Review of constraint analysis
ISHEAP2026 (13th-20th August 2026)

Pichet Vanichchapongjaroen
The Institute for Fundamental Study, Naresuan University, Thailand

13th August 2026

1 Dirac-Bergmann algorithm

Consider a system with generalised coordinates $(q^i) = (q^1, q^2, \cdots, q^n)$. The dynamics of this system is encoded in the Lagrangian[1] $L(q^i, \dot{q}^i)$. The Euler-Lagrange equations are

$$\begin{aligned} 0 &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} \\ &= \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\ddot{q}^j + \frac{\partial^2 L}{\partial \dot{q}^i \partial q^j}\dot{q}^j - \frac{\partial L}{\partial q^i}. \end{aligned} \tag{1.1}$$

Denote

$$H_{ij} \equiv \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}. \tag{1.2}$$

We think of H_{ij} as matrix element of the matrix (H_{ij}) called the Hessian matrix.

- If $\det(H_{ij}) \neq 0$, then we can invert the equation (1.1) to be of the form

  $$\ddot{q}^i = \ddot{q}^i(q^j, \dot{q}^j), \qquad i = 1, 2, \cdots, n. \tag{1.3}$$

  In principle, these equations can be integrated giving

  $$q^i(t) = Q^i(q^j(0), \dot{q}^j(0); t), \tag{1.4}$$

  i.e. the generalised coordinates be completely determined from the $2n$ independent initial variables $(q^j(0), \dot{q}^j(0))$. We say that the system has n degrees of freedom.

- If $\det(H_{ij}) = 0$, then the equations (1.1) cannot be inverted. The equations can be rearranged such that at least one equation does not contain acceleration. That equation gives the relationship between generalised coordinates and generalised velocities at any time, including the initial time. Therefore, the dynamics of the system is determined by less than $2n$ independent initial variables. The system has less than n degrees of freedom, and is said to be a constrained system. More analysis is required.

[1] We suppose that the system is isolated. The Lagrangian does not depend explicitly on time.

A standard approach to analyse constrained systems is to follow Dirac-Bergman algorithm. The starting point is to Legendre transform and consider the Hamiltonian.

The definition of conjugate momenta

$$p_i = \frac{\partial L}{\partial \dot{q}^i} \tag{1.5}$$

can be geometrically interpreted as a map F from a tangent bundle TM to a cotangent bundle T^*M (phase space) as

$$\begin{aligned} F : TM &\to T^*M \\ (q^i, \dot{q}^i) &\mapsto \left(q^i, p_i = \frac{\partial L}{\partial \dot{q}^i} \right). \end{aligned} \tag{1.6}$$

The tangent space $T_{q,\dot{q}}TM$ of the tangent bundle is spanned by

$$E \equiv \left\{ \frac{\partial}{\partial q^1}, \cdots, \frac{\partial}{\partial q^n}, \frac{\partial}{\partial \dot{q}^1}, \cdots, \frac{\partial}{\partial \dot{q}^n} \right\}, \tag{1.7}$$

which is linearly independent. This is transported to

$$F_* E \equiv \left\{ F_* \frac{\partial}{\partial q^1}, \cdots, F_* \frac{\partial}{\partial q^n}, F_* \frac{\partial}{\partial \dot{q}^1}, \cdots, F_* \frac{\partial}{\partial \dot{q}^n} \right\}, \tag{1.8}$$

where

$$F_* \frac{\partial}{\partial q^i} = \frac{\partial}{\partial q^i} + \frac{\partial^2 L}{\partial q^i \partial \dot{q}^j} \frac{\partial}{\partial p_j}, \tag{1.9}$$

$$F_* \frac{\partial}{\partial \dot{q}^i} = \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \frac{\partial}{\partial p_j}. \tag{1.10}$$

If the Hessian matrix is not invertible, the set F_*E is now linearly dependent. It cannot then be used as a basis.

Suppose that F_*E spans an m-dimensional space with $n \leq m < 2n$ everywhere in the phase space, then $F(TM)$ describes an m-dimensional submanifold of the phase space. In this case the Hessian matrix does not have the full rank: $\mathrm{rank}(H_{ij}) = r = m - n < n$. Physical interpretation is that dynamics occur only on $F(TM)$.

The embedding of $F(TM)$ in the phase space is described by $k = 2n - m$ phase space functions $\phi_I(q,p) = 0$, $I = 1, 2, \cdots, k$. The number k is the same as the nullity of (H_{ij}), i.e. the number of linearly independent zero modes, the eigenvectors of (H_{ij}) corresponding to eigenvalue 0. The functions $\phi_I(q,p)$ are called **primary constraints**. The surface $F(TM)$ is called the **primary constraint surface**. They can be obtained by rearranging the definition of conjugate momenta $p_i = \partial L / \partial \dot{q}^i$.

