - 1 &= -2Kr^2, \\ e^{-2B} - r^2K - 1 &= -2Kr^2, \\ -\cancel{Kr^2} + C - \cancel{r^2K} - 1 &= -\cancel{2Kr^2}, \\ C &= 1. \end{aligned} \tag{1.16}$$

Thus, one has

$$e^{-2B} = 1 - Kr^2, \tag{1.17}$$

and the line element in Eq. (1.11) will then be

$$ds^2 = -dt^2 + S^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]. \tag{1.18}$$

To classify the geometry of this spacetime, one can consider the value of the curvature K. One finds that the geometry of the spacetime can be distinguished by the sign of the curvature $K > 0$, $K < 0$, and $K = 0$. To see explicitly, one may redefine the curvature in terms of the dimensionless one as follows

$$k = \frac{K}{|K|}, \qquad \bar{r} = |K|^{\frac{1}{2}} r, \qquad R^2(t) = \frac{S^2(t)}{|K|}. \tag{1.19}$$

Therefore, the line element in Eq. (1.18) can be expressed as

$$ds^2 = -dt^2 + R^2(t)\left[\frac{d\bar{r}^2}{1-k\bar{r}^2} + \bar{r}^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right], \tag{1.20}$$

where the curvature of the maximally symmetric space can be classified very simply as

$$\begin{aligned} k &= -1 \quad \rightarrow \quad \text{open}, \\ k &= 0 \quad \rightarrow \quad \text{flat}, \\ k &= 1 \quad \rightarrow \quad \text{closed}. \end{aligned} \tag{1.21}$$

It is important to note that the metric explicitly obeys the rotational invariance. This is also by construction of the metric we have used. For the translational invariance, it is a hidden symmetry. One cannot see it explicitly from the metric. However, we can see that one can choose the origin of the radial coordinate completely arbitrarily. Therefore, it means that the metric contains a translational invariance.

In order to obtain the geometric properties of FLRW metric, let us consider the line element of the three-dimensional hypersurface as follows

$$d\sigma^2 = \frac{d\bar{r}^2}{1-k\bar{r}^2} + \bar{r}^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{1.22}$$

By using the coordinate transformation such that

$$d\chi = \frac{d\bar{r}}{\sqrt{1-k\bar{r}^2}}, \tag{1.23}$$

the line element in Eq. (1.22) can be written as

$$d\sigma^2 = d\chi^2 + \underline{S}^2(\chi)\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{1.24}$$

where $\underline{S}$ can be classified according to different kinds of curvature as

$$\underline{S}^2(\chi) = \sinh^2\chi \qquad \text{for } k = -1, \tag{1.25}$$

$$\underline{S}^2(\chi) = \chi^2 \qquad \text{for } k = 0, \tag{1.26}$$

$$\underline{S}^2(\chi) = \sin^2\chi \qquad \text{for } k = 1. \tag{1.27}$$

By embedding these three-dimensional surfaces into four-dimensional Euclidean space, one finds that for $k = 1$ the surface is the three-dimensional sphere, for $k = 0$ the surface is flat, and for $k = -1$ the surface is a hyperboloid. This is the reason why we call closed, flat, and open geometry for $k = 1$, $k = 0$, and $k = -1$, respectively. We leave the explicit calculations for this embedding procedure in the exercise.

From Eq. (1.20), the radial coordinate $\bar{r}$ is dimensionless, and the scale factor $\underline{S}$ has the dimension of length. It is convenient to consider the scale factor dimensionless since it determines the how much the three-surface will be magnified. Conventionally, most of cosmologists redefine the radial coordinate $\bar{r}$, the scale factor and the curvature as follows

$$a(t) = \frac{R(t)}{R_0}, \quad r = R_0 \bar{r}, \quad \kappa = \frac{k}{R_0^2}. \tag{1.28}$$

Therefore, the line element in Eq. (1.20) can be written in terms of new variables as

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - \kappa r^2} + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2 \right) \right], \tag{1.29}$$

The metric corresponding to this line element is commonly used in cosmology and the Friedmann-Lemaître-Robertson-Walker (FLRW) metric is usually referred to as this metric. It is important to note that the curvature κ is now dimensionful and can be chosen as an arbitrary number. The geometry of the spacetime can be classified by $\kappa > 0$, $\kappa = 0$ and $\kappa < 0$. However, it is convenient to choose this number as $\kappa = 1$, $\kappa = 0$ and $\kappa = -1$ for closed, flat and open geometry of the three-dimensional hypersurface.

1.4 Exercise

1. From the line element $d\sigma^2 = \gamma_{ij} dx^i dx^j = e^{2B} dr^2 + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2 \right)$, find non-zero components of Γ^i_{jk} and R_{ij}.

2. By embedding the three-dimensional hypersurface $d\sigma^2 = \frac{dr^2}{1 - k r^2} + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2 \right)$ into four-dimensional Euclidean space, show that for $k = 1$, $k = 0$ and $k = -1$ the geometry of the space corresponds to a sphere, flat and hyperboloid, respectively.

Chapter 2

Background evolution

In order to study the dynamics of the universe, one has to use the Einstein field equation with FLRW metric. In the left-hand side of the Einstein equation, there is only one function to determine, which is the scale factor $a(t)$, and only one number, which is the curvature κ. The scale factor will provide us the evolution behavior of the universe, and the curvature will provide us the geometry of the three-dimensional hypersurface. For the right-hand side of the Einstein equation, one needs to find the form of the fluid to satisfy the homogeneity and isotropy of the universe.

2.1 Perfect fluid

The perfect fluid is the fluid without viscosity and heat transfer between its elements in its rest frame. Since there is no heat transfer, the components T^{0i} will vanish. This satisfies the same requirement for the metric, where $g_{0i} = 0$. Moreover, no viscosity leads to no shear between elements of the fluid, so that the components $T^{ij}, (i \neq j)$ vanish. This also satisfies the same requirement for the metric where $g_{ij} = 0$ for $i \neq j$. As a result, for the perfect fluid there are two functions to determine, which are the energy density, ρ, and the pressure, p, of the fluid. Note that the pressure p serves as the the pressure in every direction, satisfying the isotropic condition. Moreover, to satisfy the homogeneity and isotropy, one has to require that the energy density and pressure must be independent of the spatial coordinates, so that one obtains $\rho = \rho(t)$ and $p = p(t)$. As a result, the EMT can be written in the matrix form as

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}. \tag{2.1}$$

Note that this form of the EMT is valid only in its rest frame. One needs a covariant form of the EMT in order to use it in the field equation. Let us find this covariant form by using the Lorentz transformation with the Lorentz matrix as

$$\Lambda^i_j = \delta^i_j + \frac{\gamma - 1}{v^2} v^i v_j, \quad \Lambda^i_0 = \gamma v^i, \quad \Lambda^0_0 = \gamma. \tag{2.2}$$

By applying these to the energy-momentum tensor above, one obtains

$$\begin{aligned} T^{ij} &= p\delta^{ij} + v^i v^j \gamma^2(\rho + p), && (2.3) \\ T^{0i} &= v^i \gamma^2 (\rho + p), && (2.4) \\ T^{00} &= (\gamma - 1)p + \gamma^2 \rho. && (2.5) \end{aligned}$$

Note that calculation details are left in the exercise. By using the form of the 4-velocity as

$$U^0 = \frac{dt}{d\tau} = \gamma, \quad U^i = \frac{dx^i}{d\tau} = \gamma v^i. \tag{2.6}$$

The components of the EMT above can be written as

$$T^{\mu\nu} = (\rho + p)U^\mu U^\nu + p\eta^{\mu\nu}. \tag{2.7}$$

Now, we already have the form of the perfect fluid, which can be expressed in any frame. However, this form can be used only in the flat spacetime. We need to express it in a more general form that includes curved spacetime. This can be generalized by replacing $\eta^{\mu\nu}$ with the general metric tensor $g^{\mu\nu}$ and remembering that the indices can be raised or lowered by using $g^{\mu\nu}$. As a result, the generally covariant form of the EMT for the perfect fluid can be expressed as

$$T^{\mu\nu} = (\rho + p)U^\mu U^\nu + pg^{\mu\nu}. \tag{2.8}$$

2.2 Friedmann equation dynamics of the universe

We already know the general form of both the metric tensor and EMT. Now, we will perform calculations to find the exact solutions. In order to solve for the solution, let us find the components of the Einstein tensor first. By using the FLRW metric in Eq. (1.29) and the definition of the connection, each component of the connection can be calculated, such as

$$\begin{aligned}
\Gamma^0_{ij} &= \frac{1}{2}g^{00}\left(\cancel{\partial_i g_{0j}} + \cancel{\partial_j g_{0i}} - \partial_0 g_{ij}\right), && (2.9)\\
&= \frac{1}{2}(-1)\left(-\partial_t(a^2\gamma_{ij})\right), && (2.10)\\
&= \gamma_{ij}a\dot{a} = \frac{\dot{a}}{a}a^2\gamma_{ij} = Hg_{ij}, && (2.11)
\end{aligned}$$

where dot denote the derivative with respect to the coordinate t and $H = \dot{a}/a$ is the Hubble parameter. Other non-zero components can be calculated in the same way and then left for the student to perform for the exercise. The results can be expressed as

$$\begin{aligned}
&\Gamma^i_{0j} = H\delta^i_j, \quad \Gamma^1_{11} = \frac{\kappa r}{1-\kappa r^2}, \quad \Gamma^1_{22} = -r(1-\kappa r^2), \quad \Gamma^1_{33} = \sin^2\theta\,\Gamma^1_{22},\\
&\Gamma^2_{33} = -\sin\theta\,\cos\theta, \quad \Gamma^2_{12} = \Gamma^2_{21} = \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}, \quad \Gamma^3_{23} = \Gamma^3_{32} = \cot\theta. \qquad (2.12)
\end{aligned}$$

By using these components of the connection and the definition of the Ricci tensor, the non-zero components of the Ricci tensor and the Ricci scalar can be written as

$$\begin{aligned}
R_{00} &= -3(\dot{H} + H^2), && (2.13)\\
R_{ij} &= \left(\dot{H} + 3H^2 + \frac{2\kappa}{a^2}\right)g_{ij}, && (2.14)\\
R = g^{\mu\nu}R_{\mu\nu} &= g^{00}R_{00} + g^{ij}R_{ij} = 6\left(\dot{H} + 2H^2 + \frac{\kappa}{a^2}\right). && (2.15)
\end{aligned}$$

Consequently, the non-zero components of the Einstein tensor can be written as

$$\begin{aligned}
G^0_0 &= -3\left(H^2 + \frac{\kappa}{a^2}\right), && (2.16)\\
G^i_j &= -\left(2\dot{H} + 3H^2 + \frac{\kappa}{a^2}\right)\delta^i_j. && (2.17)
\end{aligned}$$

The non-zero components of EMT can be expressed as $T^0_0 = -\rho, T^i_j = p\delta^i_j$. Now we have all components of ones in both sides of the Einstein equation. By putting these into the Einstein equation, one obtains $(0,0)$ and (i,j) components respectively as

$$\begin{aligned}
3\left(H^2 + \frac{\kappa}{a^2}\right) &= 8\pi G\,\rho, && (2.18)\\
-\left(2\dot{H} + 3H^2 + \frac{\kappa}{a^2}\right) &= 8\pi G\,p. && (2.19)
\end{aligned}$$

Now, we have two equations and three variables, a, ρ, p. It might not be possible to solve the exact solutions for them. However, we actually have one more equation, coming from the conservation of the EMT, as follows

$$\begin{aligned}
\nabla_\mu T^\mu_\nu &= \partial_\mu T^\mu_\nu + \Gamma^\mu{}_{\mu\rho} T^\rho_\nu - \Gamma^\rho{}_{\mu\nu} T^\mu_\rho = 0, \\
\nabla_\mu T^\mu_0 &= \partial_\mu T^\mu_0 + \Gamma^\mu{}_{\mu\rho} T^\rho_0 - \Gamma^\rho{}_{\mu 0} T^\mu_\rho, \\
&= \partial_0 T^0_0 + \Gamma^i{}_{i0} T^0_0 - \Gamma^i{}_{j0} T^j_i, \\
&= -\dot\rho - 3H\rho - H\delta^i_j p\, \delta^j_i, \\
&= -(\dot\rho + 3H(\rho + p)) = 0, \\
\dot\rho + 3H(\rho + p) &= 0.
\end{aligned} \tag{2.20}$$

Now we have 3 equations and 3 variables to solve. It seems like we can use these equations to completely solve for the solutions. However, one finds that only two of them are independent. Actually, one can show that ∂_t(2.18)+ $3H$((2.18) + (2.19)) = 0 gives Eq. (2.20). In order to solve the equations exactly, one has to impose one more condition. It is convenient to impose the condition between ρ and p to characterize the properties of the perfect fluid. The well-known condition is the "equation of state",

$$p = w\rho, \tag{2.21}$$

where w is the equation of state parameter. Substituting this into continuity equation Eq. (2.20), and then solving ρ in terms of a, one obtains

$$\rho = \rho_0 a^{-3(1+w)}, \tag{2.22}$$

where ρ_0 is the integration constant representing the energy density at the present time at $a = 1$. Now let us consider the universe filled with one of the well-known matter/energy

- Non-relativistic matter

Non-relativistic matter is also known by other names such as dust or matter. As we know dust is the fluid with pressureless. Therefore, the equation of state parameter vanishes, $w = 0$. Substituting $w = 0$ into Eq. (2.22), we have $\rho \propto a^{-3}$. This also makes our sense, since the number of particles is conserved; then the energy density is scaled by its volume, which is proportional to a^3.

- Relativistic matter

Relativistic matter is also known as radiation. From the exercise, we found that the energy-momentum of radiation (gauge field) is traceless. This corresponds to $\rho + 3p = 0, \to\ w = 1/3$. Substituting $w = 1/3$ into Eq. (2.22), we have $\rho \propto a^{-4}$. This also make our sense since the energy of the photon (or radiation) is also inversely proportional to wavelength, which is scaled by a. Together with volume scaling, the energy density is proportional to a^{-4}.

Now we can go further to find the exact solution. Let substituting $\rho = \rho_0 a^{-3(1+w)}$ into Eq. (2.18) and then we have

$$3\left(H^2 + \frac{\kappa}{a^2}\right) = 8\pi G\, \rho_0 a^{-3(1+w)}. \tag{2.23}$$

For simplicity, let us solve this equation in the case of a flat universe $k = 0$. As a result, the solution can be written as

$$a = \left(\frac{t}{t_0}\right)^{\frac{2}{3(1+w)}}, \quad t_0 = \sqrt{12\pi G}(1 + w) \tag{2.24}$$

As a result, for matter and radiation we have

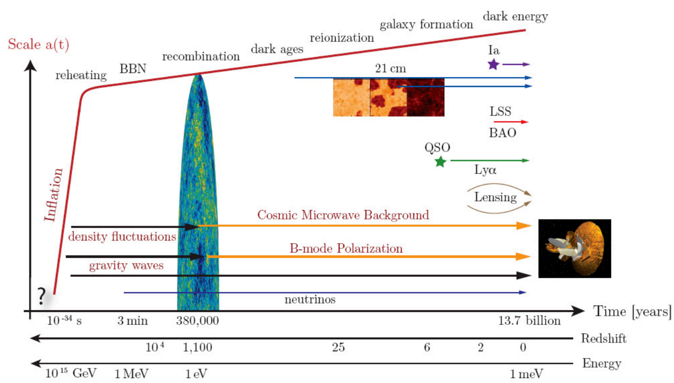
- matter: $a \propto t^{2/3}$.

- radiation: $a \propto t^{1/2}$.

Before we finish this section, let us consider one of the important equations called the acceleration equation. Eliminating κ term in Eq. (2.18) and Eq. (2.19), one obtains

$$2\dot{H} + 2H^2 = -\frac{8\pi G}{3}(1+3w)\rho, \quad \rightarrow \quad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(1+3w)\rho. \tag{2.25}$$

From this equation, one finds that for the well-known matter/energy (all non-relativistic, $w = 0$, and relativistic, $w = 1/3$ ones), it is not possible to provide the universe with accelerated expansion. Note that this is possible for the cosmological constant, and leave it for students in the exercise. Finally, I will briefly review the standard picture of the universe, including the evolution of the universe and the major events in the history of the universe. This is illustrated roughly in Fig. 2.1 and I will explain in more detail in the classroom.



	Time	Energy	
Planck Epoch?	$< 10^{-43}$ s	10^{18} GeV	
String Scale?	$\gtrsim 10^{-43}$ s	$\lesssim 10^{18}$ GeV	
Grand Unification?	$\sim 10^{-36}$ s	10^{15} GeV	
Inflation?	$\gtrsim 10^{-34}$ s	$\lesssim 10^{15}$ GeV	
SUSY Breaking?	$< 10^{-10}$ s	> 1 TeV	
Baryogenesis?	$< 10^{-10}$ s	> 1 TeV	
Electroweak Unification	10^{-10} s	1 TeV	
Quark-Hadron Transition	10^{-4} s	10^2 MeV	
Nucleon Freeze-Out	0.01 s	10 MeV	
Neutrino Decoupling	1 s	1 MeV	
BBN	3 min	0.1 MeV	
			Redshift
Matter-Radiation Equality	10^4 yrs	1 eV	10^4
Recombination	10^5 yrs	0.1 eV	1,100
Dark Ages	$10^5 - 10^8$ yrs		> 25
Reionization	10^8 yrs		$25 - 6$
Galaxy Formation	$\sim 6 \times 10^8$ yrs		~ 10
Dark Energy	$\sim 10^9$ yrs		~ 2
Solar System	8×10^9 yrs		0.5
Albert Einstein born	14×10^9 yrs	1 meV	0

Figure 2.1: The left panel shows the standard evolution of the universe. The right panel shows the major events in the history of the universe [5].

2.3 Inflation

The inflationary model is a model providing the enormous expansion of the universe at very early times. This model is constructed in order to solve some cosmological problems. It is worthwhile to first give how the problems are and then how the inflationary model can solve them.

2.3.1 Cosmological problems

- **Flatness problem**

It is convenient to change variables to characterize the dynamics of the universe as follows:

$$\Omega = \frac{8\pi G}{3}\rho, \tag{2.26}$$

and the dynamics of Ω is scaled by $N = \ln a$ instead of t. As a result, the quantity with a time derivative can be changed in terms of the derivative with respect to N as

$$\dot{X} = HX' = H\frac{dX}{dN}. \tag{2.27}$$

As a result, the Friedmann equation, Eq. (2.18), becomes

$$1 = \Omega + \Omega_k, \quad \Omega_k = -\frac{\kappa}{(aH)^2}. \tag{2.28}$$

Note that the observation suggest that $-0.0175 < \Omega_k^{(0)} < 0.085$ which is very tiny and then one can approximated as $\Omega^{(0)} \sim 1$. Also the acceleration equation, Eq. (2.25), and continuity equation, Eq. (2.20), respectively becomes

$$2\frac{H'}{H} = -2 - (1+3w)\Omega, \tag{2.29}$$

$$\rho' = -3(1+w)\rho. \tag{2.30}$$

By using the definition of the density parameter, Ω, one obtains

$$\begin{aligned} \Omega' &= \Omega\left(\frac{\rho'}{\rho} - 2\frac{H'}{H}\right), \\ &= -\Omega\left(3(1+w) - 2 - (1+3w)\Omega\right), \\ &= (1+3w)\Omega\left(\Omega - 1\right) \end{aligned} \tag{2.31}$$

From this equation, one finds that, as long as $1 + 3w > 0$, the solution for $\Omega = 1$ is a repelling solution. In order to obtain the total density at the present approximately unity, $\Omega^{(0)} \sim 1$, one has to enormously tune the initial conditions. For example, to obtain $|\Omega^{(0)} - 1| < 0.02$ one has to put the initial value at the nucleosynthesis as $\Omega^{(nuc)} \sim 1 \pm 10^{-12}$. This is known as the flatness problem: "Why is the universe so flat?".

- **Horizon problem**

To understand this problem, let us consider the null geodesic of the massless particle (photon) as

$$|d\vec{x}| = \int_0^t \frac{dt'}{a(t')}. \tag{2.32}$$

This distance gives us how long/ how far the photon travels from the past. Moreover, one can also find the new time coordinate which gives the metric conformally flat, as $dt^2 = a^2 d\tau^2$, where τ is the conformal time. As a result, we have

$$|d\vec{x}| = \tau = \int_0^t \frac{dt'}{a(t')}. \tag{2.33}$$

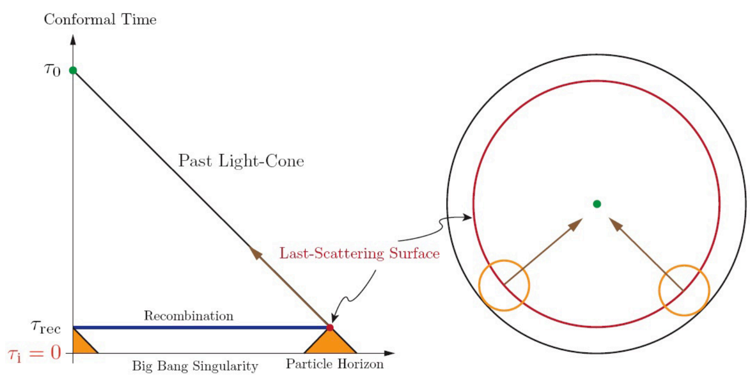


Figure 2.2: The conformal diagram shows that a photon in the past is not causally connected [5].

The conformal diagram can be shown in Fig. 2.2. From this figure, one can see that most spots in the CMB have non-overlapping past light cones. This means that they never were in causal contact. However, from observation, the CMB temperature is very smooth, $\frac{\Delta T}{T} \sim 10^{-5}$. This is the Horizon problem: "Why is the CMB temperature very smooth even though they never were in causal contact?". To see this problem mathematically, let us consider two distances as follows;

- physical distance, d_p, is proportional to the scale factor, $d_p \propto a$.

- horizon distance, d_H, is proportional to H^{-1}, $d_H \propto H^{-1}$.

Therefore, the ratio of these distances can be written as

$$\left(\frac{d_p}{d_H}\right)^2 = (aH)^2. \tag{2.34}$$

By using $a \propto t^{2/(3(1+w))}$, one has

$$\frac{d}{dN}\left(\frac{d_p}{d_H}\right)^2 = \frac{d}{dN}(aH)^2 = -(1+3w)(aH)^2 = -(1+3w)\left(\frac{d_p}{d_H}\right)^2. \tag{2.35}$$

From this expression, we found that for $1+3w > 0$, d_H always grows faster than d_p. That is, more and more space falls into the horizon or becomes causally disconnected at the late times.