In fact, there are many ways to describe the same surface. Even for a simple example surface $p_1 = 0$, we may also try $(p_1)^2 = 0$. It turns out that describing the surface by $p_1 = 0$ is better than $(p_1)^2 = 0$. This is because the gradient of the surface in the former case is dp_1 whereas that in the latter case is $2p_1 dp_1$ which vanishes on the surface. In general, we require that functions that can be used as the primary constraints ϕ_I should be such that the set $\{d\phi_1, d\phi_2, \cdots, d\phi_k\}$ of gradients is linearly independent everywhere. This requirement is called the **regularity condition**.

Next, we work out the Hamiltonian as the Legendre transformation of the Lagrangian L:

$$H_c = p_i \dot{q}^i - L. \tag{1.11}$$

The Hamiltonian is a phase space function. To see this, we consider

$$dH_c = \dot{q}^i dp_i + \cancelto{0}{\left(p_i - \frac{\partial L}{\partial \dot{q}^i}\right)} d\dot{q}^i - \frac{\partial L}{\partial q^i} dq^i = \dot{q}^i dp_i - \frac{\partial L}{\partial q^i} dq^i, \tag{1.12}$$

which means that $H_c = H_c(q, p)$. Applying differential directly gives

$$dH_c = \frac{\partial H_c}{\partial q^i} dq^i + \frac{\partial H_c}{\partial p_i} dp_i. \tag{1.13}$$

Therefore, comparing the two expressions of dH_c gives

$$-\left(\frac{\partial H_c}{\partial q^i} + \frac{\partial L}{\partial q^i}\right) dq^i + \left(-\frac{\partial H_c}{\partial p_i} + \dot{q}^i\right) dp_i = 0. \tag{1.14}$$

If we only restrict to the constraint surface, the set

$$\{dq^1, \cdots, dq^n, dp_1, \cdots, dp_n\} \tag{1.15}$$

is not linearly independent. For any vector v on the constraint surface $F(TM)$, we have

$$d\phi_I(v) = 0, \qquad I = 1, 2, \cdots, k, \tag{1.16}$$

or we write simply that

$$d\phi_I = 0, \qquad I = 1, 2, \cdots, k, \tag{1.17}$$

on the constraint surface. Therefore,

$$\begin{aligned} 0 &= u^I d\phi_I \\ &= u^I \frac{\partial \phi_I}{\partial q^i} dq^i + u^I \frac{\partial \phi_I}{\partial p_i} dp_i \end{aligned} \tag{1.18}$$

on the constraint surface. Therefore, comparing eq.(1.14) with eq.(1.18) and using Euler-Lagrange equations gives

$$\dot{q}^i = \frac{\partial H_c}{\partial p_i} + u^I \frac{\partial \phi_I}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H_c}{\partial q^i} - u^I \frac{\partial \phi_I}{\partial q^i}. \tag{1.19}$$

Let us define the **total Hamiltonian**

$$H_T = H_c + u^I \phi_I. \tag{1.20}$$

Then

$$\frac{\partial H_T}{\partial q^i} = \frac{\partial H_c}{\partial q^i} + \frac{\partial u^I}{\partial q^i} \phi_I + \frac{\partial \phi_I}{\partial q^i} u^I, \tag{1.21}$$

$$\frac{\partial H_T}{\partial p_i} = \frac{\partial H_c}{\partial p_i} + \frac{\partial u^I}{\partial p_i} \phi_I + \frac{\partial \phi_I}{\partial p_i} u^I. \tag{1.22}$$

On the constraint surface, we have $\phi_I = 0$ and hence

$$\frac{\partial H_T}{\partial q^i} = \frac{\partial H_c}{\partial q^i} + \frac{\partial \phi_I}{\partial q^i} u^I, \qquad \frac{\partial H_T}{\partial p_i} = \frac{\partial H_c}{\partial p_i} + \frac{\partial \phi_I}{\partial p_i} u^I. \tag{1.23}$$

Comparing with eq.(1.19) gives

$$\dot{q}^i = \frac{\partial H_T}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H_T}{\partial q^i}. \tag{1.24}$$

From now on, it is convenient to symbolically distinguish the two kinds of equalities. If two phase space functions $f(q,p)$ and $g(q,p)$ are equal everywhere on the phase space, we say that f and g are **strongly equal** and we denote $f = g$. On the other hand, if f and g are equal on the constrained surface, we say that they are **weakly equal**, and we denote $f \approx g$.