2.3.2 Solution and Inflationary model

In order to solve these problems, let us summarize the main equations associated to the problem as follows:

- flatness problem

$$\Omega' = (1+3w)\Omega\left(\Omega - 1\right), \tag{2.36}$$

- horizon problem

$$\frac{d}{dN}\left(\frac{d_p}{d_H}\right)^2 = -(1+3w)\left(\frac{d_p}{d_H}\right)^2, \tag{2.37}$$

- acceleration equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(1+3w)\rho. \tag{2.38}$$

From these three equations, there is one common factor, which is $1+3w$. From our discussion before, we found that the known matter gives the equation of state parameter $w > -1/3$, which makes the universe expand with deceleration, $\ddot{a} < 0$. In order to mathematically solve these problems, we have to impose that the universe must expand with acceleration, $\ddot{a} > 0 \rightarrow w < -1/3$. So this is the main idea of the inflationary model: "the universe expands with rapid acceleration at very early times". From this model, it is found that the equation in the flatness problem provides the attractor solution at $\Omega = 1$. This means that whatever the density parameter is, it will approach 1 at late times. For the horizon problem, it is found that the photons actually are in the same place initially (the place where they have enough time with causal contact to make thermal equilibrium), and then the inflation make them separated to the regions which are causally disconnected. The conformal diagram with inflation is illustrated in Fig. 2.3. Now we are in a state to ask how much the expansion rate accelerates during inflation. It is possible to separate this answer in two ways; qualitative and approximated numerical method. Let us first calculate a rough number of e-foldings, $N = \ln(a_f/a_i)$. Considering the horizon problem, in order to solve it, one can at least require that the ratio $(d_p/d_H)_i$ during inflation must be equal to the ratio $(d_p/d_H)_0$ during the present period. Let a_E and H_E be the scale factor and the Hubble parameter at the end of inflation, and the Hubble parameter during the radiation-dominated period can be approximated as $H \propto a^{-2}$. As a result, we have

$$\frac{a_0 H_0}{a_E H_E} = \frac{a_E}{a_0} = \frac{T_0}{T_E} = \frac{10^{-4} eV}{10^{15} GeV} = 10^{-28}, \tag{2.39}$$

where we have used T_0 as the CMB temperature nowadays and T_0 is the temperature at the Grand Unified scale. By using this approximated number, one can calculate the number of e-foldings as

$$a_i H_i = 10^{-28} a_E H_E, \quad \rightarrow \quad \frac{a_E}{a_i} = 10^{28}, \quad \rightarrow \quad N = \ln\frac{a_E}{a_i} = \ln 10^{28} \sim 64, \tag{2.40}$$

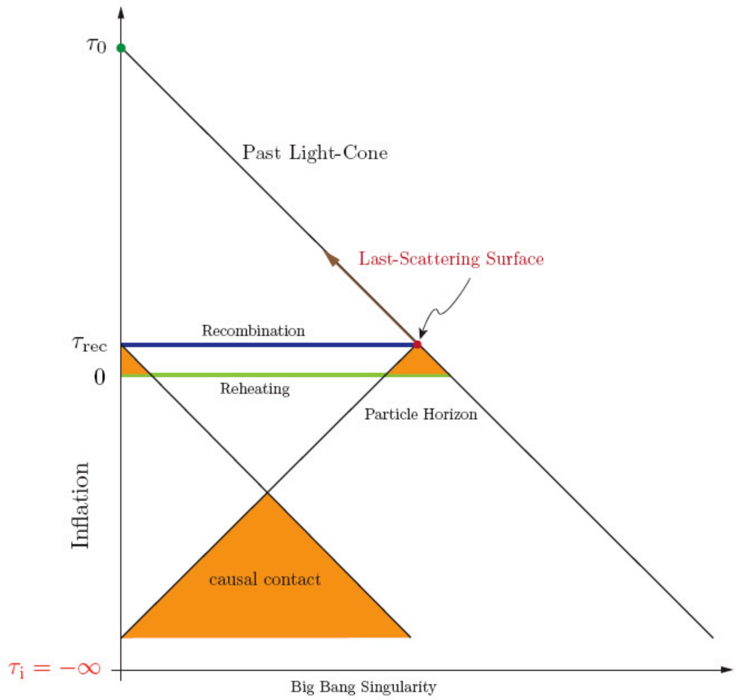


Figure 2.3: The conformal diagram with the inflationary period shows that it is possible for the photons in the past to be causally connected [5].

where we have used the fact that H is almost constant during inflation.

Now let us find the qualitative approximation during inflation called the "slow-roll approximation". Since we need the acceleration of the universe, one may find the dimensionless parameter to characterize the acceleration as follows,

$$\dot{H} = \frac{\ddot{a}}{a} - H^2, \quad \rightarrow \quad \frac{\ddot{a}}{aH^2} = 1 + \frac{\dot{H}}{H^2} = 1 - \epsilon, \tag{2.41}$$

where we have defined the slow-roll parameter ϵ to characterize the acceleration as

$$\epsilon = -\frac{\dot{H}}{H^2}. \tag{2.42}$$

Now we can require that $\epsilon \ll 1$ during inflation and then the inflation ends at $\epsilon \sim 1$. Moreover, one may require that the inflation must last long enough to guarantee solving the problems. This means that $\epsilon \ll 1$ must hold long enough. Therefore, one can define one more slow-roll parameter as

$$\eta = \frac{\epsilon'}{\epsilon} = \frac{\dot{\epsilon}}{\epsilon H}, \tag{2.43}$$

and the requirement for inflation is $\eta \ll 1$.

2.3.3 Dynamics of inflation

In the previous subsection, we know how to solve the cosmological problem, which is making the universe expand with acceleration. Now we can ask a bit more deeply: how can the universe expand with acceleration? Actually, this question is just want to know that how can we construct the dynamical model of inflation. The simple model is just add a scalar field into the model and then check how this scalar field can make the universe accelerate. We will perform this procedure in this subsection. Let us consider the scalar field model as follows:

$$S = \int d^4x \sqrt{-g} \left(\frac{M_P^2}{2} R - \frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi - V(\phi) \right), \tag{2.44}$$

where $M_P = (8\pi G)^{-1/2}$ is the Planck mass and the scalar field depends only on t to satisfy the cosmological principle.

For the energy-momentum tensor of the matter field, it can be expressed as

$$
\begin{aligned}
T_{\mu\nu} &\equiv -\frac{2}{\sqrt{-g}}\frac{\delta \mathcal{L}_m}{\delta g^{\mu\nu}} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}L_m)}{\delta g^{\mu\nu}}, \qquad (2.45)\\
&= -\frac{2}{\sqrt{-g}}\left(\sqrt{-g}\frac{\delta L_m}{\delta g^{\mu\nu}} + L_m\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}}\right),\\
&= -2\left(\frac{\delta L_m}{\delta g^{\mu\nu}} - \frac{1}{2}L_m g_{\mu\nu}\right), \qquad (2.46)
\end{aligned}
$$

Therefore, for the scalar field, we have

$$
L_m = L_\phi = -\frac{1}{2}g^{\rho\sigma}\nabla_\rho\phi\nabla_\sigma\phi - V(\phi), \tag{2.47}
$$

then the energy-momentum tensor for the scalar field can be found as follows,

$$
\begin{aligned}
\frac{\delta L_\phi}{\delta g^{\mu\nu}} &= -\frac{1}{2}\frac{\delta g^{\rho\sigma}}{\delta g^{\mu\nu}}\nabla_\rho\phi\nabla_\sigma\phi,\\
&= -\frac{1}{2}\nabla_\mu\phi\nabla_\nu\phi,\\
T_{\mu\nu} &= \nabla_\mu\phi\nabla_\nu\phi - \left(\frac{1}{2}g^{\rho\sigma}\nabla_\rho\phi\nabla_\sigma\phi + V(\phi)\right)g_{\mu\nu}. \qquad (2.48)
\end{aligned}
$$

By substituting the FLWR metric into the energy-momentum tensor in Eq. (2.48), the non-zero components of the scalar field can be written as

$$
\begin{aligned}
T^0_0 &= -\left(\frac{1}{2}\dot{\phi}^2 + V\right) = -\rho_\phi, \quad \rightarrow \quad \rho_\phi = \frac{1}{2}\dot{\phi}^2 + V, \qquad (2.49)\\
T^i_j &= -\left(-\frac{1}{2}\dot{\phi}^2 + V\right)\delta^i_j = p_\phi\delta^i_j, \quad \rightarrow \quad p_\phi = \frac{1}{2}\dot{\phi}^2 - V. \qquad (2.50)
\end{aligned}
$$

As a result, the equation of state parameter of the scalar field can be written as

$$
w_\phi = \frac{p_\phi}{\rho_\phi} = \frac{\frac{1}{2}\dot{\phi}^2 - V}{\frac{1}{2}\dot{\phi}^2 + V}. \tag{2.51}
$$

In order to provide the acceleration, we have

$$
w_\phi < -\frac{1}{3} \quad \rightarrow \quad \frac{1}{2}\dot{\phi}^2 < V. \tag{2.52}
$$

However, this does not guarantee that it is sufficient for inflation. So one needs to check the slow-roll conditions. In order to check this, we have to use the equations of motion, which are the components $(0,0)$ and (i,j) as well as the equation of the scalar field;

$$
\begin{aligned}
3H^2M_P^2 &= \frac{1}{2}\dot{\phi}^2 + V, \qquad (2.53)\\
(2\dot{H} + 3H^2)M_P^2 &= -\frac{1}{2}\dot{\phi}^2 + V, \qquad (2.54)\\
\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} &= 0. \qquad (2.55)
\end{aligned}
$$

By performing $((2.53) - (2.54))/(2.53)$, one obtains

$$
\epsilon = -\frac{\dot{H}}{H^2} = \frac{\dot{\phi}^2}{2M_P^2H^2}. \tag{2.56}
$$

Therefore, to obtain the inflation, one requires

$$\frac{\dot{\phi}^2}{2M_P^2 H^2} \ll 1. \tag{2.57}$$

Note that by using Eq. (2.53), this condition also satisfies

$$\frac{1}{2}\dot{\phi}^2 \ll V. \tag{2.58}$$

This means that the kinetic term must be much less than the potential of the scalar field. In this case, the scalar field moves very slowly on the potential. One also sees that $w_\phi \sim -1$. In order to keep the scalar field slowly moving long enough, one needs the second slow-roll condition as follows

$$\eta = \frac{\epsilon'}{\epsilon} = \frac{1}{2M_P^2}\frac{d}{dN}\left(\frac{\dot{\phi}^2}{H^2}\right) = 2(\epsilon - \delta), \quad \delta \equiv -\frac{\ddot{\phi}}{H\dot{\phi}} \ll 1. \tag{2.59}$$

Note that we defined another slow-roll parameter δ. Note also that in some books/lecture notes, this parameter is denoted by η. From this condition, we found that the first term in the equation of motion of the scalar field is much less than the second term. Since the second term corresponds to the friction term, it means that the scalar field moves very slowly with large friction. From this condition, one can see that the scalar field must move very slowly. However, from the equation of motion of the scalar field, one can see that the shape of the potential will control the dynamics of the scalar field. So it is more convenient to write the slow-roll condition in terms of the potential. By using the approximation of the equation of motion, the slow-roll parameter can be written as

$$\epsilon_V = \frac{M_P^2}{2}\left(\frac{V_{,\phi}}{V}\right)^2 \ll 1. \tag{2.60}$$

This condition implies that the scalar field must move on the flat potential (its slope is much less than its value). Then, by using the same strategy, the second slow-roll condition can be written as

$$\eta_V = \frac{\epsilon_V'}{\epsilon_V} = 2(2\epsilon_V - \delta_V), \quad \delta_V \equiv M_P^2 \frac{V_{,\phi\phi}}{V} \ll 1. \tag{2.61}$$

Note that the detailed calculation is left for the student in the exercise. From this condition, one can see that this controls the slope of the potential to be much less than its value. This guarantees that the scalar field moves slowly long enough. Now, let us calculate the number of e-foldings in terms of the slow-roll parameter as follows:

$$N = \int H dt = \int \frac{H}{\dot{\phi}} d\phi = -\int \frac{3H^2}{V_{,\phi}} d\phi = -\frac{1}{M_P^2}\int \frac{V}{V_{,\phi}} d\phi = \frac{1}{M_P^2}\int_{\phi_e}^{\phi_i} \frac{1}{\sqrt{2\epsilon_V}} d\phi. \tag{2.62}$$

From this expression, if we know the shape of the potential, it is possible to find the number of e-foldings.

2.4 Dark energy

2.4.1 Cosmological constant

- The simplest candidate to explain the accelerated expansion of the universe nowadays is known as ΛCDM where Λ stands for the dark energy contributed by the cosmological constant and *CDM* stands for the contribution of dark matter, namely Cold dark matter. These two contributions are about 95% of the content in our universe, while the contribution from dark energy is about 70% and the contribution from dark matter is about 25%.

- The cosmological constant is the promising candidate for dark energy to drive the accelerated expansion of the universe nowadays

The action for such the model can be expressed as

$$S = \frac{1}{16\pi G} \int d^4x\sqrt{-g}\,(R - 2\Lambda) + S_m, \tag{2.63}$$

where S_m is the action for the matter. Varying the action with respect to the metric tensor, one obtains

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{2.64}$$

By using the FLRW metric, the Friedmann equation and the acceleration equation can be written as

$$\left(H^2 + \frac{\kappa}{a^2}\right) = \frac{8\pi\, G}{3}\rho + \frac{\Lambda}{3}, \tag{2.65}$$

$$\frac{\ddot{a}}{a} = \frac{8\pi\, G}{3}(\rho + 3p) + \frac{\Lambda}{3}. \tag{2.66}$$

- When Einstein constructed the Einstein equation, he believed that the universe was static. Therefore, he put the cosmological constant into the Einstein equation to make the universe static.

To achieve such a goal, one can find the solution such that $\dot{a} = 0 = \ddot{a}$. Let us examine what the cosmological constant can be. Firstly, let us consider the universe filled with dust so that the pressure vanishes. From Eq. (2.66) and Eq. (2.65), one obtains

$$\frac{8\pi\, G}{3}(\rho + 3p) + \frac{\Lambda}{3} = 0, \;\Rightarrow\; \rho = \frac{\Lambda}{4\pi}, \tag{2.67}$$

$$\frac{8\pi\, G}{3}\rho + \frac{\Lambda}{3} = \frac{\kappa}{a^2} \;\Rightarrow\; \frac{\kappa}{a^2} = \Lambda. \tag{2.68}$$

- After that, Lemaitre showed that even though this is a solution to the equations of motion, it is not stable due to the small perturbation.
- In 1917, de Sitter found that there exists the solution in the empty space with $H = \sqrt{\Lambda/3}$.
- During 1910-1920, Slipler observed the spectra of the galaxies and found that they are redshifted.
- In 1922, Friedmann found the evolving solution with the expanding universe.
- In 1927, Lermaitre proposed the "hot Big Bang" model of the universe. This is may be the first model to explain how the universe evolves associated with general relativity. In such the model, the evolution of the universe can be divided into 3 states as follows

1. Fundamental elements are formed, and the universe expands from the point source. It dominated by pressureless matter, $a \propto t^n (0 < n < 1)$.
2. Nebulae and galaxies are formed. The phase is represented by the static universe proposed by Einstein, $a \propto const$
3. A period of fast expansion of the universe, $a \propto t^n (1 < n)$. This phase is realized by the de Sitter solution.

- In 1929, Hubble formulated Hubble's law by combining the results of Sliphe r. The existence of a cosmological constant was clearly not required to give rise to the cosmic expansion of the universe.

- In 1945, in the book "The Meaning of Relativity" written by Einstein, he said that "if the Hubble's expansion had been discovered at the time of creation of the general theory of relativity, the cosmological constant would never have been introduced".

- In 1970, Gamov recalled that "when I was discussing cosmological problems with Einstein, he remarked that the introduction of the cosmological constant term was the biggest blunder he ever made in his life". Remarkably, the cosmological constant has become the main ingredient in the standard cosmology nowadays.

2.5 Exercise

1. By using the metric in Eq. (1.29), find non-zero components of ${\Gamma^\rho}_{\mu\nu}$ and $R_{\mu\nu}$.

2. From the action of the gauge field

$$S = \int d^4x\sqrt{-g}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right), \tag{2.69}$$

   find the energy-momentum tensor and then show that it is traceless.

3. From Einstein-Hilbert with cosmological constant

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\,(R-2\Lambda)\,, \tag{2.70}$$

   Find the energy-momentum tensor for Λ and the equation of state parameter, as well as find the solution for $a(t)$ in the case $\kappa = 0$.

4. By using the approximated equations of motion, show that the slow-roll parameter can be written as

$$\epsilon_V = \frac{M_P^2}{2}\left(\frac{V_{,\phi}}{V}\right)^2, \tag{2.71}$$

$$\delta_V = M_P^2\frac{V_{,\phi\phi}}{V}. \tag{2.72}$$

Chapter 3

Cosmological perturbations

3.1 Perturbed spacetime and matter/energy

previous lecture, we studied a model of the very early Universe called inflation. This model is built on the cosmological principle, i.e., that the Universe is homogeneous and isotropic on macroscopic scales. However, measurements of the CMB anisotropies show that the Universe is not perfectly homogeneous and isotropic: the CMB temperature measured in different directions differs by about 0.001%. Moreover, structures in the Universe such as stars, galaxies, and clusters of galaxies—which represent inhomogeneities and anisotropies—cannot arise from a spacetime that is exactly homogeneous and isotropic. Nevertheless, the cosmological principle remains a good approximation, because if we average the matter–energy content over regions larger than about 100 Mpc, the distribution is fairly homogeneous and isotropic.

The inhomogeneities and anisotropies in the present-day Universe developed from tiny inhomogeneities and anisotropies in the early Universe. In the next lecture, we will study how the small primordial inhomogeneities and anisotropies originate from the quantum fluctuations of the inflaton that drives inflation. Since the amplitude of these primordial inhomogeneities and anisotropies is small, we can treat them as perturbations about a homogeneous and isotropic Universe. The evolution and properties of these perturbations can be studied using cosmological perturbation theory.

In general, linear perturbation theory, also called first-order perturbation theory, is sufficient to study the evolution of the small inhomogeneities and anisotropies in the early Universe, because their amplitudes are small. However, to study certain properties—for example, whether the statistics are Gaussian — we need second-order perturbation theory.

As the Universe evolves through different epochs, the perturbations also evolve. On large scales (larger than the Hubble radius), present-day perturbations do not change much compared to their primordial values. Therefore, linear theory can be used to study their evolution on these scales. In the matter-dominated era, however, on small scales (smaller than the scale of galaxy clusters), perturbations grow with time. This growth increases the inhomogeneity of the gravitational field, and when it becomes sufficiently large, it causes matter–energy in underdense regions to flow into overdense regions.

As a result, overdense regions gradually become more inhomogeneous and anisotropic and develop into clusters of galaxies, galaxies, stars, etc. This structure-formation process is known as gravitational instability.

Structure formation on small scales is a nonlinear process, with perturbations too large for linear theory. The formation of structure can be studied using nonlinear cosmological perturbation theory, the spherical collapse model, and computer N- body simulations; we will not discuss these topics here.

In this lecture we will study linear cosmological perturbation theory, starting by splitting the metric and the energy–momentum tensor into homogeneous background parts plus perturbations. After deriving the evolution equations for the perturbations, we will discuss the gauge problem. We will show that on large scales there exist conserved quantities for linear perturbations. Since observations indicate that CMB anisotropies and the (averaged) matter–energy distribution on large scales are sufficiently small, linear perturbation theory can be used to study the inhomogeneities and anisotropies observed today.

Therefore, conserved quantities for linear perturbations allow us to connect CMB measurements and the matter–energy distribution to perturbations generated in the early Universe and at later times, such as at last scattering. For this reason, cosmological perturbation theory is an important framework for studying the nature of the Universe.

3.1.1 Perturbations of the Einstein tensor

In cosmological perturbation theory, we treat small inhomogeneities and anisotropies in the spacetime of the Universe as perturbations about the FLRW metric, i.e.,

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \,, \tag{3.1}$$

where $h_{\mu\nu}$ is the perturbed metric. It is symmetric, i.e., $h_{\mu\nu} = h_{\nu\mu}$, and $\bar{g}_{\mu\nu}$ is the background metric, taken to be the FLRW metric in conformal time τ,

$$ds^a = a^2(\tau)\left(-d\tau^2 + \delta_{ij} dx^i\, dx^j\right) , \tag{3.2}$$

whose components can be written as

$$\bar{g}_{00} = -a^2(\tau) \,, \quad \bar{g}_{ij} = a^2(\tau)\delta_{ij} \,, \quad \bar{g}_{i0} = \bar{g}_{0i} = 0 \,. \tag{3.3}$$

In this lecture we will focus on perturbations in a spatially flat Universe. The extension to the case with nonzero spatial curvature is straightforward: one simply replaces δ_{ij} in the above expressions by the appropriate three-metric corresponding to the general FLRW line element. For the inverse perturbed metric, we have

$$\begin{aligned} g_{\mu\alpha} g^{\alpha\nu} = \delta^\nu_\mu = \left(\bar{g}_{\mu\alpha} + h_{\mu\alpha}\right)\left(\bar{g}^{\alpha\nu} + h^{\alpha\nu}\right) = \delta^\nu_\mu + \bar{g}_{\mu\alpha} h^{\alpha\nu} + h_{\mu\alpha}\bar{g}^{\alpha\nu} + \mathcal{O}\left(h^2\right) \,, \\ \Rightarrow \quad \bar{g}_{\mu\alpha} h^{\alpha\nu} = -\bar{g}^{\beta\nu} h_{\mu\beta} \,, \quad \Rightarrow \quad h^{\mu\nu} = -\bar{g}^{\mu\alpha}\bar{g}^{\nu\beta} h_{\alpha\beta} \,. \end{aligned} \tag{3.4}$$

The last equation above shows that we cannot directly raise the indices of $h_{\mu\nu}$ using the background metric unless we redefine $h^{\mu\nu} \to -h^{\mu\nu}$, i.e., define $g^{\mu\nu} = \bar{g}^{\mu\nu} - h^{\mu\nu}$. However, we will not adopt this definition. From the relation above, the components of $h^{\mu\nu}$ can be written as follows:

$$h^{00} = -a^{-4} h_{00} \,, \qquad h^{0i} = a^{-4} h_{0i} \,, \qquad h^{ij} = -a^{-4} h_{ij} \,. \tag{3.5}$$

For the background Christoffel connection, the nonzero components are

$$\Gamma^\nu_{\mu 0} = \delta^\nu_\mu \frac{a'}{a} = \delta^\nu_\mu \mathcal{H} \,, \quad \Gamma^0_{ij} = \delta_{ij}\mathcal{H} \,, \tag{3.6}$$

Using the metric in Eq. (3.1), we can compute the perturbed Christoffel connection $\delta\Gamma^\lambda_{\mu\nu}$ as follows:

$$\begin{aligned} \Gamma^\lambda_{\mu\nu} &= \bar{\Gamma}^\lambda_{\mu\nu} + \delta\Gamma^\lambda_{\mu\nu} \\ &= \bar{\Gamma}^\lambda_{\mu\nu} + \frac{h^{\lambda\alpha}}{2}\left(\partial_\mu \bar{g}_{\alpha\nu} + \partial_\nu \bar{g}_{\alpha\mu} - \partial_\alpha \bar{g}_{\mu\nu}\right) + \frac{\bar{g}^{\lambda\alpha}}{2}\left(\partial_\mu h_{\alpha\nu} + \partial_\nu h_{\alpha\mu} - \partial_\alpha h_{\mu\nu}\right) \\ \delta\Gamma^\lambda_{\mu\nu} &= -\bar{g}^{\lambda\beta} h_{\beta\gamma}\bar{\Gamma}^\gamma_{\mu\nu} + \frac{\bar{g}^{\lambda\alpha}}{2}\left(\partial_\mu h_{\alpha\nu} + \partial_\nu h_{\alpha\mu} - \partial_\alpha h_{\mu\nu}\right) \,, \end{aligned} \tag{3.7}$$

In the third line we used $h^{\mu\nu} = -\bar{g}^{\mu\alpha}\bar{g}^{\nu\beta} h_{\alpha\beta}$. From the previous lecture, the expression above can be written in terms of covariant derivatives as

$$\delta\Gamma^\rho_{\mu\nu} = \frac{\bar{g}^{\rho\alpha}}{2}\left(\nabla_\mu h_{\alpha\nu} + \nabla_\nu h_{\alpha\mu} - \nabla_\alpha h_{\mu\nu}\right) \,, \tag{3.8}$$

From Eqs. (3.8) and (3.6) we can show that

$$\delta\Gamma^\mu_{\mu\nu} = \frac{1}{2}\partial_\nu\left[\bar{g}^{\mu\alpha} h_{\alpha\mu}\right] = \partial_\nu\left[h^0_0 + h^i_i\right] = \partial_\nu\left[\frac{h_{ii}}{2a^2} - \frac{h_{00}}{2a^2}\right] \,, \tag{3.9}$$

which will be very useful when we compute perturbations of the curvature tensor in the next step. Using Eqs. (3.7) and (3.6), the nonzero components of $\delta\Gamma^\lambda_{\mu\nu}$ are

$$\begin{aligned}
\delta\Gamma^k_{ij} &= -\frac{\mathcal{H}}{a^2}\delta_{ij}h_{k0} + \frac{1}{2a^2}\left(\partial_i h_{kj} + \partial_j h_{ki} - \partial_k h_{ij}\right) , && (3.10)\\
\delta\Gamma^j_{i0} &= -\frac{\mathcal{H}}{a^2}h_{ji} + \frac{1}{2a^2}\left(\partial_i h_{j0} + h'_{ji} - \partial_j h_{i0}\right) , && (3.11)\\
\delta\Gamma^0_{ij} &= \frac{\mathcal{H}}{a^2}h_{00}\delta_{ij} - \frac{1}{2a^2}\left(\partial_i h_{0j} + \partial_j h_{0i} - h'_{ij}\right) , && (3.12)\\
\delta\Gamma^i_{00} &= -\frac{\mathcal{H}}{a^2}h_{i0} + \frac{1}{2a^2}\left(2h'_{i0} - \partial_i h_{00}\right) , && (3.13)\\
\delta\Gamma^0_{i0} &= \frac{\mathcal{H}}{a^2}h_{0i} - \frac{1}{2a^2}\partial_i h_{00} , && (3.14)\\
\delta\Gamma^0_{00} &= \frac{\mathcal{H}}{a^2}h_{00} - \frac{1}{2a^2}h'_{00} . && (3.15)
\end{aligned}$$

Using the background expression for the Ricci tensor, we can write the perturbed Ricci tensor in terms of $\delta\Gamma^\lambda_{\mu\nu}$ as

$$\delta R_{\mu\nu} = \partial_\lambda\delta\Gamma^\lambda_{\mu\nu} - \partial_\nu\delta\Gamma^\lambda_{\mu\lambda} + \delta\Gamma^\alpha_{\mu\nu}\bar{\Gamma}^\lambda_{\alpha\lambda} + \bar{\Gamma}^\alpha_{\mu\nu}\delta\Gamma^\lambda_{\alpha\lambda} - \delta\Gamma^\alpha_{\mu\lambda}\bar{\Gamma}^\lambda_{\nu\alpha} - \bar{\Gamma}^\alpha_{\mu\lambda}\delta\Gamma^\lambda_{\nu\alpha} . \quad (3.16)$$

Here we are interested in the perturbation of R^ν_μ, which can be obtained from $\delta R_{\mu\nu}$ as

$$\delta R^\nu_\mu = \delta\left(g^{\nu\alpha}R_{\alpha\mu}\right) = h^{\nu\alpha}\bar{R}_{\alpha\mu} + \bar{g}^{\nu\alpha}\delta R_{\alpha\mu} . \quad (3.17)$$

where the nonzero components of the background Ricci tensor and the background Ricci scalar are

$$R_{00} = -3\mathcal{H}' , \quad R_{ij} = \delta_{ij}\left(\mathcal{H}' + 2\mathcal{H}^2\right) , \quad R = \frac{6}{a^2}\mathcal{H}' + \frac{6}{a^2}\mathcal{H}^2 . \quad (3.18)$$

Using Eqs. (3.9)–(3.15) and (3.18), we obtain the following nonzero components of δR^ν_μ:

$$\begin{aligned}
\delta R^j_i &= h^{ji}\left(\mathcal{H}' + 2\mathcal{H}^2\right) + \frac{1}{a^2}\partial_0\left[\frac{\mathcal{H}}{a^2}h_{00}\delta_{ij} - \frac{1}{2a^2}\left(\partial_i h_{0j} + \partial_j h_{0i} - h'_{ij}\right)\right]\\
&\quad + \frac{1}{a^2}\partial_k\left[-\frac{\mathcal{H}}{a^2}h_{k0}\delta_{ji} + \frac{1}{2a^2}\left(\partial_i h_{kj} + \partial_j h_{ki} - \partial_k h_{ij}\right)\right] - \frac{1}{a^2}\partial_j\partial_i\left(\frac{h_{kk}}{2a^2} - \frac{h_{00}}{2a^2}\right)\\
&\quad + 4\frac{\mathcal{H}}{a^2}\delta\Gamma^0_{ij} + \frac{\mathcal{H}}{a^2}\delta_{ij}\delta\Gamma^\lambda_{0\lambda} - 2\frac{\mathcal{H}}{a^2}\delta\Gamma^0_{ij} - \frac{\mathcal{H}}{a^2}\delta\Gamma^j_{i0} - \frac{\mathcal{H}}{a^2}\delta\Gamma^i_{j0}\\
&= \frac{1}{2a^4}\left[\partial_k\left(\partial_i h_{kj} + \partial_j h_{ki} - \partial_k h_{ij}\right) - \partial_j\partial_i\left(h_{kk} - h_{00}\right) - \left(\partial_i h'_{0j} + \partial_j h'_{0i} - h''_{ij}\right)\right]\\
&\quad + \frac{\mathcal{H}}{2a^4}\left[h'_{00}\delta_{ij} - 2\partial_k h_{k0}\delta_{ji} + \delta_{ij}h'_{kk} - 2h'_{ji}\right]\\
&\quad + \frac{1}{a^4}\left(\mathcal{H}' + \mathcal{H}^2\right)h_{00}\delta_{ij} - \frac{1}{a^4}\mathcal{H}'h_{ij} - \frac{\mathcal{H}^2}{a^4}\delta_{ij}h_{kk} . \qquad (3.19)
\end{aligned}$$

$$\begin{aligned}
\delta R^i_0 &= \frac{1}{2a^4}\left[\partial_j\left(\partial_i h_{j0} + h'_{ji} - \partial_j h_{i0}\right) - \partial_i h'_{jj}\right] + \frac{\mathcal{H}}{a^4}\left[\partial_i h_{jj} - \partial_j h_{ij} - \partial_i h_{00}\right]\\
&\quad + \frac{2}{a^4}\left(\mathcal{H}^2 - \mathcal{H}'\right)h_{0i} , \qquad (3.20)\\
\delta R^0_i &= \frac{1}{a^4}h_{0j}\bar{R}_{ji} - \frac{1}{a^2}\delta R_{0i} = -\delta R^i_0 + \frac{2}{a^4}h_{0i}\left(\mathcal{H}^2 - \mathcal{H}'\right) , \qquad (3.21)
\end{aligned}$$

$$\begin{aligned}
\delta R^0_0 &= \frac{1}{2a^4}\left[h''_{ii} - 2\partial_i h'_{i0} + \nabla^2 h_{00}\right] + \frac{\mathcal{H}}{a^4}\left[\partial_i h_{i0} - \frac{3}{2}\left(h_{ii} - h_{00}\right)'\right]\\
&\quad - \frac{1}{a^4}\left(\mathcal{H}' - \mathcal{H}^2\right)h_{ii} + \frac{3}{a^4}\left(\mathcal{H}' - \mathcal{H}^2\right)h_{00} , \qquad (3.22)
\end{aligned}$$

Note that we have used $\nabla^2 = \partial_k\partial^k$. Note also that the position of the index is not matched in the above equations. This is due to the background metric, which is in Cartesian coordinates. Therefore,

$\delta^{jk} = \delta^j_k = \delta_{jk}$ and the partial derivative ∂_i is performs with coordinate x^i can be written as $\partial_i = \partial^i$ From the above equations, the Ricci scalar can be obtained as

$$\begin{aligned} \delta R = \delta R^0_0 + \delta R^i_i &= \frac{1}{a^4}\left[h''_{ii} - 2\partial_i h'_{i0} + \nabla^2 h_{00} + \partial_k \partial_i h_{ki} - \nabla^2 h_{ii}\right] \\ &+ \frac{\mathcal{H}}{a^4}\left[3h'_{00} - 2\partial_k h_{k0} - h'_{ii}\right] - 2\frac{1}{a^4}\left(\mathcal{H}' + \mathcal{H}^2\right) h_{ii} + \frac{6}{a^4}\mathcal{H}' h_{00}\,. \end{aligned} \tag{3.23}$$

The evolution of the perturbation in the metric can be followed through the Einstein equation as

$$\delta G^\nu_\mu = \delta R^\nu_\mu - \frac{1}{2}\delta^\nu_\mu \delta R = \delta T^\nu_\mu\,, \tag{3.24}$$

where δG^ν_μ is the perturbed Einstein tensor and δT^ν_μ is the perturbed energy–momentum tensor. From now on we work in units where the reduced Planck mass M_p is set to unity, i.e., $M_p = 1/\sqrt{8\pi G} = 1$. To solve the above equation, we need to know the form and evolution of the perturbation in the energy–momentum tensor, which we study next.

3.1.2 Perturbations of the energy–momentum tensor

From the conservation of the energy–momentum tensor,

$$\nabla_\mu T^\mu_\nu = \partial_\mu T^\mu_\nu + \Gamma^\mu_{\mu\alpha} T^\alpha_\nu - \Gamma^\alpha_{\mu\nu} T^\mu_\alpha = 0\,, \tag{3.25}$$

we can derive the evolution equations for perturbations of the energy–momentum tensor by splitting T^ν_μ into a background part $\bar{T}^\nu_\mu$ and a perturbed part δT^ν_μ as follows:

$$\begin{aligned} \partial_\mu\left(\bar{T}^\mu_\nu + \delta T^\mu_\nu\right) + \left(\bar{\Gamma}^\mu_{\mu\alpha} + \delta\Gamma^\mu_{\mu\alpha}\right)\left(\bar{T}^\alpha_\nu + \delta T^\alpha_\nu\right) - \left(\bar{\Gamma}^\alpha_{\mu\nu} + \delta\Gamma^\alpha_{\mu\nu}\right)\left(\bar{T}^\mu_\alpha + \delta T^\mu_\alpha\right) = 0\,, \\ \Rightarrow \partial_\mu \delta T^\mu_\nu + \bar{\Gamma}^\mu_{\mu\alpha}\delta T^\alpha_\nu + \delta\Gamma^\mu_{\mu\alpha}\bar{T}^\alpha_\nu - \bar{\Gamma}^\alpha_{\mu\nu}\delta T^\mu_\alpha - \delta\Gamma^\alpha_{\mu\nu}\bar{T}^\mu_\alpha = 0\,. \end{aligned} \tag{3.26}$$

Because the background spacetime is homogeneous and isotropic, the background energy–momentum tensor $\bar{T}^\nu_\mu$ takes the perfect-fluid form (as discussed earlier). The $\nu = 0$ component of the above equation gives the energy-conservation equation, namely

$$\partial_\mu \delta T^\mu_0 + 3\mathcal{H}\delta T^0_0 - \mathcal{H}\delta T^i_i - \left(\frac{h_{ii}}{2a^2}\right)'(\bar{\rho} + \bar{p}) = 0\,. \tag{3.27}$$

The momentum-conservation equation follows from the $\nu = i$ component of Eq. (3.26), i.e.,

$$\begin{aligned} \partial_\mu \delta T^\mu_i + 3\mathcal{H}\delta T^0_i - \frac{\partial_i h_{00}}{2a^2}(\bar{\rho} + \bar{p}) - \mathcal{H}\delta_{ij}\delta T^j_0 + \frac{\mathcal{H}}{a^2} h_{0i}(\bar{\rho} + \bar{p}) = 0\,, \\ \partial_\mu \delta T^\mu_i + 4\mathcal{H}\delta T^0_i - \frac{\partial_i h_{00}}{2a^2}(\bar{\rho} + \bar{p}) = 0\,, \end{aligned} \tag{3.28}$$

In the second line we used the relation

$$\delta T^i_0 = -\delta T^0_i + \frac{1}{a^2} h_{0i}(\bar{\rho} + \bar{p})\,. \tag{3.29}$$

<u>Exercise1</u> Using the energy–momentum tensor of a scalar field,

$$T_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi + \left(\frac{1}{2}\nabla_\rho\nabla^\rho\phi + V(\phi)\right) g_{\mu\nu} \tag{3.30}$$

find the linear perturbation of the scalar-field energy–momentum tensor, $\delta T_{\mu\nu}$.

3.2 Scalar, vector, and tensor modes of perturbations

In general, the components of the perturbed metric and the perturbed $T_{\mu\nu}$ can be written in terms of scalar, vector, and tensor functions. In linear perturbation theory, these three types of functions evolve independently. Therefore, we refer to them as the scalar mode, vector mode, and tensor mode of the perturbations. These modes are related to scalar-type perturbations.

3.2.1 Scalar, vector, and tensor modes of the perturbed metric

In this subsection, we decompose the components of the metric perturbation into scalar, vector, and tensor modes. When we separate the time components from the spatial components, we see that the component h_{00} can be written as a scalar function, denoted here by ϕ. Before proceeding, we emphasize that the components of the perturbations in spacetime and in matter–energy depend on both time and space; in particular, ϕ is a function of position and time. In general, the components h_{i0} and h_{0i} can be written as a three-dimensional vector. Any three-vector can be decomposed into two independent vectors: a divergence-free (transverse) vector and a curl-free (longitudinal) vector. Since a longitudinal vector can be written as the gradient of a scalar function, we can write h_{0i} and h_{i0} in terms of scalar and vector functions as

$$h_{i0} = h_{0i} = a^2 \left(\partial_i B - S_i\right) , \tag{3.31}$$

where $\partial^i S_i = 0$ since S_i are transverse. From the background metric in (3.3) we have $\partial_i = \partial^i$ for the flat space. For curved space, we need to change the derivative to be the covariant derivative with the metric $\bar{g}_{ij}/a^2$. By using δ^i_j and partial derivatives, one can write h_{ij} in terms of a scalar function. For the vector, one can write it in terms of partial derivatives. In order to make the vector and scalar independent, the vector must be transverse. In addition, we can write h_{ij} in terms of a three-dimensional tensor. In order for this tensor to be independent of the scalar and vector modes, it must be traceless and divergence-free. With these considerations, we can write $h_{\mu\nu}$ as

$$h_{\mu\nu} = a^2(\tau) \begin{pmatrix} -2\phi & \partial_i B - S_i \\ \partial_j B - S_j & -2\psi\delta_{ij} + 2\partial_i\partial_j E + \partial_j F_i + \partial_i F_j + H_{ij} \end{pmatrix} , \tag{3.32}$$

where $\partial^i S_i = \partial^i F_i = H^i_i = \partial^i H_{ij} = 0$. Using Eq. (3.5) we obtain

$$h^{\mu\nu} = a^{-2}(\tau) \begin{pmatrix} 2\phi & \partial^i B - S^i \\ \partial^j B - S^j & 2\psi\delta^{ij} - 2\partial^i\partial^j E - \partial^j F^i - \partial^i F^j + H^{ij} \end{pmatrix} . \tag{3.33}$$

In the above expression, $a^{-2}H^{ij} = -\bar{g}^{ik}\bar{g}^{jl}a^2 H_{kl} = -a^{-2}H_{ij}$, i.e., $H^{ij} = -H_{ij}$. The decomposition of the perturbed metric into scalar, vector, and tensor functions given above is quite general. However, both ψ and E contribute to the trace of $h_{\mu\nu}$. Therefore, in some cases the component h_{ij} is written in terms of two scalar functions as follows:

$$h_{ij} = h_L\delta_{ij} + \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right) h_T , \tag{3.34}$$

where h_L and h_T are scalar functions and $\nabla^2 = \partial_k\partial^k$. We see that the trace of h_{ij} depends only on h_L, while (for the scalar mode) the traceless part depends only on h_T.

From this decomposition, we see that $h_{\mu\nu}$ contains 4 degrees of freedom from the 4 scalar functions. Because the vectors S_i and F_i are three-vectors, each has 3 components. However, the components are not all independent due to the divergence-free conditions; each vector therefore has 2 degrees of freedom. Thus $h_{\mu\nu}$ contains an additional 4 degrees of freedom from the vectors S_i and F_i.

Moreover, since H_{ij} is a three-dimensional tensor, it has 9 components. Because H_{ij} is symmetric (since $h_{\mu\nu}$ is symmetric), only 6 components are independent. The condition $H^i_i = 0$ reduces this by 1, and the condition $\partial^i H_{ij} = 0$ reduces it by 3 (since $j = 1, 2, 3$), leaving 2 degrees of freedom. Hence $h_{\mu\nu}$ contains 2 additional degrees of freedom from the tensor H_{ij}.

In summary, $h_{\mu\nu}$ has 4 scalar-mode degrees of freedom, 4 vector-mode degrees of freedom, and 2 tensor-mode degrees of freedom, consistent with a symmetric rank-2 tensor in four dimensions having 10 independent components.

3.2.2 Scalar, vector, and tensor modes of the energy–momentum tensor

In this subsection we decompose the perturbed energy–momentum tensor into scalar, vector, and tensor parts. Because the background space is homogeneous and isotropic, the background energy–momentum tensor has the perfect-fluid form.

In a perturbed spacetime, the energy–momentum tensor also acquires perturbations. Small inhomogeneities lead to perturbations in the energy density and pressure, which we write as

$$\rho(\tau,\mathbf{x}) = \bar{\rho}(\tau) + \delta\rho(\tau,\mathbf{x})\,, \quad p(\tau,\mathbf{x}) = \bar{p}(\tau) + \delta p(\tau,\mathbf{x})\,, \tag{3.35}$$

where $\mathbf{x}$ is the three-dimensional position, $\bar{\rho}$ and $\bar{p}$ are the background energy density and pressure (time-dependent but position-independent), and $\delta\rho$ and δp are the perturbations in the energy density and pressure.

A small anisotropy in the three-space implies that the fluid three-velocity is no longer zero; equivalently, there is a perturbation in the three-velocity around a vanishing background value. As in the case of the perturbed metric, we decompose the three-velocity perturbation into a curl-free part and a divergence-free part, i.e.,

$$\partial^i v + v^i\,, \tag{3.36}$$

where v^i is the transverse part of the three-velocity, satisfying $\partial_i v^i = 0$. Therefore, the four-velocity can be written as

$$u^\mu = \frac{1}{a}\left(1-\phi, \partial^i v + v^i\right)\,, \tag{3.37}$$

The term $-\phi$ in the time component appears because the four-velocity must satisfy the normalization condition $u_\mu u^\mu = -1$, which implies

$$g_{\mu\nu}u^\mu u^\nu = a^2\left(-1-2\phi\right)\left(u^0\right)^2 + \text{second-order terms in the perturbations} = -1\,, \tag{3.38}$$

i.e., $u^0 = (1-\phi)/a$. Using $u_\mu = g_{\mu\nu}u^\nu$ we obtain

$$u_\mu = \left(\bar{g}_{\mu\nu} + h_{\mu\nu}\right)u^\nu = a\left(-1-\phi, \partial_i B + v_i + \partial_i v - S_i\right)\,. \tag{3.39}$$

In addition to a nonzero three-velocity, in general a small anisotropy in the matter content can appear as an anisotropic-stress perturbation. This can be described by a traceless tensor, which can be decomposed into a scalar function ${}^{(S)}\Pi$, a transverse vector ${}^{(V)}\Pi$, and a traceless, divergence-free tensor ${}^{(T)}\Pi$ as

$$\Pi^i{}_j = \left(\partial^i\partial_j - \frac{1}{3}\delta^i_j\nabla^2\right){}^{(S)}\Pi + \frac{1}{2}\left(\partial_j{}^{(V)}\Pi^i + \partial^{i(V)}\Pi_j\right) + {}^{(T)}\Pi^i{}_j\,. \tag{3.40}$$

Here we take $\Pi^0_0 = \Pi^0_i = \Pi^i_0 = 0$. We see that, at linear order, an energy–momentum tensor of perfect-fluid form cannot generate anisotropic-stress perturbations. Moreover, isotropy implies $\Pi^j_i = 0$ in the background spacetime.

With the above considerations, we write the energy–momentum tensor as

$$T^\mu_{\ \nu} = (\rho+p)\,u^\mu u_\nu + p\delta^\mu{}_\nu + \Pi^\mu{}_\nu\,, \tag{3.41}$$

and from Eqs. (3.35), (3.37), and (3.39) we can write $\delta T_{\mu\nu}$ as

$$\begin{aligned}
\delta T^0_0 &= -\delta\rho\,, &(3.42)\\
\delta T^0_i &= \left(\bar{\rho}+\bar{p}\right)\left(\partial_i B + \partial_i v + v_i - S_i\right)\,, &(3.43)\\
\delta T^i_0 &= -\left(\bar{\rho}+\bar{p}\right)\left(\partial^i v + v^i\right)\,, &(3.44)\\
\delta T^i_j &= \delta p\ \delta^i_j + \Pi^i_j\,. &(3.45)
\end{aligned}$$

We see that the above expressions for δT^i_0 and δT^0_i are consistent with Eq. (3.29).

3.2.3 Evolution equations for scalar, vector, and tensor modes

The evolution equation of the perturbation for scalar, vector, and tensor modes can be obtained by substituting (3.32), (3.33), (3.42) - (3.45) in Einstein equation (3.24) and conservation equations (3.27) and (3.28)

The purpose of decomposing the perturbed metric and perturbed $T_{\mu\nu}$ into scalar, vector, and tensor modes is to make them independent. In fact, we take a vector perturbation to be a three-vector with vanishing divergence because such a vector cannot be written as a linear combination of scalar functions (unlike a longitudinal vector, which can be written as the gradient of a scalar).

Similarly, we write the tensor mode as a traceless, divergence-free tensor because such a tensor cannot be expressed in terms of scalars and vectors.

Therefore, when we substitute $h_{\mu\nu}$ and δT^{ν}_{μ}—with their components written in terms of scalar, vector, and tensor parts—into the Einstein equations and the conservation equations, the terms associated with scalar, vector, and tensor perturbations are linearly independent. Consequently, we can derive the evolution equation for one type of perturbation without considering the others; for example, we can obtain the evolution of vector perturbations by keeping only the vector part of the equations.

However, the independence of the scalar, vector, and tensor modes holds only at linear order.

As a simple example, consider the term $\partial_\mu \delta T^{\mu}_{0}$ in Eq. (3.27). In linear perturbation theory, the vector mode in this term vanishes because it is acted on by a partial derivative, i.e.,

$$\partial_\mu \delta T^{\mu}_{0} = -\delta\rho' - (\bar{\rho}+\bar{p})\,\partial_i\left(\partial^i v + v^i\right) = -\delta\rho' - (\bar{\rho}+\bar{p})\,\partial_i\partial^i v\,. \tag{3.46}$$

But if we consider nonlinear perturbations, we obtain

$$\begin{aligned}\partial_\mu \delta T^{\mu}_{0} &= -\delta\rho' - \partial_i\left[(\rho+p)\left(\partial^i v + v^i\right)\right] \\ &= -\delta\rho' - \partial_i\left(\delta\rho+\delta p\right)\left(\partial^i v + v^i\right) - (\rho+p)\,\partial_i\partial^i v\,.\end{aligned} \tag{3.47}$$

From this expression we see that the divergence-free condition on v^i does not make the terms involving v^i independent of those involving v. We will understand this point better when we derive the equations for the scalar, vector, and tensor modes in what follows.

Scalar mode

First, we consider scalar perturbations. From the $\mu\nu = 00$ component of the Einstein equation (3.24), we obtain

$$\begin{aligned}&\delta R^0_0 - \frac{1}{2}\delta R = -\delta\rho\,, \\ &\Rightarrow \frac{1}{2}\left(\nabla^2 h_{ii} - \partial_k\partial_i h_{ki}\right) + \mathcal{H}\left(2\partial_i h_{i0} - h'_{ii}\right) + \mathcal{H}^2\left(2h_{ii} - 3h_{00}\right) = -a^4\delta\rho\,,\end{aligned} \tag{3.48}$$

$$\Rightarrow 3\mathcal{H}\left(\psi' + \mathcal{H}\phi\right) - \mathcal{H}\nabla^2\sigma - \nabla^2\psi = -\frac{1}{2}a^2\delta\rho\,, \tag{3.49}$$

where $\sigma = E' - B$ is the scalar shear of spacetime. From the $\mu\nu = 0i$ component of the Einstein equation we obtain

$$\begin{aligned}&\delta R^i_0 = -\left(\bar{\rho}+\bar{p}\right)\partial^i v\,, \\ &\Rightarrow \psi' + \mathcal{H}\phi = -\frac{1}{2}a^2\left(\bar{\rho}+\bar{p}\right)(v+B)\;.\end{aligned} \tag{3.50}$$

In deriving the above equation, we used the relation

$$\left(\mathcal{H}' - \mathcal{H}^2\right) = a^2\left(\frac{\ddot{a}}{a} - H^2\right) = a^2\dot{H} = -4\pi G a^2\left(\bar{\rho}+\bar{p}\right)\;. \tag{3.51}$$

In the case $i \neq j$, the components $\mu\nu = ij$ of Einstein equation are

$$\begin{aligned}&R^j_i = \partial_i\partial^{j\,(S)}\Pi\,, \\ &\Rightarrow \partial_k\partial_i h_{kj} + \partial_k\partial_j h_{ki} - \nabla^2 h_{ij} - \partial_j\partial_i\left(h_{kk} - h_{00}\right) - \left(\partial_i h'_{0j} + \partial_j h'_{0i} - h''_{ij}\right) \\ &\quad - 2\mathcal{H}h'_{ji} - 2\mathcal{H}'h_{ij} = 2a^4\partial_i\partial^{j\,(S)}\Pi\,,\end{aligned} \tag{3.52}$$

$$\Rightarrow \psi - \phi + \sigma' + 2\mathcal{H}\sigma = a^{2\,(S)}\Pi\,, \tag{3.53}$$

Taking the trace of the $\mu\nu = ij$ component of the Einstein equation gives

$$
\begin{aligned}
&\delta R^i_i - \frac{3}{2}\delta R = 3\delta p\,, \\
&\Rightarrow \frac{1}{2}\left[\nabla^2 h_{kk} - \partial_k\partial_i h_{ki} - 2\nabla^2 h_{00} + 4\partial_i h'_{0i} - 2h''_{ij}\right] + \frac{\mathcal{H}}{2}\left[-6h'_{00} + 4h'_{kk}\right] \\
&\quad - 3\left(2\mathcal{H}' - \mathcal{H}^2\right) h_{00} + 2\mathcal{H}' h_{ii} = 3a^4\delta p\,, \qquad (3.54)\\
&\Rightarrow \nabla^2\left[\phi - \psi - \sigma' - 2\mathcal{H}\sigma\right] + 3\psi'' + 6\mathcal{H}\psi' + 3\mathcal{H}\phi' + 3\left(2\mathcal{H}' + \mathcal{H}^2\right)\phi \\
&\quad = \frac{3}{2}a^2\delta p\,, \\
&\Rightarrow \psi'' + 2\mathcal{H}\psi' + \mathcal{H}\phi' + \left(2\mathcal{H}' + \mathcal{H}^2\right)\phi = \frac{1}{2}a^2\left(\delta p + \frac{2}{3}\nabla^{2(S)}\Pi\right)\,. \qquad (3.55)
\end{aligned}
$$

In deriving the last line of the above equation, we used Eq. (3.53).