The Hamilton's equation (1.24) can then be written as

$$\dot{q}^i \approx \frac{\partial H_T}{\partial p_i}, \qquad \dot{p}_i \approx -\frac{\partial H_T}{\partial q^i}. \tag{1.25}$$

Vector field X corresponding to the integral curves with parameter t satisfies

$$\dot{f} \approx Xf, \tag{1.26}$$

where f is any function on the phase space. From the Hamilton's equation (1.25), we obtain

$$\dot{f} \approx \left(\frac{\partial H_T}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial H_T}{\partial q^i} \frac{\partial}{\partial p_i} \right) f, \tag{1.27}$$

and hence

$$X = \left(\frac{\partial H_T}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial H_T}{\partial q^i} \frac{\partial}{\partial p_i} \right). \tag{1.28}$$

The vector field X should remain on the constraint surface otherwise the time evolution would allow the trajectory to leave the constraint surface. Since X remains on the constraint surface, it should be annihilated by all $d\phi_I$, i.e.

$$d\phi_I(X) \approx 0, \qquad I = 1, 2, \cdots, k, \tag{1.29}$$

which is equivalent to

$$X\phi_I \approx 0, \qquad I = 1, 2, \cdots, k, \tag{1.30}$$

or[2]

$$0 \approx \{\phi_I, H_T\} \approx \dot{\phi}_I, \qquad I = 1, 2, \cdots, k. \tag{1.31}$$

Here $\{\cdot, \cdot\}$ is the Poisson's bracket defined as

$$\{f, g\} = \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i}, \tag{1.32}$$

where f, g are any phase space functions. Note that

$$\dot{f} \approx \{f, H_T\}. \tag{1.33}$$

Let us consider the requirements $\dot{\phi}_I \approx 0$ for $I = 1, 2, \cdots, k$. From direct calculation, we obtain

$$\dot{\phi}_I \approx \{\phi_I, H_c\} + u^J \{\phi_I, \phi_J\}, \tag{1.34}$$

and hence we demand

$$\{\phi_I, H_c\} + u^J \{\phi_I, \phi_J\} \approx 0. \tag{1.35}$$

After rearranging these equations, non-trivial equations could arise either as

[2] This is known colloquially as "primary constraints are preserved in time". On the other hand, the geometrical interpretation that we discussed is "vector field corresponding to time evolution should remain on the primary constraint surface".

- Conditions on Lagrange's multiplier u^J.
- Existence of extra constraints $\chi_{I'} \approx 0$. We call the new functions $\chi_{I'}$ as **secondary constraints**.

The existence of secondary constraints redefines the constraint surface to

$$\phi_I \approx 0, \qquad \chi_{I'} \approx 0, \qquad I = 1, 2, \cdots, k, \;\; I' = 1, 2, \cdots, k'. \tag{1.36}$$

The vector field X should be on this new constraint surface, i.e.

$$d\chi_{I'}(X) \approx 0, \qquad I' = 1, 2, \cdots, k', \tag{1.37}$$

which is equivalent to

$$0 \approx \dot{\chi}_{I'} \approx \{\chi_{I'}, H_c\} + u^J\{\chi_{I'}, \phi_J\}. \tag{1.38}$$

We should check whether these consistency conditions lead to further constraints. If this is such a case, we check whether the extra consistency conditions lead to even further constraints. We repeat the process until there is no more constraint generated.

Let us collect all the constraints and denote them as $\Phi_{\mathcal{I}}$ where $\mathcal{I} = 1, 2, \cdots, \hat{k}$. By construction, $\Phi_{\mathcal{I}}$ are all independent. This means that $\{d\Phi_1, d\Phi_2, \cdots, d\Phi_{\hat{k}}\}$ is linearly independent everywhere on the constraint surface. This corresponds to that constraint surface is a $(2n - \hat{k})$-dimensional submanifold of the phase space. Sometimes it is useful for the calculation to write constraints that are not all independent. However, we will not consider those cases here.

Let f be a function on phase space. If its Poisson bracket with every constraint vanishes on the constraint surface:

$$\{f, \Phi_{\mathcal{I}}\} \approx 0, \qquad \mathcal{I} = 1, 2, \cdots, \hat{k}, \tag{1.39}$$

then f is said to be a **first-class function**. On the other hand, if f is not a first-class function, then it is said to be a **second-class function**.