The evolution equations for the scalar perturbations in the energy density and velocity can be obtained from Eqs. (3.27) and (3.28) as follows:

$$
\delta\rho' + 3\mathcal{H}\left(\delta\rho + \delta p\right) - \left(\bar{\rho} + \bar{p}\right)\left(3\psi' - \nabla^2 E' - \nabla^2 v\right) = 0\,, \qquad (3.56)
$$

$$
\left[\left(\bar{\rho} + \bar{p}\right)\left(v + B\right)\right]' + \delta p + \frac{2}{3}\nabla^{2(S)}\Pi + \left(\bar{\rho} + \bar{p}\right)\left(\phi + 4\mathcal{H}\left(v + B\right)\right) = 0\,. \qquad (3.57)
$$

For scalar perturbations, we see that there are seven variables whose evolution we would like to determine: $\phi, \psi, B, E, \delta\rho, \delta p$, and v[1]. However, there are only four independent evolution equations, because Eqs. (3.50) and (3.57) can be used to derive Eq. (3.55), and Eq. (3.53) can be obtained from Eq. (3.49) together with Eqs. (3.56) and (3.57).

Nevertheless, as we will see in Sec. 3.3, two of the perturbation variables are not physical degrees of freedom associated with physical phenomena. These two degrees of freedom arise from the freedom to choose the hypersurface (equivalently, the freedom to choose coordinates) used to describe the system; we refer to them as gauge degrees of freedom. We discuss gauge issues in cosmological perturbation theory in Sec. 3.3.

Therefore, in the scalar sector there are five physical degrees of freedom, which is one more than the number of independent evolution equations. To solve the evolution equations we need one additional relation among $\phi, \psi, E, B, \delta\rho, \delta p$, and v in order to reduce the number of degrees of freedom to match the number of independent equations. As in solving the Friedmann equations, we can reduce the degrees of freedom by specifying a relation between $\delta\rho$ and δp, which determines the properties of the matter content at the perturbative level. We will study this in Sec. 3.4.

Vector mode

From the $\mu\nu = 0i$ component of the Einstein equation we obtain

$$
\begin{aligned}
&\delta R^i_0 = -\left(\bar{\rho} + \bar{p}\right)v^i\,, \\
&\Rightarrow \frac{1}{2a^4}\left(\partial_j h'_{ij} - \nabla^2 h_{i0}\right) - \frac{\mathcal{H}}{a^4}\partial_j h_{ij} + \frac{2}{a^4}\left(\mathcal{H}^2 - \mathcal{H}'\right)h_{0i} = -\left(\bar{\rho} + \bar{p}\right)v^i\,, \\
&\Rightarrow \nabla^2\left(S_i + F'_i\right) = -2a^2\left(\bar{\rho} + \bar{p}\right)\left(v_i - S_i\right)\,, \qquad (3.58)
\end{aligned}
$$

and from the $i \neq j$ component of the Einstein equation we obtain

$$
\begin{aligned}
&\delta R^j_i = \frac{1}{2}\left(\partial^{j\,(V)}\Pi_i + \partial_i{}^{(V)}\Pi^j\right)\,, \\
&\Rightarrow \mathcal{V}'_{ij} + 2\mathcal{H}\mathcal{V}_{ij} = a^2\left(\partial_j{}^{(V)}\Pi_i + \partial_i{}^{(V)}\Pi_j\right)\,, \qquad (3.59)
\end{aligned}
$$

where

$$
\mathcal{V}_{ij} = \partial_i F'_j + \partial_j F'_i + \partial_i S_j + \partial_j S_i\,. \qquad (3.60)
$$

[1]For simplicity in this introductory discussion, we set $\Pi^j_i = 0$.

Because vector perturbations are divergence-free, there are no vector perturbations in the other components of the Einstein equation. For the same reason, only the $\nu = i$ component of the conservation equation contains vector perturbations, and it takes the form

$$[(\bar{\rho}+\bar{p})(v_i - S_i)]' + \frac{1}{2}\nabla^{2(V)}\Pi_i + 4\mathcal{H}(\bar{\rho}+\bar{p})(v_i - S_i) = 0\,. \tag{3.61}$$

Differentiating both sides of Eq. (3.58) with respect to τ and using Eq. (3.61) to rewrite $v_i - S_i$ in terms of $S_i + F_i'$ yields Eq. (3.59). Hence, only two of the above three equations are independent for vector perturbations.

We will see in the discussion of gauge that, for vector perturbations, $\mathcal{V}$ and $v_i - S_i$ represent physical degrees of freedom. In the case ${}^{(V)}\Pi = 0$, Eq. (3.59) shows that both degrees of freedom decay as the Universe expands. In general, one often assumes that vector perturbations decay and become negligible from the early Universe onward. However, for some forms of matter or in some modified-gravity theories, the amplitude of vector perturbations may not decay with time, or may affect metric perturbations. We will not consider such cases in this school.

Tensor mode

Because tensor perturbations are traceless and divergence-free, only the $i \neq j$ component of the Einstein equation contains tensor modes. From this component we obtain

$$\begin{aligned} &\delta R_i^j = {}^{(T)}\Pi_i^j\,, \\ &\Rightarrow H_{ij}'' + 2\mathcal{H}H_{ij}' - \nabla^2 H_{ij} = 2a^{2(T)}\Pi_{ij}\,, \end{aligned} \tag{3.62}$$

We see that the evolution equation for tensor perturbations has the form of a wave equation. Therefore, tensor perturbations propagate as waves, which are generally called gravitational waves.

3.3 Gauge in cosmological perturbation theory

Although Einstein's gravitational equations and the conservation equations for the energy–momentum tensor are tensor equations—so their form does not depend on the choice of reference frame or coordinates—the explicit form of their components does depend on the coordinates used. Therefore, we should choose coordinates that are suitable for describing the system, e.g., coordinates adapted to the symmetries of the system under consideration. This can simplify the equations of motion obtained from the various components of the tensor equations and may also make the underlying physics more transparent.

For the background Universe, the homogeneity and isotropy of three-dimensional space are associated with invariance under translations and rotations (three of each). These symmetries are sufficient to select a coordinate system appropriate for describing the background Universe. Put simply, for the background we should use coordinates in which the metric tensor $\bar{g}_{\mu\nu}$ is purely diagonal, consistent with the homogeneity and isotropy of three-space.

However, inhomogeneity and anisotropy break the translation and rotation symmetries of the spatial slices. Consequently, in a perturbed spacetime there is no symmetry principle that singles out one preferred coordinate system over another. Nevertheless, by choosing certain coordinates we can set some components of the perturbed metric to zero. This indicates that some degrees of freedom of the perturbations arise from the freedom to choose coordinates rather than from physical degrees of freedom. We call such degrees of freedom gauge degrees of freedom.

In this section, we study how to eliminate gauge degrees of freedom by using the transformation properties of perturbation variables under a gauge transformation. (gauge transformation)

3.3.1 Gauge transformation

Gauge transformation can be written in terms of an infinitesimal coordinate transformation by using the vector ξ^μ as follows

$$\tilde{x}^\mu = x^\mu + \xi^\mu(x)\,. \tag{3.63}$$

Under this transformation, the components of a tensor $\mathcal{T}_{\mu\nu}$ transform as follows

$$\widetilde{\mathcal{T}}_{\mu\nu}\left(\tilde{x}\right)=\frac{\partial x^{\alpha}}{\partial\tilde{x}^{\mu}}\frac{\partial x^{\beta}}{\partial\tilde{x}^{\nu}}\mathcal{T}_{\alpha\beta}(x)\,. \tag{3.64}$$

We see that the tensor components on the left-hand side of this equation are evaluated at the point $\tilde{x}^{\mu}$, whereas the tensor components on the right-hand side are evaluated at the point x^{μ}. To determine how the tensor components change under a gauge transformation, we must compare the components on both sides at the same point. We therefore define

$$\Delta\left(\mathcal{T}_{\mu\nu}\right)(x)=\widetilde{\mathcal{T}}_{\mu\nu}\left(x\right)-\mathcal{T}_{\mu\nu}\left(x\right)\,. \tag{3.65}$$

Using a Taylor expansion, we have

$$\widetilde{\mathcal{T}}_{\mu\nu}\left(\tilde{x}\right)=\widetilde{\mathcal{T}}_{\mu\nu}\left(x\right)+\frac{\partial\bar{\mathcal{T}}_{\mu\nu}(x)}{\partial x^{\alpha}}\xi^{\alpha}+\text{nonlinear perturbation terms}\,, \tag{3.66}$$

where $\bar{\mathcal{T}}_{\mu\nu}$ is the background tensor. Substituting this expansion into Eq. (3.65), we obtain

$$\begin{aligned}\Delta\left(\mathcal{T}_{\mu\nu}\right)(x) &= \widetilde{\mathcal{T}}_{\mu\nu}\left(\tilde{x}\right)-\frac{\partial\bar{\mathcal{T}}_{\mu\nu}(x)}{\partial x^{\alpha}}\xi^{\alpha}-\mathcal{T}_{\mu\nu}\left(x\right)\\ &= \frac{\partial x^{\alpha}}{\partial\tilde{x}^{\mu}}\frac{\partial x^{\beta}}{\partial\tilde{x}^{\nu}}\mathcal{T}_{\alpha\beta}(x)-\frac{\partial\bar{\mathcal{T}}_{\mu\nu}(x)}{\partial x^{\alpha}}\xi^{\alpha}-\mathcal{T}_{\mu\nu}\left(x\right)\\ &= \mathcal{T}_{\mu\nu}\left(x\right)-\frac{\partial\xi^{\alpha}}{\partial x^{\mu}}\bar{\mathcal{T}}_{\alpha\nu}\left(x\right)-\frac{\partial\xi^{\alpha}}{\partial x^{\nu}}\bar{\mathcal{T}}_{\mu\alpha}\left(x\right)-\frac{\partial\bar{\mathcal{T}}_{\mu\nu}(x)}{\partial x^{\alpha}}\xi^{\alpha}-\mathcal{T}_{\mu\nu}\left(x\right)\\ &= -\frac{\partial\xi^{\alpha}}{\partial x^{\mu}}\bar{\mathcal{T}}_{\alpha\nu}\left(x\right)-\frac{\partial\xi^{\alpha}}{\partial x^{\nu}}\bar{\mathcal{T}}_{\mu\alpha}\left(x\right)-\frac{\partial\bar{\mathcal{T}}_{\mu\nu}(x)}{\partial x^{\alpha}}\xi^{\alpha}\,,\end{aligned} \tag{3.67}$$

In the above calculation, we neglected perturbations of higher than first order and used the fact that ξ^{μ} is a first-order perturbation. The quantity $\Delta\left(\mathcal{T}_{\mu\nu}\right)(x)$ therefore represents the change of the perturbed tensor components $\delta\mathcal{T}_{\mu\nu}$ under the gauge transformation in Eq. (3.63). Since the right-hand side depends only on first-order quantities, the background tensor $\bar{\mathcal{T}}_{\mu\nu}$ does not change under the gauge transformation.

To compute how the scalar, vector, and tensor modes of the perturbations in the metric tensor and the energy–momentum tensor transform under a gauge transformation, we decompose the gauge-transformation vector ξ^{μ} into independent scalar and vector parts as follows:

$$\widetilde{\tau}=\tau+\xi^{0}\left(\tau,\mathbf{x}\right)\,,\qquad \widetilde{x}^{i}=x^{i}+\partial^{i}\xi\left(\tau,\mathbf{x}\right)+\xi^{i}\left(\tau,\mathbf{x}\right)\,, \tag{3.68}$$

with $\partial_i\xi^i=0$. Taking $\mathcal{T}_{\mu\nu}=g_{\mu\nu}=\bar{g}_{\mu\nu}+h_{\mu\nu}$, we can show that $h_{\mu\nu}$ transforms under a gauge transformation as follows:

$$\widetilde{h}_{ij} = h_{ij}-2a^2\partial_i\partial_j\xi-a^2\left(\partial_i\xi^j+\partial_j\xi^i\right)-2\mathcal{H}a^2\xi^0\delta_{ij}\,, \tag{3.69}$$

$$\widetilde{h}_{0i} = h_{0i}-a^2\left(\partial_i\xi'+\xi_i'\right)+a^2\partial_i\xi^0\,, \tag{3.70}$$

$$\widetilde{h}_{00} = h_{00}+2a^2\xi^{0\prime}+2\mathcal{H}a^2\xi^0\,. \tag{3.71}$$

Substituting Eq. (3.32) into the above expressions, we obtain

$$\widetilde{\phi} = \phi-\mathcal{H}\xi^0-\xi^{0\prime} \tag{3.72}$$

$$\widetilde{\psi} = \psi+\mathcal{H}\xi^0\,, \tag{3.73}$$

$$\widetilde{B} = B+\xi^0-\xi' \tag{3.74}$$

$$\widetilde{E} = E-\xi\,, \tag{3.75}$$

$$\widetilde{F}_i = F_i-\xi_i\,, \tag{3.76}$$

$$\widetilde{S}_i = S_i+\xi_i'\,. \tag{3.77}$$

The tensor H_{ij} does not change under a gauge transformation, because it has no background part. Using the same method as in the derivation of Eq. (3.67), we can show that

$$\Delta \mathcal{T}_{\mu}^{\nu}(x) = \frac{\partial \xi^{\nu}}{\partial x^{\alpha}} \bar{\mathcal{T}}_{\mu}^{\alpha}(x) - \frac{\partial \xi^{\alpha}}{\partial x^{\mu}} \bar{\mathcal{T}}_{\alpha}^{\nu}(x) - \frac{\partial \bar{\mathcal{T}}_{\mu}^{\nu}(x)}{\partial x^{\alpha}} \xi^{\alpha} \,. \tag{3.78}$$

$\mathcal{T}_{\mu}^{\nu} = T_{\mu}^{\nu}$

$$\widetilde{\delta T}_{i}^{j} = \delta T_{i}^{j} - \bar{p}' \xi^{0} \delta_{i}^{j} \,, \qquad \widetilde{\delta T}_{0}^{0} = \delta T_{0}^{0} + \bar{\rho}' \xi^{0} \,, \tag{3.79}$$

$$\widetilde{\delta T}_{i}^{0} = \delta T_{i}^{0} + (\bar{\rho} + \bar{p}) \partial_{i} \xi^{0} \,, \qquad \widetilde{\delta T}_{0}^{i} = \delta T_{0}^{i} - (\bar{\rho} + \bar{p}) \left(\partial^{i} \xi' + \xi^{i\prime} \right) . \tag{3.80}$$

Using Eqs. (3.42)–(3.45) together with the above relations, we obtain

$$\widetilde{\delta \rho} = \delta \rho - \bar{\rho}' \xi^{0} \,, \qquad \widetilde{\delta p} = \delta p - \bar{p}' \xi^{0} \,, \tag{3.81}$$

$$\widetilde{v} = v + \xi' \,, \qquad \widetilde{v^{i}} = v^{i} + \xi^{i\prime} \,. \tag{3.82}$$

From the above calculation we also see that, since Π_{i}^{j} vanishes in the background Universe, this quantity is gauge-invariant.

3.3.2 Eliminating gauge degrees of freedom

Since the gauge-transformation vector ξ^{μ} can be written in terms of two scalar functions and one transverse vector, we can use a gauge transformation to remove two scalar degrees of freedom and two vector degrees of freedom from the perturbations. This shows that two degrees of freedom in the scalar sector and two degrees of freedom in the vector sector are gauge degrees of freedom.

In Secs. 3.2.3 and 3.2.3 we saw that, after eliminating two scalar and two vector gauge degrees of freedom, the number of degrees of freedom in the scalar and vector perturbations matches the number of independent evolution equations (assuming we also know the relation between δp and $\delta \rho$). Therefore, to solve the evolution equations and study the properties of perturbations, we must eliminate gauge degrees of freedom.

There are two main ways to eliminate gauge degrees of freedom. In the first approach, we perform calculations using linear combinations of perturbation variables chosen so that the resulting variables are gauge-invariant. We call such variables gauge-invariant combinations, and we will study them in the next subsection. The second approach is to fix a gauge, i.e., choose a hypersurface (or coordinate condition) on which to compute the perturbations. Gauge fixing can be implemented by expressing the gauge-transformation vector ξ^{μ} in terms of the perturbation variables. We will study this method in several cases after introducing gauge-invariant combinations.

3.3.3 Gauge-invariant combinations

From Eqs. (3.72)–(3.75) we see that the variables Φ and Ψ, defined in terms of the perturbation variables by

$$\Phi = \phi - \mathcal{H} \sigma - \sigma' \,, \tag{3.83}$$

$$\Psi = \psi + \mathcal{H} \sigma \,, \tag{3.84}$$

are gauge-invariant. The variables Φ and Ψ are called the Bardeen potentials.

For matter perturbations, we can construct the following gauge-invariant combinations:

$$\delta \rho^{\mathrm{GI}} = \delta \rho + \frac{\bar{\rho}'}{\mathcal{H}} \psi = \delta \rho - 3 (1 + w) \bar{\rho} \psi \,, \tag{3.85}$$

$$v^{\mathrm{GI}} = v + E' \,. \tag{3.86}$$

For vector perturbations, we can write gauge-invariant combinations as follows:

$$\mathcal{V}_{i} = S_{i} + F_{i}' \,, \tag{3.87}$$

$$v_{i}^{\mathrm{GI}} = v_{i} - S_{i} \,. \tag{3.88}$$

One can see that we can construct other gauge-invariant variables by using this procedure.

3.3.4 Gauge fixing

In this subsection we eliminate gauge degrees of freedom by fixing a gauge, i.e., by choosing a hypersurface. We will study five commonly used gauges, starting with the conformal Newtonian gauge (conformal Newtonian gauge).

Conformal Newtonian gauge

From Eqs. (3.74) and (3.75) we see that, under a gauge transformation, the shear σ transforms as

$$\widetilde{\sigma} = \sigma - \xi^0 \,. \tag{3.89}$$

Therefore, shifting from an arbitrary hypersurface to a hypersurface with vanishing shear allows us to fix the time component of the gauge vector ξ^μ as

$$\xi^0 = \sigma \,, \tag{3.90}$$

where the perturbation variable on the right-hand side is evaluated in an arbitrary gauge.

Fixing ξ^0 corresponds to fixing the gauge; once the gauge is fixed, the gauge degrees of freedom are removed. Alternatively, we may view this as follows: if we choose to perform calculations on a suitable hypersurface (e.g., one with $\widetilde{\sigma} = 0$), then we obtain a relation between the gauge-transformation vector and the perturbation variables. This reduces the number of degrees of freedom of the perturbation variables, and the degrees of freedom removed are precisely the gauge degrees of freedom.

To fix ξ^i, we impose $\widetilde{E} = 0$ on the hypersurface where $\widetilde{\sigma} = 0$, which gives

$$\xi = E \,. \tag{3.91}$$

From Eqs. (3.90) and (3.91) we see that, for scalar perturbations, the gauge-transformation vector is completely determined by the variables σ and E. This means that the gauge degrees of freedom are completely eliminated. For this gauge, the remaining perturbation variables can be written as

$$\phi_n = \phi - \mathcal{H}\sigma - \sigma' \,, \qquad \psi_n = \psi + \mathcal{H}\sigma \,, \tag{3.92}$$

$$\delta\rho_n = \delta\rho - \bar{\rho}'\sigma \,, \qquad v_n = v + E' \,, \tag{3.93}$$

Here the subscript $_n$ denotes quantities in the conformal Newtonian gauge, while the variables on the right-hand side of the above equations are evaluated in an arbitrary gauge. We call this gauge the conformal Newtonian gauge because, on sufficiently small scales, the $\mu\nu = 00$ component of the Einstein equation takes a form analogous to the Poisson equation for Newtonian gravity, with ϕ_n and ψ_n playing the role of gravitational potentials. In some references, this gauge is also called the longitudinal gauge or the orthogonal zero-shear gauge.

Synchronous gauge

The synchronous gauge was the first gauge proposed together with the development of cosmological perturbation theory by Lifshitz. In this gauge, we choose to work on hypersurfaces such that perturbations appear only in the spatial part, i.e., we impose $\widetilde{\phi} = \widetilde{B} = 0$. Therefore, we can write ξ^0 and ξ in terms of the perturbation variables as

$$\left(a\xi^0\right)' = a\phi \,, \quad \Rightarrow \quad \xi^0 = \frac{1}{a}\int d\tau a\phi \,, \tag{3.94}$$

$$\xi' = B + \xi^0 \,, \quad \Rightarrow \quad \xi = \int d\tau \left(B + \xi^0\right) \,. \tag{3.95}$$

From the above equations we see that the synchronous-gauge conditions $\widetilde{\phi} = \widetilde{B} = 0$ do not completely fix ξ^0 and ξ, because the conditions remain true even if ξ^0 and ξ are shifted by

$$\widetilde{\widetilde{\xi}}{}^0 = \xi^0 + \frac{f(\mathbf{x})}{a} \,, \quad \text{and} \quad \widetilde{\widetilde{\xi}} = \xi + f(\mathbf{x})\int \frac{d\tau}{a} \,, \tag{3.96}$$

where $f(\mathbf{x})$ is an arbitrary function of position $\mathbf{x}$. We call this the residual gauge freedom. This shows that the conditions $\widetilde{\phi} = \widetilde{B} = 0$ are not sufficient to eliminate the gauge degrees of freedom completely.

To find additional conditions that completely remove the gauge freedom, we first write the perturbation variables using the gauge choice in Eqs. (3.94) and (3.95), and then perform an additional gauge transformation given by Eq. (3.96), obtaining

$$\widetilde{\widetilde{\psi}} = \left[\psi + \frac{\mathcal{H}}{a} \int d\tau a\phi\right] + \mathcal{H}\frac{f(\mathbf{x})}{a}\,, \tag{3.97}$$

$$\widetilde{\widetilde{E}} = \left[E - \int d\tau \left(B + \frac{1}{a} \int d\bar{\tau} a\phi\right)\right] - f(\mathbf{x}) \int \frac{d\tau}{a}\,, \tag{3.98}$$

$$\widetilde{\widetilde{\delta\rho}} = \left[\delta\rho - \frac{\bar{\rho}'}{a} \int d\tau a\phi\right] - \bar{\rho}'\frac{f(\mathbf{x})}{a}\,, \tag{3.99}$$

$$\widetilde{\widetilde{v}} = \left[v + B + \frac{1}{a} \int d\tau a\phi\right] + \frac{f(\mathbf{x})}{a}\,. \tag{3.100}$$

These equations show that the perturbation variables depend on the residual gauge freedom, and the terms in parentheses give the expressions for the perturbation variables in the synchronous gauge when $f(\mathbf{x}) = 0$. From the above equation we see that, if we perform calculations in a frame in which the velocity perturbation of some matter component vanishes ($\widetilde{\widetilde{v}} = 0$), then

$$v + B + \frac{1}{a} \int d\tau a\phi = v_{\rm syn} = -\frac{f(\mathbf{x})}{a}\,. \tag{3.101}$$

That is, the residual gauge freedom can be written in terms of the velocity perturbation in the synchronous gauge. This means that, if we specify $v_{\rm syn}$, we can completely eliminate the residual gauge freedom. Substituting the relation between $f(\mathbf{x})$ and $v_{\rm syn}$ from Eq. (3.101) into Eqs. (3.97)–(3.99) shows that the condition $\widetilde{\widetilde{v}} = 0$ corresponds to the rest frame of the fluid (see Sec. 3.3.4). In practice, when using the synchronous gauge one often works in the frame where the dark-matter velocity perturbation vanishes (i.e., $v_{\rm syn}$ for dark matter is set to zero) [**?**].

Uniform curvature gauge

In the uniform curvature gauge, we set $\widetilde{\psi} = \widetilde{E} = 0$, i.e., there are no perturbations in the three-dimensional spatial metric. Therefore, we can fix the gauge-transformation vector as

$$\xi^0 = -\frac{\psi}{\mathcal{H}}\,, \qquad \xi = E\,. \tag{3.102}$$

With this choice, the other perturbation variables can be written as

$$\phi_f = \phi + \psi + \left(\tfrac{\psi}{\mathcal{H}}\right)'\,, \qquad B_f = B - \frac{\psi}{\mathcal{H}} - E'\,, \tag{3.103}$$

$$\delta\rho_f = \delta\rho - 3\,(1+w)\,\bar{\rho}\psi\,, \qquad v_f = v + E'\,. \tag{3.104}$$

This gauge is sometimes called the spatially flat gauge. It is commonly used to compute perturbations during inflation.

Comoving orthogonal gauge

In the three gauges studied above, the gauge degrees of freedom were eliminated by setting certain components of the perturbed metric to zero. In the next two gauges, we eliminate gauge degrees of freedom by imposing conditions on the matter perturbations.

If we work in the rest frame of the matter, the fluid three-velocity vanishes; i.e., in that frame $\widetilde{v} = 0$. This condition can be used to eliminate one gauge degree of freedom. To eliminate the other gauge degree of freedom, we require the matter four-velocity to be orthogonal to hypersurfaces of constant time, which implies $\widetilde{v + B} = 0$. These two conditions define the comoving orthogonal gauge (often simply called the comoving gauge). With these conditions, the gauge transformation is fixed by

$$\begin{aligned} \xi^0 &= -(v+B) \,, \\ \xi' &= -v \,, \quad \Rightarrow \quad \xi = -\int v d\tau + g(\mathbf{x}) \,, \end{aligned} \tag{3.105}$$

where $g(\mathbf{x})$ is a residual gauge freedom. However, this residual freedom does not appear in the definitions of other perturbation variables that are relevant for the physics of the perturbations; i.e., $g(\mathbf{x})$ does not appear in

$$\phi_m = \phi + \mathcal{H}(v+B) + (v+B)' \,, \qquad \psi_m = \psi - \mathcal{H}(v+B) \,, \tag{3.106}$$

$$\sigma_m = \sigma + (v+B) \,, \qquad \delta\rho_m = \delta\rho + \bar{\rho}'(v+B) \,. \tag{3.107}$$

Although E in this gauge depends on $g(\mathbf{x})$, a more careful examination shows that the residual gauge freedom has no effect on the physics or the evolution of the perturbations. This is because physical quantities such as the spacetime shear, as well as the perturbation evolution equations, depend on E' only through σ. Hence, the gauge freedom relevant for the physics is completely removed in this gauge.

The rest frame considered here may be the rest frame of a single matter component or the total-matter rest frame. The total-matter rest frame is the frame in which the total three-velocity vanishes. The total three-velocity can be obtained by summing the energy–momentum tensors of the various components as

$$\left(\delta T^i_0\right)_{\rm tot} = \sum_\alpha \left(\delta T^i_0\right)_\alpha \,, \quad \Rightarrow \quad v_{\rm tot} = \frac{\sum_\alpha (\bar{\rho}_\alpha + \bar{p}_\alpha)\, v_\alpha}{\bar{\rho}_{\rm total} + \bar{p}_{\rm total}} \,, \tag{3.108}$$

where the subscript $_{\rm tot}$ denotes total quantities and the index α labels the different matter components.

Uniform density gauge

The uniform density gauge (uniform density gauge) is defined on hypersurfaces where the energy-density perturbation vanishes ($\widetilde{\delta\rho} = 0$). On such hypersurfaces we have

$$\xi^0 = \frac{\delta\rho}{\bar{\rho}'} \,. \tag{3.109}$$

The remaining gauge freedom can be eliminated by imposing $\widetilde{E} = 0$, i.e.,

$$\xi = E \tag{3.110}$$

The other perturbation variables in this gauge can be written as

$$\phi_{\delta\rho} = \phi - \mathcal{H}\tfrac{\delta\rho}{\bar{\rho}'} - \left(\tfrac{\delta\rho}{\bar{\rho}'}\right)' \,, \qquad \psi_{\delta\rho} = \psi + \mathcal{H}\frac{\delta\rho}{\bar{\rho}'} \,, \tag{3.111}$$

$$B_{\delta\rho} = B + \tfrac{\delta\rho}{\bar{\rho}'} - E' \,, \qquad v_{\delta\rho} = v + E' \,. \tag{3.112}$$

As in the comoving orthogonal gauge, the hypersurface in this gauge may be defined either by setting $\delta\rho$ of a single matter component to zero or by setting the total $\delta\rho$ to zero.

3.4 Conservation of perturbations on large scales

In this section we study the conservation of perturbations on large scales. Here, "large scales" means length scales larger than the Hubble radius. The Hubble radius is defined as the inverse of the Hubble parameter, H^{-1}. In the following subsections, we will see that the Hubble radius sets the characteristic scale that controls the evolution of perturbations: perturbations evolve differently on scales smaller or larger than the Hubble radius.

Before studying the conservation of perturbations, we first introduce the fact that scalar perturbations can be divided into two types: adiabatic perturbations and entropy perturbations. These two types are independent of each other and play a direct role in the conservation of perturbations on large scales.

3.4.1 adiabatic and entropy perturbation

In general, the adiabatic perturbation and entropy perturbation can be rooted in the separation of the pressure perturbation δp into a part determined by the energy-density perturbation $\delta\rho$ and a part associated with entropy perturbations. This idea follows from the thermodynamic relation between changes in pressure, density, and entropy,

$$\mathrm{d}p = \left(\frac{\partial p}{\partial \rho}\right)\bigg|_{\rho_0,S_0} \mathrm{d}\rho + \left(\frac{\partial p}{\partial S}\right)\bigg|_{\rho_0,S_0} \mathrm{d}S\,, \tag{3.113}$$

where S denotes the entropy. Since the Universe contains multiple forms of matter–energy, adiabatic and entropy perturbations may refer either to perturbations associated with a single component or to perturbations of the total pressure; we will discuss this below.

In the previous discussion, we described matter–energy in terms of a fluid with energy–momentum tensor given by Eq. (3.41). For convenience, in the rest of this chapter we will refer to the matter–energy content as a "fluid" unless stated otherwise.

Adiabatic and entropy perturbations within a single fluid

In this subsection we study adiabatic and entropy perturbations within a single fluid. This will help us determine the relation between entropy perturbations and the energy-density perturbation of an arbitrary fluid. This relation is needed to solve the evolution equations studied in Sec. 3.2.3.

We assume an equation of state

$$P = w\rho\,, \tag{3.114}$$

which we allow to hold in the general inhomogeneous case where ρ, p, and w depend on both position and time.

First, consider the case where the equation-of-state parameter is constant. From the equation above, we obtain

$$\delta p = w\delta\rho\,, \tag{3.115}$$

i.e., in this case δp depends only on $\delta\rho$. Equivalently, the pressure perturbation is purely adiabatic.

Next we consider the general case in which w need not be constant. To find the relation between δp and $\delta\rho$ we can use the transformation of δp and $\delta\rho$ in (3.81) to construct the gauge invariant as

$$\widetilde{\delta p} - c_a^2\widetilde{\delta\rho} = \delta p - c_a^2\delta\rho\,, \tag{3.116}$$

where

$$c_a = \sqrt{\frac{\bar{p}'}{\bar{\rho}'}}\,, \tag{3.117}$$

is the adiabatic sound speed.

If we impose $\widetilde{\delta\rho} = 0$, i.e., consider a gauge transformation from an arbitrary hypersurface to a hypersurface on which the energy-density perturbation vanishes, then $\widetilde{\delta p}$ on that hypersurface depends only on entropy perturbations (since there is no energy-density perturbation there). Hence $\widetilde{\delta p}$ represents the entropy part of the pressure perturbation.

In general we define the intrinsic (single-fluid) entropy perturbation Γ_{int} by

$$\bar{\rho}\Gamma_{\text{int}} = \delta p - c_a^2\delta\rho\,, \tag{3.118}$$

At this point we see that, in the absence of intrinsic entropy perturbations, we have

$$\delta p = c_a^2\delta\rho\,. \tag{3.119}$$

We conclude that the equation above gives the relation between δp and $\delta\rho$ in the case where δp is purely adiabatic (denoted δp_{adi}). It is easy to see that if w is constant then $c_a^2 = w$ and Eq. (3.119) reduces to Eq. (3.115).

To obtain an expression for the entropy part of δp, we use the conclusion from Eq. (3.116) that the entropy pressure perturbation is gauge-invariant. In principle, δp of the entropy type can be computed once the fluid model is specified; but to keep the discussion general, we assume the following suitable general form.

As discussed above, we need a relation between δp and $\delta\rho$ in order to solve the perturbation evolution equations. From Eq. (3.119) we see that δp is adiabatic when it depends only on $\delta\rho$. Therefore, for entropy perturbations δp should be a function of both $\delta\rho$ and v. Moreover, Eq. (3.116) implies that the entropy part is gauge-invariant. For these two reasons, we write the entropy pressure perturbation as

$$\delta p_{\rm ent} \propto \delta\rho - 3\mathcal{H}\left(\bar\rho + \bar p\right)\left(v + B\right) \, . \tag{3.120}$$

We choose this form because the right-hand side is the simplest combination of $\delta\rho$ and v that is gauge-invariant. It also has a physical meaning: it corresponds to $\delta\rho$ evaluated in the fluid rest frame.

For the proportionality coefficient we use $(c_e^2 - c_a^2)$ so that c_e becomes the sound speed in the fluid rest frame. This definition becomes clear when we combine Eqs. (3.119) and (3.120) as follows:

$$\delta p = \delta p_{\rm adi} + \delta p_{\rm ent} = c_e^2\delta\rho - 3\mathcal{H}\left(c_e^2 - c_a^2\right)\left(\bar\rho + \bar p\right)\left(v + B\right) \, . \tag{3.121}$$

This is a general expression for δp in terms of $\delta\rho$ and v for an arbitrary fluid. The quantities c_e^2 and c_a^2 characterize the perturbative properties of the fluid. We see that, in the fluid rest frame, the second term on the right-hand side of the second equality vanishes, so that $\delta p = c_e^2\delta\rho$. Therefore c_e is the sound speed in the fluid rest frame. We also see that if w is constant then $c_e^2 = c_a^2 = w$, and if δp is purely adiabatic then $c_e^2 = c_a^2$.

Adiabatic and entropy perturbations between two fluids

When we study perturbations in a system containing more than one fluid, we find that another type of entropy perturbation can arise. For simplicity, we consider two fluids as an example.

The perturbation in the total pressure of the two-fluid system can be obtained by adding the energy–momentum tensors of the two fluids, giving

$$\delta p_{\rm tot} = \delta p_1 + \delta p_2 \, , \tag{3.122}$$

where the subscripts $_1$, $_2$, and $_{\rm tot}$ denote fluid 1, fluid 2, and the total fluid, respectively.

We can compute the entropy perturbation of the total fluid in the same way as in Eq. (3.118):

$$\bar\rho_{\rm tot}\Gamma_{\rm tot} = \delta p_{\rm tot} - c_a^2\delta\rho_{\rm tot} \, , \tag{3.123}$$

where

$$c_a^2 = \frac{\bar p'_{\rm tot}}{\bar\rho'_{\rm tot}} = \frac{\bar p'_1 + \bar p'_2}{\bar\rho'_1 + \bar\rho'_2} \, , \tag{3.124}$$

is the adiabatic sound speed squared of the total fluid.

Subtracting the intrinsic entropy perturbations of each fluid from the total entropy perturbation, we obtain

$$\begin{aligned} \bar\rho_{\rm tot}\Gamma_{\rm rel} &= \bar\rho_{\rm tot}\Gamma_{\rm tot} - \bar\rho_1\Gamma_{{\rm int}\,1} - \bar\rho_2\Gamma_{{\rm int}\,2} \\ &= \delta p_{\rm tot} - c_a^2\delta\rho_{\rm tot} - \delta p_1 - \delta p_2 + c_{a\,1}^2\delta\rho_1 + c_{a\,2}^2\delta\rho_2 \\ &= -\left(c_{a\,1}^2 - c_{a\,2}^2\right)\frac{\bar\rho'_1\bar\rho'_2}{\bar\rho'_1 + \bar\rho'_2}\left(\frac{\delta\rho_1}{\bar\rho'_1} - \frac{\delta\rho_2}{\bar\rho'_2}\right) \, , \end{aligned} \tag{3.125}$$

Here $\Gamma_{\rm rel}$ is an entropy perturbation generated by the relative evolution of the perturbations in the two fluids, and it vanishes when

$$S_{12} = -\left(\frac{\delta\rho_1}{\bar\rho'_1} - \frac{\delta\rho_2}{\bar\rho'_2}\right) = \frac{1}{3\mathcal{H}}\left(\frac{\delta\rho_1}{(1+w_1)\,\bar\rho_1} - \frac{\delta\rho_2}{(1+w_2)\,\bar\rho_2}\right) = 0 \, . \tag{3.126}$$

In general, we define the relative entropy perturbation as

$$S_{\alpha\beta} = \frac{\delta\rho_\alpha}{(1+w_\alpha)\,\bar{\rho}_\alpha} - \frac{\delta\rho_\beta}{(1+w_\beta)\,\bar{\rho}_\beta}\,. \tag{3.127}$$

The relative entropy perturbation between fluids α and β vanishes when $S_{\alpha\beta} = 0$.

In summary, for a two-fluid system the entropy perturbation contains both (i) intrinsic entropy perturbations within each fluid (which characterize the perturbative properties of that fluid) and (ii) entropy perturbations arising from the relative evolution between perturbations in the two fluids. The concept of relative entropy perturbations introduced here can be straightforwardly generalized to systems with more than two fluids.

Definitions of adiabatic and isocurvature perturbations

In the previous two subsections we studied the decomposition of perturbations into adiabatic and entropy parts. To ensure that entropy perturbations are independent of adiabatic perturbations, we need to impose an additional condition on the entropy perturbations.

Suppose the Universe contains N matter–energy components. We define an isocurvature perturbation as an entropy perturbation for which only one pair of fluids has a nonzero relative entropy perturbation, while the relative entropy perturbations for all other pairs vanish, and the curvature perturbation in the total-fluid rest frame, ψ_m, also vanishes. This can be written as

$$\psi_m = 0\,, \quad \text{and} \quad S_{\alpha\beta} = 0\,, \quad \text{except for the chosen pair of fluids } \alpha,\beta\,. \tag{3.128}$$

We see that N fluids can have $N(N-1)/2$ independent isocurvature modes. These isocurvature perturbations are independent of adiabatic perturbations, which in this case are defined by

$$S_{\alpha\beta} = 0\,, \text{for all}\,\alpha,\beta\,, \quad \text{and} \quad \psi_m \neq 0\,. \tag{3.129}$$

However, it is not all $N(N-1)/2$ mode of the isocurvature perturbation are physically interesting

3.4.2 conservation of curvature perturbation at large scale

In this subsection, we will show the the curvature perturbation in the uniform fluid frame at large-scale perturbation is conserved. The perturbation can be defined from (3.106) as

$$\mathcal{R} \equiv \psi_m = \psi - \mathcal{H}\,(v_{\rm tot} + B) = \psi - \mathcal{H}\frac{\sum_\alpha (\bar{\rho}_\alpha + \bar{p}_\alpha)\,(v_\alpha + B)}{\bar{\rho}_{\rm tot} + \bar{p}_{\rm tot}}\,, \tag{3.130}$$

where we have used (3.108) in order to compute the right-hand side and used $\mathcal{R}$ to denote the curvature perturbation in the uniform fluid frame. To find the evolution equation for $\mathcal{R}$, let us take the derivative with respect to τas follows

$$\mathcal{R}' = \psi' - \mathcal{H}'\frac{\sum_\alpha (\bar{\rho}_\alpha + \bar{p}_\alpha)\,(v_\alpha + B)}{\bar{\rho}_{\rm tot} + \bar{p}_{\rm tot}} - \mathcal{H}\left[\frac{\sum_\alpha (\bar{\rho}_\alpha + \bar{p}_\alpha)\,(v_\alpha + B)}{\bar{\rho}_{\rm tot} + \bar{p}_{\rm tot}}\right]'\,. \tag{3.131}$$

Using the Friedmann equation in conformal time for a spatially flat Universe,

$$\mathcal{H}^2 = \frac{1}{3}a^2\sum_\alpha \bar{\rho}_\alpha\,, \tag{3.132}$$

we obtain

$$\mathcal{H}' = -\frac{1}{2}a^2\,(\bar{\rho}_{\rm tot} + \bar{p}_{\rm tot}) + \mathcal{H}^2\,. \tag{3.133}$$

In deriving the above equation, we used Eq. (**??**). The third term on the right-hand side of Eq. (3.131) can be computed using Eqs. (**??**) and (3.57) as follows:

$$\begin{aligned}\mathcal{H}\left[\frac{\sum_\alpha (\bar{\rho}_\alpha + \bar{p}_\alpha)\,(v_\alpha + B)}{\bar{\rho}_{\rm tot} + \bar{p}_{\rm tot}}\right]' &= 3\mathcal{H}^2\left(1 + c_a^2\right)(v_{\rm tot} + B) - \mathcal{H}\phi - 4\mathcal{H}^2\,(v_{\rm tot} + B) \\ &\quad - \frac{\mathcal{H}}{\bar{\rho}_{\rm tot} + \bar{p}_{\rm tot}}\sum_\alpha\left(\delta p_\alpha + \frac{2}{3}\nabla^{2(S)}\Pi_\alpha\right)\,, \end{aligned} \tag{3.134}$$

In the above calculation we expressed $\bar{p}'_{\rm tot}$ in terms of $\bar{\rho}'_{\rm tot}$ using Eq. (3.124). The term ψ' in Eq. (3.131) can be obtained from Eq. (3.50) as

$$\psi' = -\mathcal{H}\phi - \frac{1}{2}a^2\left(\bar{\rho}_{\rm tot} + \bar{p}_{\rm tot}\right)\left(v_{\rm tot} + B\right) . \tag{3.135}$$

Substituting the results from Eqs. (3.133), (3.135), and (3.134) into Eq. (3.131), we obtain

$$\mathcal{R}' = \frac{\mathcal{H}}{\bar{\rho}_{\rm tot} + \bar{p}_{\rm tot}}\sum_\alpha \left(\delta p_\alpha + \frac{2}{3}\nabla^{2(S)}\Pi_\alpha\right) - 3\mathcal{H}^2 c_a^2\left(v_{\rm tot} + B\right) . \tag{3.136}$$

To study the evolution of $\mathcal{R}$ more conveniently, we rewrite $v_{\rm tot} + B$ in terms of $\delta\rho_{\rm tot}$. Using the equation obtained by subtracting Eq. (3.50) from Eq. (3.49), we have

$$\mathcal{H}\nabla^2\sigma + \nabla^2\psi = \frac{1}{2}a^2\left[\delta\rho_{\rm tot} - 3\mathcal{H}\left(\bar{\rho}_{\rm tot} + \bar{p}_{\rm tot}\right)\left(v_{\rm tot} + B\right)\right] , \tag{3.137}$$

Using this relation, we can write Eq. (3.136) as

$$\mathcal{R}' = \frac{\mathcal{H}}{\bar{\rho}_{\rm tot} + \bar{p}_{\rm tot}}\left[\delta p_{\rm tot} - c_a^2\delta\rho_{\rm tot} + \sum_\alpha \frac{2}{3}\nabla^{2(S)}\Pi_\alpha + 2c_a^2 a^{-2}\left(\mathcal{H}\nabla^2\sigma + \nabla^2\psi\right)\right] . \tag{3.138}$$

On large scales, the terms involving ∇^2 are small and can be neglected, because they scale as the inverse length scale squared, $1/L^2$. Therefore, on large scales we obtain

$$\mathcal{R}' = \frac{\mathcal{H}}{\bar{\rho}_{\rm tot} + \bar{p}_{\rm tot}}\left(\delta p_{\rm tot} - c_a^2\delta\rho_{\rm tot}\right) = \frac{\mathcal{H}}{\bar{\rho}_{\rm tot} + \bar{p}_{\rm tot}}\bar{p}_{\rm tot}\Gamma_{\rm tot} = \frac{\mathcal{H}}{\bar{\rho}_{\rm tot} + \bar{p}_{\rm tot}}\delta p_{\rm non} , \tag{3.139}$$

where $\delta p_{\rm non}$ is the non-adiabatic pressure perturbation. We see from this equation that if the entropy perturbation vanishes (equivalently, the pressure perturbation is adiabatic), then $\mathcal{R}$ is conserved on large scales.

3.5 scalar field perturbation

Previously, we derived the perturbation of the energy–momentum tensor for a general fluid. In this subsection, we derive the perturbation of the energy–momentum tensor for the inflaton field, which is given by

$$T_{\mu\nu} = \partial_\mu\varphi\,\partial_\nu\varphi - g_{\mu\nu}\left(\frac{1}{2}g^{\alpha\beta}\,\partial_\alpha\varphi\,\partial_\beta\varphi + V(\varphi)\right) , \tag{3.140}$$

Substituting the FLRW metric, we obtain the following nonzero components:

$$\begin{aligned} T_{00} &= \frac{1}{2}\varphi'^2 + V(\varphi)\,a^2, \\ T_{0i} &= 0, \\ T_{ij} &= \left(\frac{1}{2}\varphi'^2 - V(\varphi)\,a^2\right)\delta_{ij}. \end{aligned} \tag{3.141}$$

For the perturbations, we use the perturbed metric and expand to linear order. Finally, we obtain

$$\begin{aligned} \delta T_{\mu\nu} &= \partial_\mu\delta\varphi\,\partial_\nu\varphi + \partial_\mu\varphi\,\partial_\nu\delta\varphi - \delta g_{\mu\nu}\left(\frac{1}{2}g^{\alpha\beta}\,\partial_\alpha\varphi\,\partial_\beta\varphi + V(\varphi)\right) \\ &- g_{\mu\nu}\left(\frac{1}{2}\delta g^{\alpha\beta}\,\partial_\alpha\varphi\,\partial_\beta\varphi + g^{\alpha\beta}\,\partial_\alpha\delta\varphi\,\partial_\beta\varphi + \frac{\partial V}{\partial\varphi}\,\delta\varphi + \frac{\partial V}{\partial\varphi}\delta\varphi\right) . \end{aligned} \tag{3.142}$$

The individual components can be written as follows:

$$\delta T_{00} = \delta\varphi'\,\varphi' + 2\,\phi\,V(\varphi)\,a^2 + a^2\,\frac{\partial V}{\partial\varphi}\,\delta\varphi, \tag{3.143}$$

$$\delta T_{0i} \,=\, \partial_i \,\delta\varphi\,\varphi' \,+\, \frac{1}{2}\,\partial_i B\,\varphi'^2 \,-\, \partial_i B\, V(\varphi)\,a^2, \tag{3.144}$$

$$\begin{aligned}\delta T_{ij} \;&=\; \left(\delta\varphi'\,\varphi' \,-\, \phi\,\varphi'^2 \,-\, a^2\,\frac{\partial V}{\partial\varphi}\,\delta\varphi \,-\, \psi\,\varphi'^2 \,+\, 2\,\psi\,V(\varphi)\,a^2\right)\delta_{ij}\\ &+\; \frac{1}{2}\,\partial_i\partial_j E\,\varphi'^2 \,-\, \partial_i\partial_j E\,V(\varphi)\,a^2.\end{aligned} \tag{3.145}$$

For later convenience, we compute the mixed-index components as follows:

$$\begin{aligned}\delta T^\mu_\nu \;&=\; \delta(g^{\mu\alpha}\,T_{\alpha\nu})\\ &=\; \delta g^{\mu\alpha}\,T_{\alpha\nu} \,+\, g^{\mu\alpha}\,\delta T_{\alpha\nu},\end{aligned} \tag{3.146}$$

or

$$\begin{aligned}\delta T^0_0 \;&=\; \phi\,\varphi'^2 \,-\, \delta\varphi'\,\varphi' \,-\, \delta\varphi\,\frac{\partial V}{\partial\varphi}\,a^2,\\ \delta T^i_0 \;&=\; \partial^i B\,\varphi'^2 \,+\, \partial^i\delta\varphi\,\varphi',\\ \delta T^0_i \;&=\; -\,\partial^i\delta\varphi\,\varphi',\\ \delta T^i_j \;&=\; \left(-\,\phi\,\varphi'^2 \,+\, \delta\varphi'\,\varphi' \,-\, \delta\varphi\,\frac{\partial V}{\partial\varphi}\,a^2\right)\delta^i_j.\end{aligned} \tag{3.147}$$

Since a scalar field has its own equation of motion, we can perturb the Klein–Gordon equation. The background equation can be written as

$$\frac{1}{\sqrt{-g}}\,\partial_\nu\left(\sqrt{-g}\,g^{\mu\nu}\,\partial_\nu\varphi\right) \,=\, \frac{\partial V}{\partial\varphi}, \tag{3.148}$$

which, in the FLRW metric, becomes

$$\varphi'' \,+\, 2\,\frac{a'}{a}\,\varphi' \;=\; -\,\frac{\partial V}{\partial\varphi}\,a^2. \tag{3.149}$$

The variation of Eq. (3.148) can be performed using $\frac{1}{\sqrt{-g}}$, $\sqrt{-g}$, $g^{\mu\nu}$, and φ. We then obtain

$$\delta g \;=\; g\,g^{\mu\nu}\,\delta g_{\nu\mu}, \tag{3.150}$$

and at the linear order, we obtain

$$\begin{aligned}\delta\sqrt{-g} \;&=\; -\,\frac{\delta g}{2\,\sqrt{-g}},\\ \delta\,\frac{1}{\sqrt{-g}} \;&=\; \frac{\delta\sqrt{-g}}{g}.\end{aligned} \tag{3.151}$$

Substituting this result into (3.149, we have

$$\begin{aligned}\delta\partial_\mu\partial^\mu\,\varphi \;&=\; -\,\delta\varphi'' \,-\, 2\,\frac{a'}{a}\delta\varphi' \,+\, \partial_i\partial^i\delta\varphi \,+\, 2\,\phi\,\varphi'' \,+\, 4\,\frac{a'}{a}\phi\,\varphi' \,+\, \phi'\varphi'\\ &+\; 3\,\psi'\varphi' \,+\, \partial_i\partial^i B\,\varphi'\\ &=\; \delta\varphi\,\frac{\partial^2 V}{\partial\varphi^2}\,a^2.\end{aligned} \tag{3.152}$$

Multiplying ϕ in background equation (3.149), we obtain

$$2\,\phi\,\varphi'' \,+\, 4\,\frac{a'}{a}\phi\varphi' \;=\; 2\,\phi\,\frac{\partial V}{\partial\varphi}a^2, \tag{3.153}$$

By using this equation, the equation (3.152) becomes

$$\begin{aligned}\delta\varphi'' \,+\, 2\,\frac{a'}{a}\delta\varphi' \;&-\; \partial_i\partial^i\delta\varphi \,-\, \varphi'\varphi' \,-\, 3\,\psi'\varphi' \,-\, \partial_i\partial^i B\,\varphi'\\ &=\; -\delta\varphi\,\frac{\partial^2 V}{\partial\varphi^2}\,a^2 \,-\, 2\,\varphi\,\frac{\partial V}{\partial\varphi}.\end{aligned} \tag{3.154}$$

3.5.1 Gauge for scalar fields

Under a coordinate transformation, the scalar-field perturbation transforms as

$$\tilde{\delta\varphi} = \delta\varphi - \varphi'\delta\xi^0 \tag{3.155}$$

In some cases, we can choose a gauge such that the hypersurface satisfies $\tilde{\delta\varphi} = 0$.