We are interested in classifying the constraints themselves. For this, let us define the $\hat{k} \times \hat{k}$ antisymmetric matrix on phase space:

$$C_{\mathcal{I}\mathcal{J}} \equiv \{\Phi_{\mathcal{I}}, \Phi_{\mathcal{J}}\}. \tag{1.40}$$

Suppose that $v^{\mathcal{I}}$ is a zero mode, i.e.

$$v^{\mathcal{I}} C_{\mathcal{I}\mathcal{J}} \approx 0. \tag{1.41}$$

Then

$$v^{\mathcal{I}} \Phi_{\mathcal{I}} \tag{1.42}$$

is a first-class constraint, i.e.

$$\{v^{\mathcal{I}} \Phi_{\mathcal{I}}, \Phi_{\mathcal{J}}\} \approx 0, \qquad \text{for } \mathcal{J} = 1, 2, \cdots, \hat{k}. \tag{1.43}$$

If there are k_1 zero modes, then there are k_1 first-class constraints. The other $k_2 = \hat{k} - k_1$ constraints are of second-class.

Therefore, after appropriate redefinitions, the constraints are classified into

- First-class constraints $\phi_a, a = 1, 2, \cdots, k_1$

- Second-class constraints $\chi_\alpha, \alpha = 1, 2, \cdots, k_2$.

As a consistency check, the antisymmetric matrix with elements

$$\{\chi_\alpha, \chi_\beta\} \tag{1.44}$$

must be invertible. One immediate consequence is that k_2 is an even number.

Having classified the constraints, an important consequence is the calculation of the number of degrees of freedom. Recall that the number of phase space dimensions is $2n$, the number of first-class constraints is k_1, and the number of second-class constraints is k_2. The number of degrees of freedom can be obtained from the formula

$$\frac{2n - 2k_1 - k_2}{2}. \tag{1.45}$$

As we have mentioned earlier, k_2 is an even number, it is clear that the number of degrees of freedom is a non-negative integer.

2 Maxwell Theory

Consider the Maxwell theory in 4 dimensional spacetime with coordinates x^μ, $\mu = 0, 1, 2, 3$. The metric is $(\eta_{\mu\nu}) = \mathrm{diag}(-1, 1, 1, 1)$. The action is

$$S = -\frac{1}{4}\int d^4x\ F_{\mu\nu}F^{\mu\nu}, \tag{2.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

The constraint analysis of field theory follows steps similarly to what outlined in the previous section. Key modifications to make calculation simpler are for example that we consider Lagrangian density and Hamiltonian density.

The Lagrangian density is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{2.2}$$

The conjugate momenta are

$$\begin{aligned} \pi^\mu &= \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} \\ &= \frac{\partial \mathcal{L}}{\partial \partial_0 A_\mu} \\ &= -F^{\rho\sigma}\frac{\partial \partial_\rho A_\sigma}{\partial \partial_0 A_\mu} \\ &= -F^{0\mu}. \end{aligned} \tag{2.3}$$

We immediately see that the primary constraint is

$$\phi_1 \equiv \pi^0 \approx 0. \tag{2.4}$$

The Hessian is

$$\begin{aligned} \mathcal{H}^{\mu\nu} &= \frac{\partial^2 \mathcal{L}}{\partial \dot{A}_\mu \partial \dot{A}_\nu} \\ &= \eta^{\mu\nu} + \eta^{0\mu}\eta^{0\nu}. \end{aligned} \tag{2.5}$$

Suppose that v_ν is a zero mode. Then

$$\mathcal{H}^{\mu\nu} v_\nu = 0. \tag{2.6}$$

This implies

$$v^\mu + \eta^{\mu 0} v^0 = 0. \tag{2.7}$$

Therefore, there is only one independent zero mode: $(v_\mu) = (1, 0, 0, 0)$. Therefore, there is only one primary constraint.

The canonical Hamiltonian density is given by

$$\begin{aligned}
\mathcal{H}_c &= \pi^\mu \dot{A}_\mu - \mathcal{L} \\
&= \pi_i \dot{A}_i - \frac{1}{2}\pi_i\pi_i + \frac{1}{4}F_{ij}F_{ij} \\
&= \pi_i(\partial_0 A_i - \partial_i A_0) + \pi_i \partial_i A_0 - \frac{1}{2}\pi_i\pi_i + \frac{1}{4}F_{ij}F_{ij} \\
&= \pi_i\pi_i + \pi_i \partial_i A_0 - \frac{1}{2}\pi_i\pi_i + \frac{1}{4}F_{ij}F_{ij} \\
&= \frac{1}{2}\pi_i\pi_i + \frac{1}{4}F_{ij}F_{ij} + \pi_i \partial_i A_0.
\end{aligned} \tag{2.8}$$