3.5.2 The comoving curvature perturbation

One important quantity that is useful to compute during inflation is the curvature perturbation, which is directly related to the intrinsic spatial curvature on constant-time hypersurfaces, i.e.,

$${}^{(3)}R = \frac{4}{a^2}\nabla^2\,\psi.$$

Under a gauge transformation, the curvature perturbation transforms as

$$\tilde{\psi} = \psi + \mathcal{H}\,\xi^0$$

In this gauge we choose $\tilde{\delta\varphi} = 0$, which is the origin of the name "comoving": we work in the frame comoving with the scalar field. Moreover, in this frame there is no energy flux because $T_{0i} = 0$, and one may approximate T_{0i} as $\partial_i\delta\varphi(\mathbf{x},\tau)\varphi'(\tau)$.

Since $\tilde{\delta\varphi} = \delta\varphi - \varphi'\delta\xi^0$, we have

$$\delta\varphi - \varphi'\,\xi^0 = 0 \Longrightarrow \xi^0 = \frac{\delta\varphi}{\varphi'}.$$

Finally, we obtain

$$\mathcal{R} = \psi_{\rm com} = \psi + \mathcal{H}\,\xi^0 = \psi + \mathcal{H}\frac{\delta\varphi}{\varphi'}.$$

This quantity is an important one during inflation and is called the *comoving curvature perturbation*. We also find that this quantity is gauge-invariant.

3.5.3 The curvature perturbation on spatial slices of uniform energy density

In this gauge, we impose $\tilde{\delta\rho} = 0$, which means that we choose hypersurfaces on which the energy density is uniform. That is, we have

$$\tilde{\delta\rho} = \delta\rho - \rho'\,\xi^0 = 0 \Longrightarrow \xi^0 = \frac{\delta\rho}{\rho'},$$

In the same fashion as the previous, the curvature perturbation in this frame can be written as

$$\zeta = \psi_{\rm unif} = \psi + \mathcal{H}\,\xi^0 = \psi + \mathcal{H}\frac{\delta\rho}{\rho'}.$$

If we use the gauge conservation $\rho' + 3\mathcal{H}(\rho + P) = 0$, the curvature perturbation is

$$\zeta = \psi - \frac{\delta\rho}{3(\rho + P)}.$$

During inflation we have relation $\rho + P = \dot{\varphi}^2$ and then considering at super-Hubble scales, we obtain $\delta\dot{\varphi} =$ (slow roll parameters) $\times\, H\,\delta\varphi$ and $\delta\rho = \dot{\varphi}\delta\dot{\varphi} + V'\delta\varphi \simeq V'\delta\varphi \simeq -3H\dot{\varphi}\delta\varphi$, leading to curvature perturbation in this gauge as

$$\zeta \simeq \psi + \frac{3H\dot{\varphi}}{3\dot{\varphi}^2}\delta\varphi = \psi + H\frac{\delta\varphi}{\dot{\varphi}} = \mathcal{R} \qquad \text{(ON SUPER} - \text{HUBBLE SCALES)}$$

We can see that the comoving curvature perturbation and curvature perturbation at uniform energy density are equal at the super-Hubble scales

3.5.4 Scalar field perturbations in the spatially flat gauge

For the *spatially flat gauge*, $\tilde{\psi} = 0$, then we obtain

$$\psi + \mathcal{H}\,\xi^0 = 0 \Longrightarrow \xi^0 = -\frac{\psi}{\mathcal{H}},$$

The inflaton transforms under this gauge as

$$\tilde{\delta\varphi} = \delta\varphi - \varphi'\,\xi^0 = \delta\varphi + \frac{\varphi'}{\mathcal{H}}\,\psi.$$

We can define the gauge invariance which corresponds to the inflation fluctuation as

$$\boxed{Q = \delta\varphi + \frac{\varphi'}{\mathcal{H}}\,\psi = \delta\varphi + \frac{\dot{\varphi}}{H}\psi \equiv \frac{\dot{\varphi}}{H}\,\mathcal{R}}$$

Chapter 4

ADM formalism

In the previous section, we study linear perturbation in equations of motion. In this section, we will study how to obtain such equations from action with second order perturbation. In this case, it is convenient to use (1+3) formalism.

4.1 (1+3)-formalism

Let us introduce a scalar field, $t(x)$ to describe a family of nonintersecting spacelike hypersurface, Σ_t. Each hypersurfaces are specified by one value of $t(x) = const.$, so that the normal vector to the hypersurface is in the form $n \propto \partial_\mu t$.

Next let us consider congruence of curves that intersect these whole hypersurfaces specified by γ parametrized by t, then the tangent vector along the curve can be defined as

$$t^\mu = \frac{dx^\mu}{dt}, \quad \text{with} \quad t^\mu \partial_\mu t = 1. \tag{4.1}$$

This construction can be illustrated in Fig. 4.1. Now, let us introduce the coordinates on the hypersurface as y^i. By fixing these coordinates through the whole hypersurfaces at the point intersecting with the curves γ. As illustrate in Fig. 4.1, we fix coordinates y^i for P, P' and P'' for the curve γ_P. As a result, this allows us to construct the coordinate as $x^\mu = (t, y^i)$. We can use these coordinates to all events in the spacetime.

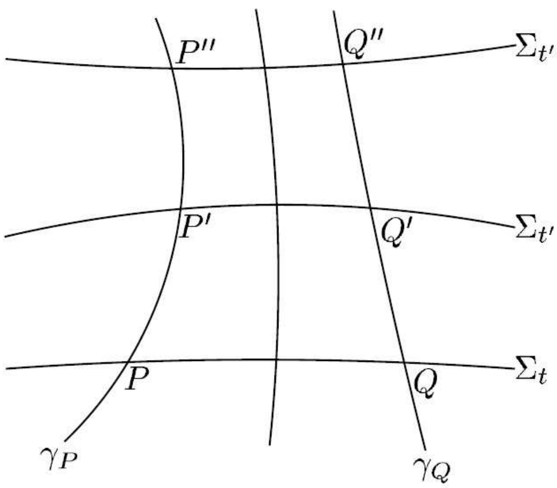


Figure 4.1: Congruence of curves through the hypersucfaces.

Now we can define the tangent vector along the curve γ and the projection tetrad to the Σ_t as

$$t^\mu = \frac{\partial x^\mu}{\partial t}|_{y^i}, \quad \text{and} \quad e^\mu_i = \frac{\partial x^\mu}{\partial y^i}|_t. \tag{4.2}$$

As a result, the proper normal vector can be defined as

$$n^{\mu} = -N\partial_{\mu}t \quad \text{with} \quad n_{\mu}e_i^{\mu} = 0. \tag{4.3}$$

The minus sign on the definition is just the convention and N is a scalar function called "lapse" function introduced to ensure the normalization, $n^{\mu}n_{\mu} = -1$. The condition on the right one is introduced to ensure that the projection of the normal vector must vanish.

Generally, the curve γ does not orthogonally intersect to the hypersurface Σ_t, so that one can decomposed the tangent vector into the normal direction and the tangent direction to the hypersurface Σ_t as shown in Fig. 4.2. Mathematically, one can write the tangent vector as

$$t^{\mu} = Nn^{\mu} + N^i e_i^{\mu}, \tag{4.4}$$

where N^i is a vector on the hypersurface Σ called "shift" function. Note that this decomposition is satisfied the condition of the tangent vector $t^{\mu}\partial_{\mu}t = 1$.

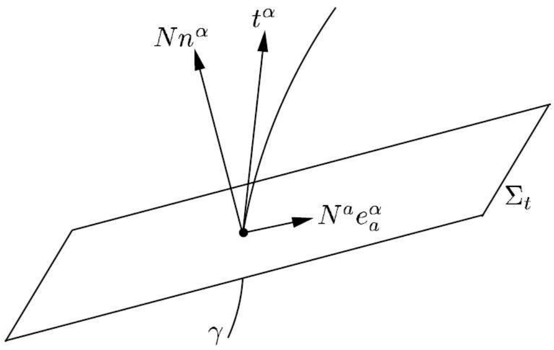


Figure 4.2: The decomposition of the tangent vector along the curve γ into the normal and tangent direction to Σ_t .

Now we are position to find the metric tensor in this formalism. In order to do that, let us consider an infinitesimal displacement constructed from the coordinates $x^{\mu} = (t, y^i)$ as

$$\begin{aligned} dx^{\mu} &= t^{\mu}\,dt + e_i^{\mu}dy^i, \\ &= (Nn^{\mu} + N^i e_i^{\mu})\,dt + e_i^{\mu}dy^i, \\ &= Nn^{\mu}\,dt + (N^i\,dt + dy^i)e_i^{\mu}. \end{aligned} \tag{4.5}$$

Therefore, the line element can be written as

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^{\mu}dx^{\nu}, \\ &= n_{\mu}n^{\mu}N^2dt^2 + g_{\mu\nu}e_i^{\mu}e_j^{\nu}(N^i dt + dy^i)(N^j dt + dy^j), \\ &= -N^2dt^2 + h_{ij}(N^i dt + dy^i)(N^j dt + dy^j), \end{aligned} \tag{4.6}$$

where we have introduced the metric on the hypersurface h_{ij} called the induced metric as follows

$$h_{ij} = g_{\mu\nu}e_i^{\mu}e_j^{\nu}. \tag{4.7}$$

As a result, the components of the metric can be written in terms of N, N^i and h_{ij} as

$$g_{00} = N^iN_i - N^2, \quad g_{0i} = N_i, \quad g_{ij} = h_{ij}. \tag{4.8}$$

The components of the inverse metric can be expressed as

$$g^{00} = -N^{-2}, \quad g^{0i} = N^{-2}N^i, \quad g^{ij} = h_{ij} - N^{-2}N^iN^j. \tag{4.9}$$

Note that the square root of the determinant of the metric can be written as $\sqrt{-g} = N\sqrt{h}$. From this splitting one can see that the total number of the independent components of the metric is still the same, one from N, three from N^i and six from symmetric tensor h_{ij} . The line element of the general hypersurface can be obtained by

$$\begin{aligned} ds^2_\Sigma &= g_{\mu\nu}dx^\mu dx^\nu, \\ &= g_{\mu\nu}\frac{\partial x^\mu}{\partial y^i}\frac{\partial x^\nu}{\partial y^j}dy^i dy^j,, \\ &= g_{\mu\nu}e^\mu_i e^\nu_j dy^i dy^j. \end{aligned} \tag{4.10}$$

In this sense, the e^μ_i plays the role of coordinate transformation from 4D to 3D surface. Also, it plays the role of the projection of the metric tensor to the induced metric. In principle one can use e^μ_i to project 4D tensor into the 3D hypersurface.

In order to construct the covariant 4D quantities, let us introduce the induced metric in 4D notation as follows

$$h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu, \tag{4.11}$$

with choosing the convention $n_0 = -N$ and $n_i = 0$. The $(0,0)$ and $(0,i)$ components of the induced metric can be written as $h_{00} = N_i N^i$ and $h_{0i} = N_i$. Even though $h_{\mu\nu}$ has non-zero components of $h_{0\mu}$, it is the metric on the hypersurface Σ. This can be checked from the fact that the contracting with the normal vector will vanish as follows

$$h_{\mu\nu}n^\mu = (g_{\mu\nu} + n_\mu n_\nu)n^\mu = n_\nu + n_\mu n^\mu n_\nu = n_\nu - n_\nu = 0. \tag{4.12}$$

The 4D induced metric $h_{\mu\nu}$ can also play the role of projection as $X_\mu = h^\nu_\mu Y^\nu$ and then $X_\mu n^\mu = 0$. Moreover, the useful properties of the normal vector can be expressed as

$$n^\mu \nabla_\nu n_\mu = 0, \quad n_{[\mu}\nabla_\nu n_{\rho]} = 0. \tag{4.13}$$

The first one comes from the fact that $\nabla_\nu(n_\mu n^\mu) = 0$, the second one comes from Frobeneus theorem and then are left to students in exercise.

Exercise2 Show that

$$n_{[\mu}\nabla_\nu n_{\rho]} = 0.$$

4.2 Extrinsic curvature

Before to move further to the Einstein equation in (1+3) formalism, let us consider the extrinsic curvature since we are now considering the 3D spacelike hypersurface embedding in 4D spacetime. This provide us the full information of the curvature of the whole spacetime since the intrinsic curvature of the hypersurface can be obtained as usual one in 4D spacetime by using the round trip parallel transport of a vector. In similar fashion as done in the intrinsic curvature, the information of the extrinsic curvature should be contained in the covariantly change of the normal vector. Since the extrinsic curvature is the properties of the hypersurface, it should be a tensor with projecting of $\nabla_\mu n_\nu$ onto the hypersurface,

$$K_{\mu\nu} = -h^\rho_\mu h^\sigma_\nu \nabla_\rho n_\sigma. \tag{4.14}$$

Note that the minus sign is just a convention we used. It can be reduced to the useful form as

$$\begin{aligned} K_{\mu\nu} &= -h^\rho_\mu(\delta^\sigma_\nu - \epsilon n^\sigma n_\nu)\nabla_\rho n_\sigma, \\ &= -h^\rho_\mu\left(\nabla_\rho n_\nu - \epsilon n_\nu \cancelto{0}{n^\sigma\nabla_\rho n_\sigma}\right), \\ &= -h^\rho_\mu \nabla_\rho n_\nu, \end{aligned} \tag{4.15}$$

$$\begin{aligned} &= -(\delta^\rho_\mu - \epsilon n^\rho n_\mu)\nabla_\rho n_\nu, \\ &= -\nabla_\mu n_\nu + \epsilon n_\mu a_\nu, \quad a_\nu \equiv n^\rho \nabla_\rho n_\nu. \end{aligned} \tag{4.16}$$

Note that we introduced the acceleration vector a_ν defined in the same way as one from the 4-velocity u_ν instead of n_ν. Since the extrinsic curvature is the quantity in the 3D hypersurface, one can check that their contacting with n^μ vanish,

$$K_{\mu\nu}n^\nu \;=\; -h^\rho_\mu \cancelto{0}{n^\nu\nabla_\rho n_\nu} = 0. \tag{4.17}$$

Also, since $n_i = 0$, then we have the fact that the contravaraint components of $K^{\mu\nu}$ are purely spatial,

$$K^{\mu\nu}n_\nu = K^{\mu 0}n_0 + \cancelto{0}{K^{\mu i}n_i} = K^{\mu 0}n_0 = 0, \quad\Rightarrow\quad K^{\mu 0} = 0. \tag{4.18}$$

The other properties of the extrinsic curvature is a symmetric tensor $K_{\mu\nu} = K_{\nu\mu}$. The proof of this property is left for the student in exercise.

<u>Exercise3</u> Show that

$$K_{\nu\mu} = K_{\mu\nu}.$$

Now let us consider especially the spatial component of the extrinsic curvature,

$$\begin{aligned}
K_{ij} &= -\nabla_i n_j = -\left(\partial_i n_j - \Gamma^\rho_{ij}n_\rho\right) = \Gamma^0_{ij}n_0 = -N\Gamma^0_{ij}, \\
&= -\frac{N}{2}\left(g^{0\rho}(\partial_i g_{j\rho} + \partial_j g_{i\rho} - \partial_\rho g_{ij})\right), \\
&= \frac{1}{2N}\Big((\partial_i N_j + \partial_j N_i - \partial_0 h_{ij}) - N^k(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij})\Big), \\
&= \frac{1}{2N}\Big((\partial_i N_j + \partial_j N_i) - N_l h^{kl}(\partial_i h_{jk} + \partial_j h_{ik} - \partial_k h_{ij}) - \partial_0 h_{ij}\Big), \\
&= \frac{1}{2N}\left(D_i N_j + D_j N_i - \partial_0 h_{ij}\right).
\end{aligned} \tag{4.19}$$

One can see that the spatial components of the extrinsic curvature K_{ij} contain only on the first time derivative of the induced metric h_{ij} and there are no time derivative on N and N^i. This is an essential property of the extrinsic curvature.

It is important to consider the extrinsic curvature in other point of view. Actually, the extrinsic curvature directly relate to the Lie derivative of the induced metric with respect to the normal vector. Applying the Lie derivative to the induced metric, one obtains

$$\begin{aligned}
L_{\hat{n}}h_{\mu\nu} &= n^\rho\nabla_\rho h_{\mu\nu} + h_{\rho\nu}\nabla_\mu n^\rho + h_{\rho\mu}\nabla_\nu n^\rho, \\
&= n^\rho\left(\cancelto{0}{\nabla_\rho g_{\mu\nu}} + \nabla_\rho(n_\mu n_\nu)\right) + \left(g_{\rho\nu} + \cancelto{0}{n_\rho n_\nu}\right)\nabla_\mu n^\rho + \left(g_{\rho\mu} + \cancelto{0}{n_\rho n_\mu}\right)\nabla_\nu n^\rho, \\
&= n^\rho n_\nu\nabla_\rho n_\mu + n^\rho n_\mu\nabla_\rho n_\nu + \delta^\rho_\nu\nabla_\mu n_\rho + \delta^\rho_\mu\nabla_\nu n_\rho, \\
&= h^\rho_\nu\nabla_\mu n_\rho + h^\rho_\mu\nabla_\nu n_\rho, \\
&= -2K_{\mu\nu}, \quad\Rightarrow\quad K_{\mu\nu} = -\frac{1}{2}L_{\hat{n}}h_{\mu\nu}.
\end{aligned} \tag{4.20}$$

4.3 Gauss-Codazzi equation

In GR, one of the most important equations is Einstein field equation which contains the curvature of the spacetime. For (1+3) formalism, the 4D curvature of the spacetime will be splitted into 3D intrinsic curvature and extrinsic curvature. So that it is possible to find the equation to connect the 4D curvature tensor to the extrinsic and intrinsic curvature in 3D case and this equation is known as "Gauss-Codazzi" equation. In order to obtain the Gauss-Codazzi equation, let us consider the 3D intrinsic curvature tensor as follows

$$
\begin{aligned}
{}^{(3)}R_{\mu\nu\rho\sigma}X^{\sigma} &= D_{\mu}D_{\nu}X_{\rho} - D_{\nu}D_{\mu}X_{\rho}, \\
&= h^{\alpha}_{\mu}h^{\beta}_{\nu}h^{\gamma}_{\rho}\nabla_{\alpha}D_{\beta}X_{\gamma} - (\mu \longleftrightarrow \nu), \\
&= h^{\alpha}_{\mu}h^{\beta}_{\nu}h^{\gamma}_{\rho}\nabla_{\alpha}\left(h^{\tau}_{\beta}h^{\sigma}_{\gamma}\nabla_{\tau}X_{\sigma}\right) - (\mu \longleftrightarrow \nu), \\
&= h^{\alpha}_{\mu}h^{\beta}_{\nu}h^{\gamma}_{\rho}\left(h^{\tau}_{\beta}h^{\sigma}_{\gamma}\nabla_{\alpha}\nabla_{\tau}X_{\sigma} + (\nabla_{\tau}X_{\sigma})\nabla_{\alpha}(h^{\tau}_{\beta}h^{\sigma}_{\gamma})\right) - (\mu \longleftrightarrow \nu), \\
&= h^{\alpha}_{\mu}h^{\tau}_{\nu}h^{\sigma}_{\rho}\nabla_{\alpha}\nabla_{\tau}X_{\sigma} + \nabla_{\tau}X_{\sigma}\left(h^{\alpha}_{\mu}h^{\tau}_{\nu}h^{\gamma}_{\rho}\nabla_{\alpha}h^{\sigma}_{\gamma} + h^{\alpha}_{\mu}h^{\sigma}_{\rho}h^{\beta}_{\nu}\nabla_{\alpha}h^{\tau}_{\beta}\right) - (\mu \longleftrightarrow \nu), \\
&= h^{\alpha}_{\mu}h^{\tau}_{\nu}h^{\sigma}_{\rho}\nabla_{\alpha}\nabla_{\tau}X_{\sigma} + \nabla_{\tau}X_{\sigma}\left(h^{\alpha}_{\mu}h^{\tau}_{\nu}h^{\gamma}_{\rho}\nabla_{\alpha}(\cancelto{0}{\delta^{\sigma}_{\gamma}} + n^{\sigma}n_{\gamma}) + h^{\alpha}_{\mu}h^{\sigma}_{\rho}h^{\beta}_{\nu}\nabla_{\alpha}(\cancelto{0}{\delta^{\tau}_{\beta}} + n^{\tau}n_{\beta})\right) - (\mu \longleftrightarrow \nu), \\
&= h^{\alpha}_{\mu}h^{\tau}_{\nu}h^{\sigma}_{\rho}\nabla_{\alpha}\nabla_{\tau}X_{\sigma} \\
&\quad +\nabla_{\tau}X_{\sigma}\left(h^{\alpha}_{\mu}h^{\tau}_{\nu}(n^{\sigma}h^{\gamma}_{\rho}\nabla_{\alpha}n_{\gamma} + \cancelto{0}{n_{\gamma}h^{\gamma}_{\rho}}\nabla_{\alpha}n^{\sigma}) + h^{\alpha}_{\mu}h^{\sigma}_{\rho}(\cancelto{0}{n_{\beta}h^{\beta}_{\nu}}\nabla_{\alpha}n^{\tau} + n^{\tau}h^{\beta}_{\nu}\nabla_{\alpha}n_{\beta})\right) - (\mu \longleftrightarrow \nu), \\
&= h^{\alpha}_{\mu}h^{\tau}_{\nu}h^{\sigma}_{\rho}\nabla_{\alpha}\nabla_{\tau}X_{\sigma} - \nabla_{\tau}X_{\sigma}\left(h^{\tau}_{\nu}n^{\sigma}h^{\gamma}_{\rho}K_{\mu\gamma} + h^{\sigma}_{\rho}n^{\tau}h^{\beta}_{\nu}K_{\mu\beta}\right) - (\mu \longleftrightarrow \nu), \\
&= h^{\alpha}_{\mu}h^{\tau}_{\nu}h^{\sigma}_{\rho}\nabla_{\alpha}\nabla_{\tau}X_{\sigma} - \nabla_{\tau}X_{\sigma}\left(h^{\tau}_{\nu}n^{\sigma}K_{\mu\rho} + h^{\sigma}_{\rho}n^{\tau}\cancelto{0\text{ cancel to anti-sym}}{K_{\mu\nu}}\right) - (\mu \longleftrightarrow \nu), \\
&= h^{\alpha}_{\mu}h^{\tau}_{\nu}h^{\sigma}_{\rho}\nabla_{\alpha}\nabla_{\tau}X_{\sigma} + X_{\sigma}h^{\tau}_{\nu}\nabla_{\tau}n^{\sigma}K_{\mu\rho} - (\mu \longleftrightarrow \nu), \quad (\text{note: } X_{\sigma}n^{\sigma} = 0) \\
&= h^{\alpha}_{\mu}h^{\tau}_{\nu}h^{\sigma}_{\rho}\nabla_{\alpha}\nabla_{\tau}X_{\sigma} - K^{\sigma}_{\nu}K_{\mu\rho}X_{\sigma} - (\mu \longleftrightarrow \nu), \\
&= h^{\alpha}_{\mu}h^{\tau}_{\nu}h^{\sigma}_{\rho}R_{\alpha\tau\sigma\beta}X^{\beta} - K_{\nu\beta}K_{\mu\rho}X^{\beta} + K_{\mu\beta}K_{\nu\rho}X^{\beta}, \\
{}^{(3)}R_{\mu\nu\rho\beta} &= h^{\alpha}_{\mu}h^{\tau}_{\nu}h^{\sigma}_{\rho}R_{\alpha\tau\sigma\beta} - K_{\nu\beta}K_{\mu\rho} + K_{\mu\beta}K_{\nu\rho}.
\end{aligned}
\tag{4.21}
$$

As a result, the intrinsic curvature in 3D hypersurface can be obtained from two parts; first by projecting the 4D curvature (first term) and the contribution from extrinsic curvature (second and third terms). Now, let us find the component $(0,0)$ of Eingstein field equation by contacting $g^{\mu\rho}g^{\nu\beta}$ to the above equation as follows

$$
\begin{aligned}
g^{\mu\rho}g^{\nu\beta}h^{\alpha}_{\mu}h^{\tau}_{\nu}h^{\sigma}_{\rho}R_{\alpha\tau\sigma\beta} &= g^{\mu\rho}g^{\nu\beta}\left({}^{(3)}R_{\mu\nu\rho\beta} + K_{\nu\beta}K_{\mu\rho} - K_{\mu\beta}K_{\nu\rho}\right), \\
h^{\alpha\sigma}h^{\tau\beta}R_{\alpha\tau\sigma\beta} &= {}^{(3)}R + K^2 - K_{\mu\nu}K^{\mu\nu}, \\
(g^{\alpha\sigma} + n^{\alpha}n^{\sigma})(g^{\tau\beta} + n^{\tau}n^{\beta})R_{\alpha\tau\sigma\beta} &= {}^{(3)}R + K^2 - K_{\mu\nu}K^{\mu\nu}, \\
R + 2n^{\alpha}n^{\sigma}R_{\alpha\sigma} &= {}^{(3)}R + K^2 - K_{\mu\nu}K^{\mu\nu}, \\
2n^{\alpha}n^{\sigma}\left(R_{\alpha\sigma} - \frac{1}{2}Rg_{\alpha\sigma}\right) &= {}^{(3)}R + K^2 - K_{\mu\nu}K^{\mu\nu}, \\
2n^{\mu}n^{\nu}G_{\mu\nu} &= {}^{(3)}R + K^2 - K_{\mu\nu}K^{\mu\nu}, \\
2\kappa n^{\mu}n^{\nu}T_{\mu\nu} &= {}^{(3)}R + K^2 - K_{\mu\nu}K^{\mu\nu} = 2\kappa\rho.
\end{aligned}
\tag{4.22}
$$

One can see that this equation is not the dynamical equation since it contains only the first time derivatives of the induced metric $h_{\mu\nu}$ and the are no time derivative for lapse N and shift N^i functions.

Next let us find the other components of the Einstein equation. In order to obtain $(0,i)$ components, one may contacting one normal vector and one projection tensor to the Einstein tensor,

$$
n^{\mu}h^{\nu}_{\rho}G_{\mu\nu} = n^{\mu}h^{\nu}_{\rho}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) = n^{\mu}h^{\nu}_{\rho}R_{\mu\nu} - \frac{1}{2}R\cancelto{0}{n_{\nu}h^{\nu}_{\rho}} = n^{\mu}h^{\nu}_{\rho}R_{\mu\nu}. \tag{4.23}
$$

This equation implies that one has to contact one normal vector to the curvature tensor. By doing such performance, we have

$$
\begin{aligned}
R_{\mu\nu\rho\sigma}n^{\sigma} &= \nabla_{\mu}\nabla_{\nu}n_{\rho} - \nabla_{\nu}\nabla_{\mu}n_{\rho}, \\
&= \nabla_{\mu}(-K_{\nu\rho} - n_{\nu}n^{\sigma}\nabla_{\sigma}n_{\rho}) - (\mu \longleftrightarrow \nu), \\
h^{\mu}_{\alpha}h^{\nu}_{\beta}h^{\rho}_{\gamma}R_{\mu\nu\rho\sigma}n^{\sigma} &= h^{\mu}_{\alpha}h^{\nu}_{\beta}h^{\rho}_{\gamma}\nabla_{\mu}(-K_{\nu\rho} - n_{\nu}n^{\sigma}\nabla_{\sigma}n_{\rho}) - (\alpha \longleftrightarrow \beta), \\
&= -D_{\alpha}K_{\beta\gamma} - h^{\mu}_{\alpha}h^{\rho}_{\gamma}\left(h^{\nu}_{\beta}n^{\sigma}\nabla_{\sigma}n_{\rho}\nabla_{\mu}n_{\nu} + \cancelto{0}{h^{\nu}_{\beta}n_{\nu}}\nabla_{\mu}(n^{\sigma}\nabla_{\sigma}n_{\rho})\right) - (\alpha \longleftrightarrow \beta), \\
&= -D_{\alpha}K_{\beta\gamma} - h^{\rho}_{\gamma}n^{\sigma}\nabla_{\sigma}n_{\rho}\cancelto{0\,\text{cancel to anti-sym}}{K_{\alpha\beta}} - (\alpha \longleftrightarrow \beta), \\
&= -D_{\alpha}K_{\beta\gamma} + D_{\beta}K_{\alpha\gamma}, \\
g^{\alpha\gamma}h^{\mu}_{\alpha}h^{\nu}_{\beta}h^{\rho}_{\gamma}R_{\mu\nu\rho\sigma}n^{\sigma} &= g^{\alpha\gamma}(-D_{\alpha}K_{\beta\gamma} + D_{\beta}K_{\alpha\gamma}), \\
h^{\nu}_{\beta}R_{\nu\sigma}n^{\sigma} &= D_{\beta}K - D_{\alpha}K^{\alpha}_{\beta}, \\
h^{\nu}_{\beta}G_{\nu\sigma}n^{\sigma} &= D_{\beta}K - D_{\alpha}K^{\alpha}_{\beta}, \\
\kappa h^{\nu}_{\beta}n^{\sigma}T_{\nu\sigma} &= D_{\beta}K - D_{\alpha}K^{\alpha}_{\beta} = \kappa j_{\beta}.
\end{aligned}
\tag{4.24}
$$

Again, these equations are just constraint equations, since there are no second time derivatives.

Next we will find the (i,j) components of Einstein equation. In order to find it we need the spatial components of Ricci tensor which can be expressed as

$$
\begin{aligned}
h^{\alpha}_{\mu}h^{\beta}_{\nu}R_{\alpha\beta} &= h^{\alpha}_{\mu}h^{\beta}_{\nu}g^{\rho\sigma}R_{\alpha\rho\beta\sigma}, \\
&= h^{\alpha}_{\mu}h^{\beta}_{\nu}(h^{\rho\sigma} - n^{\rho}n^{\sigma})R_{\alpha\rho\beta\sigma}, \\
&= h^{\alpha}_{\mu}h^{\beta}_{\nu}h^{\rho\sigma}R_{\alpha\rho\beta\sigma} - n^{\rho}n^{\sigma}h^{\alpha}_{\mu}h^{\beta}_{\nu}R_{\alpha\rho\beta\sigma}.
\end{aligned}
\tag{4.25}
$$

Therefore, our task is finding the expression for contacting two normal vectors to the curvature tensor in the second term on the right hand side of the above equation, $n^{\rho}n^{\sigma}h^{\alpha}_{\mu}h^{\beta}R_{\alpha\rho\beta\sigma}$ since the first term we have already in Eq. (4.21). Let us first find the expression for contacting one normal vectors and then contacting one more normal vector as follows,

$$
\begin{aligned}
R_{\alpha\rho\beta\sigma}n^{\sigma} &= \nabla_{\alpha}\nabla_{\rho}n_{\beta} - \nabla_{\rho}\nabla_{\alpha}n_{\beta}, \\
&= -\nabla_{\alpha}(K_{\rho\beta} + n_{\rho}a_{\beta}) + \nabla_{\rho}(K_{\beta\alpha} + n_{\beta}a_{\alpha}), \\
&= -\nabla_{\alpha}K_{\rho\beta} + \nabla_{\rho}K_{\beta\alpha} - \nabla_{\alpha}(n_{\rho}a_{\beta}) + \nabla_{\rho}(n_{\beta}a_{\alpha}), \\
R_{\alpha\rho\beta\sigma}n^{\rho}n^{\sigma} &= -n^{\rho}\nabla_{\alpha}K_{\rho\beta} + n^{\rho}\nabla_{\rho}K_{\beta\alpha} - n^{\rho}\nabla_{\alpha}(n_{\rho}a_{\beta}) + n^{\rho}\nabla_{\rho}(n_{\beta}a_{\alpha}), \\
&= -n^{\rho}\nabla_{\alpha}K_{\rho\beta} + n^{\rho}\nabla_{\rho}K_{\beta\alpha} + \nabla_{\alpha}a_{\beta} + a_{\beta}\cancelto{0}{n^{\rho}\nabla_{\alpha}n_{\rho}} + a_{\beta}a_{\alpha} + n^{\rho}n_{\beta}\nabla_{\rho}a_{\alpha}, \\
&= -\nabla_{\alpha}\cancelto{0}{(n^{\rho}K_{\rho\beta})} + K_{\rho\beta}\nabla_{\alpha}n^{\rho} + n^{\rho}\nabla_{\rho}K_{\beta\alpha} + (\delta^{\rho}_{\beta} + n^{\rho}n_{\beta})\nabla_{\rho}a_{\alpha} + a_{\beta}a_{\alpha}, \\
&= -K_{\rho\beta}(K^{\rho}_{\alpha} + a_{\alpha}\cancelto{0\,(n^{\rho}K_{\rho\beta}=0)}{n^{\rho}}) + n^{\rho}\nabla_{\rho}K_{\beta\alpha} + h^{\rho}_{\beta}\nabla_{\rho}a_{\alpha} + a_{\beta}a_{\alpha}, \\
n^{\rho}n^{\sigma}h^{\alpha}_{\mu}h^{\beta}_{\nu}R_{\alpha\rho\beta\sigma} &= -K^{\rho}_{\mu}K_{\rho\nu} + h^{\alpha}_{\mu}h^{\beta}_{\nu}n^{\rho}\nabla_{\rho}K_{\beta\alpha} + h^{\alpha}_{\mu}h^{\rho}_{\nu}\nabla_{\rho}a_{\alpha} + h^{\alpha}_{\mu}h^{\beta}_{\nu}a_{\beta}a_{\alpha}, \\
&= h^{\alpha}_{\mu}h^{\beta}_{\nu}(L_{\hat{n}}K_{\beta\alpha} + K_{\beta\rho}\nabla_{\alpha}n^{\rho} + K_{\alpha\rho}\nabla_{\beta}n^{\rho}) - K^{\rho}_{\mu}K_{\rho\nu} + D_{\nu}a_{\mu} + a_{\nu}a_{\mu}, \\
&= h^{\alpha}_{\mu}h^{\beta}_{\nu}L_{\hat{n}}K_{\beta\alpha} - K_{\nu\rho}K^{\rho}_{\mu} - K_{\mu\rho}K^{\rho}_{\nu} - K^{\rho}_{\mu}K_{\rho\nu} + D_{\nu}a_{\mu} + a_{\nu}a_{\mu}, \\
&= L_{\hat{n}}K_{\mu\nu} + K_{\mu\rho}K^{\rho}_{\nu} + D_{\nu}a_{\mu} + a_{\nu}a_{\mu}.
\end{aligned}
\tag{4.26}
$$

Note that we have used the relation of $h^{\alpha}_{\mu}h^{\beta}_{\nu}L_{\hat{n}}K_{\beta\alpha}$ and $L_{\hat{n}}K_{\mu\nu}$ as follows

$$
\begin{aligned}
L_{\hat{n}}K_{\mu\nu} &= L_{\hat{n}}(h^{\alpha}_{\mu}h^{\beta}_{\nu}K_{\beta\alpha}), \\
&= h^{\alpha}_{\mu}h^{\beta}_{\nu}L_{\hat{n}}K_{\beta\alpha} + K_{\beta\alpha}h^{\beta}_{\nu}L_{\hat{n}}h^{\alpha}_{\mu} + K_{\beta\alpha}h^{\alpha}_{\mu}L_{\hat{n}}h^{\beta}_{\nu}, \\
&= h^{\alpha}_{\mu}h^{\beta}_{\nu}L_{\hat{n}}K_{\beta\alpha} - 2K_{\nu\alpha}K^{\alpha}_{\mu} - 2K_{\beta\mu}K^{\beta}_{\nu}, \\
&= h^{\alpha}_{\mu}h^{\beta}_{\nu}L_{\hat{n}}K_{\beta\alpha} - 4K_{\nu\alpha}K^{\alpha}_{\mu}.
\end{aligned}
\tag{4.27}
$$

From Gauss-Codazzi equation in Eq. (4.21), one can expressed in proper form as

$$\begin{aligned} h^\alpha_\mu h^\beta_\nu h^{\rho\sigma} R_{\alpha\rho\beta\sigma} &= g^{\gamma\sigma} h^\alpha_\mu h^\beta_\nu h^\rho_\gamma R_{\alpha\rho\beta\sigma}, \\ &= g^{\gamma\sigma}\left({}^{(3)}R_{\mu\gamma\nu\sigma} + K_{\mu\nu}K_{\sigma\gamma} - K_{\mu\gamma}K_{\nu\sigma}\right), \\ &= {}^{(3)}R_{\mu\nu} + K_{\mu\nu}K - K_{\mu\gamma}K^\gamma_\nu. \end{aligned} \tag{4.28}$$

Substituting Eq. (4.26) and Eq. (4.28) into Eq. (4.25), one obtains

$$\begin{aligned} h^\alpha_\mu h^\beta_\nu R_{\alpha\beta} &= {}^{(3)}R_{\mu\nu} + K_{\mu\nu}K - K_{\mu\gamma}K^\gamma_\nu - (L_{\hat n}K_{\mu\nu} + K_{\mu\rho}K^\rho_\nu + D_\nu a_\mu + a_\nu a_\mu), \\ h^\alpha_i h^\beta_j R_{\alpha\beta} &= {}^{(3)}R_{ij} + K_{ij}K - 2K_{il}K^l_j - L_{\hat n}K_{ij} + N^{-1}D_jD_iN. \end{aligned} \tag{4.29}$$

Note that we have used the equation for the acceleration as follows

$$\begin{aligned} a_\mu &= n^\rho\nabla_\rho n_\mu = n^\rho\partial_\rho n_\mu + n_\rho\partial_\mu n^\rho, \\ a_i &= n_\rho\partial_i n^\rho = N^{-1}D_iN, \\ D_j a_i &= D_j(N^{-1}D_iN) = N^{-1}D_jD_iN - N^{-2}D_jND_iN, \\ &= N^{-1}D_jD_iN - a_ia_j. \end{aligned} \tag{4.30}$$

By inserting Eq. (4.29) into Einstein field equation, one obtains

$$\begin{aligned} L_{\hat n}K_{ij} &= {}^{(3)}R_{ij} + K_{ij}K - 2K_{il}K^l_j + N^{-1}D_jD_iN - h^\alpha_i h^\beta_j R_{\alpha\beta}, \\ N^{-1}(\partial_0 K_{ij} - L_{\vec N}K_{ij}) &= {}^{(3)}R_{ij} + K_{ij}K - 2K_{il}K^l_j + N^{-1}D_jD_iN, \\ &\quad -\kappa h^\alpha_i h^\beta_j\left(T_{\alpha\beta} - Tg_{\alpha\beta}\right), \end{aligned} \tag{4.31}$$

where $\vec N = (0, N^i)$. The detail calculation to obtain the relation between $L_{\hat n}K_{ij}$ and $L_{\vec N}K_{ij}$ is left for student in exercise. These are the dynamical equations according to (1+3) formalism. Moreover, this equation can be viewed as the Hamilton equation analogous to $\dot p = -\partial_q H$ in Hamiltonian formalism of classical mechanics where K_{ij} play the role of the conjugate momentum. The other Hamilton equation analogous to $\dot q = \partial_p H$ can be obtained from the relation of the Lie derivatives of h_{ij} with the extrinsic curvature in Eq. (4.20) as

$$L_{\hat n}h_{ij} = N^{-1}(\partial_0 h_{ij} - L_{\vec N}h_{ij}) = -2K_{ij}. \tag{4.32}$$

By considering Lagrangian in GR well known, $L_{EH} = R/2\kappa$, we will consider it in terms of (1+3) formalism. Therefore, our task is find the Ricci scalar in terms of (1+3) form. As we have seen, on the way to get the Gauss-Codazzi equation, the Ricci scalar can be expressed as

$$R = -2n^\mu n^\nu R_{\mu\nu} + 2n^\mu n^\nu G_{\mu\nu} = -2n^\mu n^\nu R_{\mu\nu} + {}^{(3)}R + K^2 - K_{\mu\nu}K^{\mu\nu}. \tag{4.33}$$

Therefore, our task now just find the expression of $n^\mu n^\nu R_{\mu\nu}$. Actually, we have calculate it in the previous section. However, we now separate it into surface terms and non-surface ones as follows

$$\begin{aligned} n^\mu n^\nu R_{\mu\nu} &= n^\mu n^\nu g^{\rho\sigma} R_{\rho\mu\sigma\nu}, \\ &= n^\mu g^{\rho\sigma}(\nabla_\rho\nabla_\mu n_\sigma - \nabla_\mu\nabla_\rho n_\sigma), \\ &= n^\mu(\nabla_\rho\nabla_\mu n^\rho - \nabla_\mu\nabla_\rho n^\rho), \\ &= \nabla_\rho(n^\mu\nabla_\mu n^\rho) - \nabla_\rho n^\mu\nabla_\mu n^\rho - \nabla_\mu(n^\mu\nabla_\rho n^\rho) + \nabla_\mu n^\mu\nabla_\rho n^\rho, \\ &= \nabla_\rho(a^\rho + n^\rho K) - K^\mu_\rho K^\rho_\mu + K^2. \end{aligned} \tag{4.34}$$

Substituting this result into the expression for R above, one obtains

$$\begin{aligned} R &= -2(\nabla_\rho(a^\rho + n^\rho K) - K^\mu_\rho K^\rho_\mu + K^2) + {}^{(3)}R + K^2 - K_{\mu\nu}K^{\mu\nu}, \\ &= {}^{(3)}R - K^2 + K_{\mu\nu}K^{\mu\nu} - 2\nabla_\rho(a^\rho + n^\rho K) \\ &= L_{\mathrm{ADM}} + L_{sur}. \end{aligned} \tag{4.35}$$

where we have defined $L_{\mathrm{ADM}} = {}^{(3)}R - K^2 + K_{\mu\nu}K^{\mu\nu}$ standing for Arnowit-Deser-Misner (ADM) Lagrangian.

Chapter 5

Inflation with ADM formalism

We consider an inflationary model described by the action

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{2}R + P(\phi, X)\right), \tag{5.1}$$

where R is the Ricci scalar for 4-dimensional spacetime, $P(\phi, X)$ is the Lagrangian density for scalar field ϕ, and

$$X \equiv -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\,. \tag{5.2}$$

In terms of the ADM metric, the action in Eq. (5.1) becomes

$$S = \frac{1}{2}\int dtd^3xN\sqrt{-h}\left({}^{(3)}R + K_{ij}K^{ij} - K^2 + 2P\right), \tag{5.3}$$

where we have used $\sqrt{-g} = N\sqrt{h}$. Variation of the above action with respect to N yields the Hamiltonian constraint as

$$0 =^{(3)} R - K_{ij}K^{ij} + K^2 + 2P - \frac{2}{N^2}P_{,X}\left(\dot{\phi} - N^i\partial_i\phi\right)^2. \tag{5.4}$$

To obtain the above equation, we use

$$\begin{aligned} \delta_N(NK_{ij}K^{ij}) &= \delta_N\left[\frac{1}{4N}\left(\dot{h}_{ij} - D_iN_j - D_jN_i\right)\left(\dot{h}^{ij} - D^iN^j - D^jN^i\right)\right] \\ &= -K_{ij}K^{ij}\delta N\,, \end{aligned} \tag{5.5}$$

where δ_N denotes variation with respect to N. Similarly, we have

$$\delta_N\left(NK^2\right) = -K^2\delta N\,. \tag{5.6}$$

In addition, we use

$$2\delta_N(NP) = 2P\delta N + 2N\delta_N P\,, \tag{5.7}$$

$$\begin{aligned} \delta_N P &= P_{,X}\delta_N X = \frac{1}{2}P_{,X}\delta_N\left(\frac{1}{N^2}\dot{\phi}^2 - \frac{2}{N^2}N^i\partial_i\phi\dot{\phi} + \left(\frac{N^iN^j}{N^2} - h^{ij}\right)\partial_i\phi\partial_j\phi\right) \\ &= -P_{,X}\left(\frac{1}{N^3}\dot{\phi}^2 - \frac{2}{N^3}N^i\partial_i\phi\dot{\phi} + \frac{N^iN^j}{N^3}\partial_i\phi\partial_j\phi\right)\delta N \\ &= -\frac{1}{N^3}P_{,X}\left(\dot{\phi} - N^i\partial_i\phi\right)^2\delta N\,, \end{aligned} \tag{5.8}$$

and therefore we get

$$2\delta_N(NP) = 2P\delta N - \frac{2}{N^2}P_{,X}\left(\dot{\phi} - N^i\partial_i\phi\right)^2\delta N\,. \tag{5.9}$$

Here, $P_{,X} \equiv \partial P/\partial X$. To vary the action (5.3) with respect to N^i, we use

$$\begin{aligned}
\delta_{N^k}\left(NK_{ij}K^{ij}\right) &= 2NK_{ij}\delta_{N^k}K^{ij} = -\left(K_{ik}D^i\delta N^k + K_{jk}D^j\delta N^k\right), \\
&= -\left(K^i_kD_i\delta N^k + K^j_kD_j\delta N^k\right) \\
&= \underbrace{2D_iK^i_k\delta N^k}_{\text{from } (D_iK^i_k+D_jK^j_k)\delta N^k} \underbrace{-2D_i\left(K^i_k\delta N^k\right)}_{\text{divergent term}}, \qquad (5.10)
\end{aligned}$$

where δ_{N^k} denotes variation with respect to N^k. Similarly, we have

$$\delta_{N^k}\left(NK^2\right) = 2NK\delta_{N^k}K = -2D_kK\delta N^k + \text{divergent term}. \qquad (5.11)$$

Morover, we have

$$\begin{aligned}
2\delta_{N^k}P &= P_{,X}\delta_{N^k}\left(\frac{1}{N^2}\dot{\phi}^2 - \frac{2}{N^2}\dot{\phi}N^i\partial_i\phi + \left(\frac{N^iN^j}{N^2} - h^{ij}\right)\partial_i\phi\partial_j\phi\right) \\
&= \frac{2}{N^2}P_{,X}\left(-\dot{\phi}\delta N^k\partial_k\phi + N^i\delta N^k\partial_i\phi\partial_k\phi\right) \\
&= -\frac{2}{N^2}P_{,X}\left(\dot{\phi} - N^i\partial_i\phi\right)\delta N^k\partial_k\phi. \qquad (5.12)
\end{aligned}$$

Using Eqs. (5.10) - (5.12), one can show that variation of the action (5.3) with respect to N^k gives

$$0 = D_jK^j_i - D_iK - \frac{1}{N}P_{,X}\left(\dot{\phi} - N^j\partial_j\phi\right)\partial_i\phi, \qquad (5.13)$$

which is the momentum constraint equation.

5.1 Background universe

Before we compute the action for perturbation during inflation, let us study the action for the background universe. We assume that the background spacetime is the spatially flat FLRW spacetime, so that we set

$$N = n(t), \qquad N_i = 0, \qquad h_{ij} = a^2(t)\delta_{ij}. \qquad (5.14)$$

Inserting the above expressions into Eq. (4.19), we have

$$K_{ij} = \frac{1}{2n}2a\dot{a}\delta_{ij} = \frac{1}{n}h_{ij}H, \qquad (5.15)$$

where $H \equiv \dot{a}/a$ is the Hubble parameter. For spatially flat spacetime, one can show that

$${}^{(3)}R = 0. \qquad (5.16)$$

Using Eq. (5.15), we get

$$K_{ij}K^{ij} - K^2 = \frac{1}{N^2}\left(3H^2 - 9H^2\right) = -\frac{1}{N^2}6H^2. \qquad (5.17)$$

Inserting Eqs. (5.16) and (5.17) into Eq. (5.3) we obtain

$$S = \frac{1}{2}\int dtd^3x\left(-6\frac{a\dot{a}^2}{n} + 2a^3nP\right). \qquad (5.18)$$

This is the action for the background universe. Vary this action with respect to n, we get

$$\begin{aligned}
\delta S &= \frac{1}{2}\int dtd^3x\left(6\frac{a\dot{a}^2}{n^2} + 2a^3P + 2a^3nP_{,X}\frac{\delta X}{\delta n}\right)\delta n \\
&= \frac{1}{2}\int dtd^3x\left(6\frac{a\dot{a}^2}{n^2} + 2a^3P - 4a^3P_{,X}X\right)\delta n. \qquad (5.19)
\end{aligned}$$

After setting $n = 1$, the above equation yields the Friedmann equation as

$$3H^2 = 2P_{,X}x - P\,. \tag{5.20}$$

In the calculation for Eq. (5.19), we have used

$$\frac{\delta X}{\delta n} = \frac{\delta}{\delta n}\left(-\frac{1}{2}g^{00}\dot{\phi}^2\right) = \frac{\delta}{\delta n}\left(\frac{1}{2n^2}\dot{\phi}^2\right) = -2\frac{X}{n}\,, \tag{5.21}$$

where X in this expression is evaluated at background. We now vary the action (5.18) with respect to a. The result is

$$\begin{aligned}
\delta S &= \frac{1}{2}\int dt d^3x \left(-6\frac{\dot{a}^2}{n}\delta a - 12\frac{a\dot{a}}{n}\delta\dot{a} + 6a^2 nP\right) \\
&= \frac{1}{2}\int dt d^3x \left(-6\frac{\dot{a}^2}{n} + 12\frac{d}{dt}\left(\frac{a\dot{a}}{n}\right) + 6a^2 nP\right)\delta a \\
&= \frac{1}{2}\int dt d^3x \left(6\dot{a}^2 + 12a\ddot{a} + 6a^2 P\right)\delta a\,,
\end{aligned} \tag{5.22}$$

where we have set $n = 1$ and $\dot{n} = 0$ in the third line of the above calculation. Thus variation of the action with respect to a gives

$$2\frac{\ddot{a}}{a} + H^2 = -P\,. \tag{5.23}$$

The above equation can be rewritten as

$$2\dot{H} + 3H^2 = -P\,. \tag{5.24}$$

5.2 The action of perturbations

In this note, we work in comoving gauge under the conditions,

$$\begin{aligned}
(v + B) = \frac{1}{a^2}\phi'\delta\phi P_{,X} = 0\,, \quad &\rightarrow \quad \delta\phi = 0\,, \\
& \qquad\quad E = 0\,,
\end{aligned} \tag{5.25}$$

where $\delta\phi$ denotes perturbations in field ϕ. Based on this choice of gauge fixing, we set

$$h_{ij} = a^2(1 + 2\zeta)\delta_{ij} = a^2 e^{2\zeta}\,, \quad N = 1 + \alpha\,, \quad N_i = \partial_i\beta\,. \tag{5.26}$$

Here, we consider only the scalar mode of perturbations. Note that the varibles used here are different to ones in the first part, just redefining as $\zeta \rightarrow \psi$, $\alpha \rightarrow \phi$ and $\beta \rightarrow B$. From the above expression for h_{ij}, the Christoffel symbol associated with h_{ij} is

$$\Gamma^k_{ij} = \partial_i\zeta\delta^k_j + \partial_j\zeta\delta^k_i - \partial^k\zeta h_{ij}\,. \tag{5.27}$$

Thus we have

$$\begin{aligned}
{}^{(3)}R &= \underbrace{h^{ij}\partial_k\Gamma^k_{ij}}_{(1)}\underbrace{-h^{ij}\partial_i\Gamma^k_{kj}}_{(2)}\underbrace{+h^{ij}\Gamma^k_{ij}\Gamma^l_{kl}}_{(3)}\underbrace{-h^{ij}\Gamma^l_{ik}\Gamma^k_{jl}}_{(4)} \\
&= \underbrace{-\partial^2\zeta}_{(1)}\underbrace{-3\partial^2\zeta}_{(2)}\underbrace{-3\partial_i\zeta\partial^i\zeta}_{(3)}\underbrace{+\partial_i\zeta\partial^i\zeta}_{(4)} \\
&= -4\partial^2\zeta - 2\partial_i\zeta\partial^i\zeta\,,
\end{aligned} \tag{5.28}$$

where $\partial^2 \equiv \partial_i\partial^i$. For this choice of gauge, one can show that Eq. (4.19) gives

$$\begin{aligned}
K_{ij} &= \frac{1}{1+\alpha}\left(\left(H + \dot{\zeta}\right)h_{ij} - \partial_i\partial_j\beta + \Gamma^k_{ij}\partial_k\beta\right)\,, \\
K &= \frac{1}{1+\alpha}\left(3\left(H + \dot{\zeta}\right) - \partial^2\beta - \Gamma^k_{kj}\partial^j\beta\right), \\
&= \frac{1}{1+\alpha}\left(3\left(H + \dot{\zeta}\right) - \partial^2\beta - 3\partial_i\zeta\partial^i\beta\right)\,.
\end{aligned} \tag{5.29}$$

5.3 Constraint equations for linear perturbations

We see that we now have 3 perturbation variables. Nevertheless, we have 2 constraint equations given in Eq. (5.4) and (5.13). Hence, we could use these constraint equations to write α and β in terms of ζ. We first insert the expressions in Eq. (5.29) into Eq. (5.13), and consequently we get

$$\begin{aligned} 0 &= D_j\left[\frac{1}{1+\alpha}\left(\left(H+\dot{\zeta}\right)\delta_i^j-\partial_i\partial^j\beta\right)\right]-D_i\left[\frac{1}{1+\alpha}\left(3\left(H+\dot{\zeta}\right)-\partial^2\beta\right)\right] \\ &= -2\partial_i\left[\frac{1}{1+\alpha}\left(\left(H+\dot{\zeta}\right)\right)\right]=-2\left(H-\alpha H+\dot{\zeta}\right). \end{aligned} \tag{5.30}$$

In the above calculation, we use $\partial_i\phi=0$ because the perturbed part of ϕ vanishes in this gauge, and keep the terms up to first order in perturbation. The above calculation yields

$$\alpha=\frac{\dot{\zeta}}{H}. \tag{5.31}$$

Inserting the expressions from Eq. (5.28) and (5.29) into Eq. (5.4), we obtain

$$H\partial^2\beta=-\partial^2\zeta+X\left(P_{,X}+2P_{,XX}X\right)\alpha=-\partial^2\zeta+X\rho_X\frac{\dot{\zeta}}{H}, \tag{5.32}$$

where $P_{,XX}\equiv\partial^2P/\partial X^2$, and $\rho_X\equiv P_{,X}+2P_{,XX}X$. To obtain the above equation, we use the following calculations:

$$\begin{aligned} K_{ij}K^{ij} &= \frac{1}{(1+\alpha)^2}\left(\left(H+\dot{\zeta}\right)h_{ij}-D_i\partial_j\beta\right)\left(\left(H+\dot{\zeta}\right)h^{ij}-D^i\partial^j\beta\right) \\ &= \frac{1}{(1+\alpha)^2}\left(3\left(H+\dot{\zeta}\right)^2-2\left(\partial^2\beta+\Gamma^k_{ki}\partial^i\beta\right)\beta(H+\dot{\zeta})+D_i\partial_j\beta D^i\partial^j\beta\right) \\ &= \left(1-2\alpha+3\alpha^2+\ldots\right)\Big(3H^2+6H\dot{\zeta}+3\dot{\zeta}^2-2\left(\partial^2\beta+3\partial_i\zeta\partial^i\beta\right)\left(H+\dot{\zeta}\right) \\ &\quad +D_i\partial_j\beta D^i\partial^j\beta\Big) \\ &= 3H^2+6H\dot{\zeta}-6H^2\alpha-2H\left(\partial^2\beta+3\partial_i\zeta\partial^i\beta\right)+9\alpha^2H^2-12H\alpha\dot{\zeta}+3\dot{\zeta}^2 \\ &\quad -2\partial^2\beta(-2\alpha H+\dot{\zeta})+\partial_i\partial_j\beta\partial^i\partial^j\beta \\ &= 3H^2-2H\left(\partial^2\beta+3\partial_i\zeta\partial^i\beta\right)+2\partial^2\beta\dot{\zeta}+\partial_i\partial_j\beta\partial^i\partial^j\beta, \end{aligned} \tag{5.33}$$

where in the 4th equality we have ignored the third and higher order perturbation terms, and we have used Eq. (5.31) in the 5th equality. Similarly, we have

$$\begin{aligned} K^2 &= \frac{1}{(1+\alpha)^2}\left(9\left(H+\dot{\zeta}\right)^2-6\left(H+\dot{\zeta}\right)\left(\partial^2\beta+3\partial_i\zeta\partial^i\beta\right)+\left(D_i\partial^i\beta\right)^2\right) \\ &= 9H^2+18H\dot{\zeta}-18\alpha H^2-6H\left(\partial^2\beta+3\partial_i\zeta\partial^i\beta\right)+27H^2\alpha^2-36H\alpha\dot{\zeta}+9\dot{\zeta}^2 \\ &\quad -6\left(-2\alpha H+\dot{\zeta}\right)\partial^2\beta+\left(\partial^2\beta\right)^2 \\ &= 9H^2-6H\left(\partial^2\beta+3\partial_i\zeta\partial^i\beta\right)+6\dot{\zeta}\partial^2\beta+\left(\partial^2\beta\right)^2. \end{aligned} \tag{5.34}$$

Applying the same expansion as above, one can show that

$$\begin{aligned} \delta P &= P_{,X}\delta X+\frac{1}{2}P_{,XX}(\delta X)^2 \\ &= \frac{1}{2}P_{,X}\left[\frac{1}{N^2}\left(\dot{\phi}^2-2N^i\partial_i\phi\dot{\phi}+\left(N^iN^j-N^2h^{ij}\right)\partial_i\phi\partial_j\phi\right)-\dot{\phi}^2\right] \\ &\quad +\frac{1}{8}P_{,XX}\left[\frac{1}{N^2}\left(\dot{\phi}^2-2N^i\partial_i\phi\dot{\phi}+\left(N^iN^j-N^2h^{ij}\right)\partial_i\phi\partial_j\phi\right)-\dot{\phi}^2\right]^2 \\ &= \frac{1}{2}P_{,X}\left(\frac{1}{(1+\alpha)^2}\dot{\phi}^2-\dot{\phi}^2\right)+\frac{1}{8}P_{,XX}\left(\frac{1}{(1+\alpha)^2}\dot{\phi}^2-\dot{\phi}^2\right)^2 \\ &= \frac{1}{2}P_{,X}\left(\left(1-2\alpha+3\alpha^2\right)\dot{\phi}^2-\dot{\phi}^2\right)+\frac{1}{8}P_{,XX}\left(\left(1-2\alpha+3\alpha^2\right)\dot{\phi}^2-\dot{\phi}^2\right)^2 \\ &= P_{,X}X\left(3\alpha^2-2\alpha\right)+2P_{,XX}X^2\alpha^2, \end{aligned} \tag{5.35}$$

where $X \equiv \dot{\phi}^2/2$ is the kinetic term of the background field. In the above calculation, we have used $\delta\phi = 0$ for our choice of gauge and ignored the third as well as higher order perturbation terms in the third equality. Furthermore, we have

$$\begin{aligned}
-\frac{2}{N^2}P_{,X}\left(\dot{\phi} - N^i\partial_i\phi\right)^2 &= -\frac{2}{(1+\alpha)^2}\left(P_{,X} + P_{,XX}\delta X\right)\dot{\phi}^2 \\
&= -\frac{4}{(1+\alpha)^2}\left(P_{,X} + P_{,XX}X(3\alpha^2 - 2\alpha)\right)X \\
&= -4P_{,X}X\left(1 - 2\alpha + 3\alpha^2\right) - 4P_{,XX}X^2(7\alpha^2 - 2\alpha).
\end{aligned} \tag{5.36}$$

Inserting Eqs. (5.28), (5.33), (5.34), (5.35) and (5.36), into Eq. (5.4), and then keeping the terms up to the linear perturbation, we get

$$0 = -4\partial^2\zeta \underbrace{+6H^2 + 2P - 4PXX}_{=0 \text{due to Eq. (5.20)}} -4H\partial^2\beta + 4X\left(P_{,X} + 2P_{,XX}X\right)\alpha \tag{5.37}$$

5.4 The action for linear perturbations

to compute the action for the perturbation variables, we first expand $\sqrt{-g} = N\sqrt{h}$ up to the secon order in perturbation as

$$\sqrt{-g} = (1+\alpha)a^3e^{3\zeta} = a^3\left(1 + \alpha + 3\zeta + 3\zeta\alpha + \frac{9}{2}\zeta^2\right). \tag{5.38}$$

For convenience, we defined Lagrangian density from the action (5.3) as

$$L \equiv N\sqrt{h}\left[{}^{(3)}R + K_{ij}K^{ij} - K^2 + 2P\right]. \tag{5.39}$$

Substituting Eqs. (5.38), (5.28), (5.33), (5.34) and (5.35) into Eq. (5.39), we obtain the linear Lagrangian density as

$$\begin{aligned}
a^{-3}L &= (1+\alpha+3\zeta)\left(-4\partial^2\zeta - 6H^2 + 4H\partial^2\beta + 2P - 4P_{,X}X\alpha\right) \\
&= \underbrace{-6H^2 + 2P}_{\text{background action Eq. (5.18)}} \underbrace{-4\partial^2\zeta + 4H\partial^2\beta}_{\text{divergent terms}} -4P_{,X}X\alpha - 2(\alpha+3\zeta)(3H^2 - P) \\
&= -4P_{,X}X\alpha - 2(\alpha+3\zeta)(3H^2 - P) = -12H^2\alpha - 12\zeta(3H^2 + \dot{H}) \\
L &= -12a^3H^2\alpha - 36a^3\zeta H^2 \underbrace{-12\frac{d}{dt}\left(a^3\zeta H\right) + 12a^3\left(3H + \dot{\zeta}\right)H}_{\text{rearranged } \dot{H} \text{ term}} \\
&= -12a^3H^2\alpha + 12a^3\dot{\zeta}H = 0\,.
\end{aligned} \tag{5.40}$$

The details for the above calculations are the follows: We have dropped the action for the background universe and the divergent terms in the third line and then used Eq. (5.20) for the α terms and Eq. (5.24) for the ζ terms for the second equality in this line. In the fourth line, we have written the $\dot{H}$ term in form of the "total time derivative", which can be ignored in the fifth line. Finally, Eq. (5.31) is used in the fifth line. We see that the action for the first order perturbation vanishes. Hence, in order to compute the action that governs dynamics of the perturbations, we have to expand the action up to the second order in perturbations.

5.5 The second order action for the perturbations

Substituting Eqs. (5.38), (5.28), (5.33), (5.34) and (5.35) into Eq. (5.39), we obtain the second order Lagrangian density as

$$
\begin{aligned}
L &= a^3\left(1+\alpha+3\zeta+3\zeta\alpha+\frac{9}{2}\zeta^2\right)\left[-4\partial^2\zeta-2\partial_i\zeta\partial^i\zeta-6H^2\underbrace{+4H\left(\partial^2\beta+3\partial_i\zeta\partial^i\beta\right)-4\partial^2\beta\dot{\zeta}}_{=0\ (A)}\right. \\
&\left.\underbrace{+\partial_i\partial_j\beta\partial^i\partial^j\beta-\left(\partial^2\beta\right)^2}_{=0\,\text{after integrating by parts twice}}\quad+2P+2P_{,X}X\left(3\alpha^2-2\alpha\right)+4P_{,XX}X^2\alpha^2\right]. \qquad (5.41)
\end{aligned}
$$

Before we go further, let us show that the second order perturbation in the term (A) in the above equation vanishes.

$$
\begin{aligned}
(A) &= a^3\left(1+\alpha+3\zeta\right)\left(4H\left(\partial^2\beta+3\partial_i\zeta\partial^i\beta\right)-4\partial^2\beta\dot{\zeta}\right) \\
&= a^3\left(4H\left(\partial^2\beta\alpha\quad\underbrace{+3\partial^2\beta\zeta+3\partial_i\zeta\partial^i\beta}_{=0\ afterintegrationbyparts}\right)-4\partial^2\beta\dot{\zeta}\right)=4a^3H\partial^2\beta\left(\alpha-\dot{\zeta}\right)=0 \qquad (5.42)
\end{aligned}
$$

where we have used Eq. (5.31) in the final step. Writing ∂^2 as $h^{ij}\partial_i\partial_j$, Eq. (5.41) becomes

$$
\begin{aligned}
L &= a^3\left(1+\alpha+3\zeta+3\zeta\alpha+\frac{9}{2}\zeta^2\right)\left[-2a^{-2}\epsilon^{-2\zeta}\delta^{ij}\left(2\partial_i\partial_j\zeta+\partial_i\zeta\partial_j\zeta\right)\right. \\
&\quad\left.-6H^2+2P+4P_{,X}X\left(\alpha^2-\alpha\right)+2X\rho_X\alpha^2\right] \\
&= -2a\left(1+\alpha+\zeta\right)\delta^{ij}\left(2\partial_i\partial_j\zeta+\partial_i\zeta\partial_j\zeta\right) \\
&\quad+a^3\left(1+\alpha+3\zeta+3\zeta\alpha+\frac{9}{2}\zeta^2\right)\left(-6H^2+2P+4P_{,X}X\left(\alpha^2-\alpha\right)+2X\rho_X\alpha^2\right) \\
&= -2a\delta^{ij}\left(2\frac{\dot{\zeta}}{H}\partial_i\partial_j\zeta+2\zeta\partial_i\partial_j\zeta+\partial_i\zeta\partial_j\zeta\right) \\
&\quad+a^3\left(2\left(P-3H^2\right)\left(3\zeta\frac{\dot{\zeta}}{H}+\frac{9}{2}\zeta^2\right)-12P_{,X}X\zeta\frac{\dot{\zeta}}{H}+2X\rho_X\left(\frac{\dot{\zeta}}{H}\right)^2\right), \qquad (5.43)
\end{aligned}
$$

where we have used Eq. (5.31) in the last equality. Integrating the terms those depend on $\partial_i\partial_j\zeta$ by parts, and using $\delta^{ij}\partial_i\dot{\zeta}\partial_j\zeta=(\delta^{ij}/2)d(\partial_i\zeta\partial_j\zeta)/dt$ as well as $\zeta\dot{\zeta}=(1/2)d\zeta^2/dt$, the above equation becomes

$$
\begin{aligned}
L &= -2a\delta^{ij}\left(\underbrace{-\frac{1}{H}\frac{d}{dt}(\partial_i\zeta\partial_j\zeta}_{\text{performing integration by parts}}\quad-\partial_i\zeta\partial_j\zeta\right) \\
&\quad+a^3\left(3\left(\underbrace{P-3H^2-2XP_{,X}}_{=-6H^2\text{from Eq. (5.20)}}\right)\underbrace{\frac{\dot{\zeta}^2}{H}}_{\text{integrate by parts}}\quad+9\left(P-3H^2\right)\zeta^2+2X\rho_X\alpha^2\right) \\
&= 2a\delta^{ij}\frac{\dot{H}}{H^2}\partial_i\zeta\partial_j\zeta+2a^3\frac{X\rho_X}{H^2}\dot{\zeta}^2. \qquad (5.44)
\end{aligned}
$$

Inserting the above expression into the action (5.3), and using $\dot{H}=-XP_{,X}$ which is followed from Eqs. (5.20) and (5.24), we get

$$
S^{(2)}=\frac{1}{2}\int dtd^3x\left(2a^3\frac{X\rho_X}{H^2}\dot{\zeta}^2-2a\frac{XP_{,X}}{H^2}(\partial_i\zeta)^2\right). \qquad (5.45)
$$

This is the second order action for the perturbations. Here $(\partial_i\zeta)^2 = \delta^{ij}\partial_i\zeta\partial_j\zeta$. Using definition $d\tau = dt/a$ for the conformal time τ, the above action becomes

$$S^{(2)} = \frac{1}{2}\int d\tau d^3x \left(2a^2\frac{X\rho_X}{H^2}(\zeta')^2 - 2a^2\frac{XP_{,X}}{H^2}(\partial_i\zeta)^2\right) , \tag{5.46}$$

where a prime denotes derivative with respect to conformal time.

For tensor mode, the inflaton field does not source to the tensor mode, so that one can compute the second of action directly from Einstein-Hilbert action. Recalling the metric perturbation for the tensor mode,

$$ds^2 = a^2(\tau)\left[-d\tau^2 + (\delta_{ij} + H_{ij})\,dx^i dx^j\right] ,$$

where $|H_{ij}| \ll 1$. As we have discussed before, tensor H_{ij} has 2 physical degrees of freedom since there are traceless, $\delta^{ij}H_{ij} = 0$, and transverse $\partial^i H_{ij} = 0$. As a result, one can decomposed 2 physical degrees of freedom, or polarizations, by

$$H_{ij} = h_+\, e^+_{ij} + h_\times\, e^\times_{ij},$$

where e^+ and $e^\times$ are the polarization tensors which have the following properties

$$e^\lambda_{ij} = e^\lambda_{ji}, \quad k^i e^\lambda_{ij} = 0, \quad e^\lambda_{ii} = 0,$$
$$e^\lambda_{ij}(-\mathbf{k}) = \left[e^\lambda_{ij}(\mathbf{k})\right]^*, \quad \sum_{ij}\left(e^\lambda_{ij}\right)^* e^{\lambda'}_{ij} = \delta_{\lambda\lambda'}.$$

Notice also that the tensors H_{ij} are gauge-invariant and therefore represent physical degrees of freedom. Here h_+ and $h_\times$ are two polarization modes of the tensor perturbations.

By doing the same step as we have done in scalar perturbation, the action for H_{ij} can be written as

$$S = \frac{1}{8}\int d\tau d^3x \left[a^2 H'_{ij}H'_{ij} - a^2\partial_l H_{ij}\partial_l H_{ij}\right] . \tag{5.47}$$

Bibliography

[1] Sean M. Carroll, “An Introduction to General Relativity: Spacetime and Geometry,” Addison Wesley Press, (2004).

[2] M. P. Hobson, G. P. Efatathiou and A. N. Lasenby, “General Relativity: Introduction for Physicists,” Cambridge University Press, (2006).

[3] Lewis. Ryder, “Introduction to General Relativity,” Cambridge University Press, (2009).

[4] http://ckw.phys.ncku.edu.tw/public/pub/Notes/TheoreticalPhysics/Lawrie_2/Chap04/4.5.4._EddingtonFinkelsteinCoordinates.htm

[5] D. Buamann, "TASI lecture on Inflation", arXiv:hep-th/0907.5424.