Note that we have imposed the primary constraint $\pi^0 \approx 0$ in the calculation. Next, we then define total Hamiltonian density

$$\begin{aligned}
\mathcal{H}_T &= \mathcal{H}_c + u\pi^0 \\
&= \frac{1}{2}\pi_i\pi_i + \frac{1}{4}F_{ij}F_{ij} + \pi_i \partial_i A_0 + u\pi^0.
\end{aligned} \tag{2.9}$$

The total Hamiltonian is

$$H_T = \int d^3\vec{x}\ \mathcal{H}_T. \tag{2.10}$$

Now we compute the time derivative (note that we use total Hamiltonian) of primary constraint:

$$\begin{aligned}
\dot{\pi}^0 &= \{\pi^0, H_T\} \\
&= \int d^3\vec{y} \left(\frac{\delta \pi^0(t,\vec{x})}{\delta A_\mu(t,\vec{y})} \frac{\delta H_T(t)}{\delta \pi^\mu(t,\vec{y})} - \frac{\delta H_T(t)}{\delta A_\mu(t,\vec{y})} \frac{\delta \pi^0(t,\vec{x})}{\delta \pi^\mu(t,\vec{y})} \right).
\end{aligned} \tag{2.11}$$

The equal-time functional derivatives are defined such that

$$\frac{\delta A_\mu(t,\vec{x})}{\delta A_\nu(t,\vec{y})} = \delta^\nu_\mu \delta^{(3)}(\vec{x}-\vec{y}), \qquad \frac{\delta \pi^\mu(t,\vec{x})}{\delta A_\nu(t,\vec{y})} = 0, \tag{2.12}$$

$$\frac{\delta A_\mu(t,\vec{x})}{\delta \pi^\nu(t,\vec{y})} = 0, \qquad \frac{\delta \pi^\mu(t,\vec{x})}{\delta \pi^\nu(t,\vec{y})} = \delta^\mu_\nu \delta^{(3)}(\vec{x}-\vec{y}). \tag{2.13}$$

Therefore

$$\begin{aligned}
\dot{\pi}^0 &= \int d^3\vec{y} \left(-\frac{\delta H_T(t)}{\delta A_0(t,\vec{y})} \delta^{(3)}(\vec{x}-\vec{y}) \right) \\
&= -\frac{\delta H_T(t)}{\delta A_0(t,\vec{x})} \\
&= -\frac{\delta}{\delta A_0(t,\vec{x})} \int d^3\vec{z}\, \pi_i(t,\vec{z}) \partial^z_i A_0(t,\vec{z}) \\
&= -\int d^3\vec{z}\, \pi_i(t,\vec{z}) \partial^z_i \delta^{(3)}(\vec{x}-\vec{z}) \\
&= \partial_i \pi_i.
\end{aligned} \tag{2.14}$$

There is thus one secondary constraint:

$$\phi_2 \equiv \partial_i \pi_i \approx 0. \tag{2.15}$$

From direct calculation [exercise], we see from requiring $\dot{\phi}_2 \approx 0$ that there is no more constraint.

Let us now classify the constraints. Since ϕ_1 and ϕ_2 only depend on conjugate momenta, it is easy to see that their Poisson's brackets vanish and hence both ϕ_1 and ϕ_2 are first-class constraints.

There are 8 phase space variables: A_μ, π^μ and 2 first-class constraints. Therefore, the number of degrees of freedom (at each spatial position) is

$$\frac{2 \times 4 - 2 \times 2}{2} = 2, \tag{2.16}$$

as expected from the number of polarizations of photon.

Each first-class constraint corresponds to gauge symmetry. This suggests that there are several points on the phase space that correspond to the same physical state. We may introduce gauge fixing conditions to eliminate the redundancy. As is well-known, Coulomb gauge condition completely fixes the gauge of Maxwell's theory. To do this in the language of constraint analysis, we introduce two constraints

$$\chi_1 = A_0, \qquad \chi_2 = \partial_i A_i \tag{2.17}$$

i.e. one gauge-fixing constraint per first-class constraint. It can be shown [exercise] that the consistency requirements $\dot{\chi}_1 \approx 0, \dot{\chi}_2 \approx 0$ do not introduce extra constraint. Instead, this fixes the Lagrange multiplier to $u \approx 0$.

Alternatively, [exercise] we may only introduce $\chi_2 \approx 0$, and see that $\chi_1 \approx 0$ arises from consistency requirement $\dot{\chi}_2 \approx 0$ along with the appropriate boundary condition at spatial infinity.

Let us now classify the constraints. Ideally, we expect that all the original first-class constraints become second-class constraints and that the gauge-fixing constraints are also second-class. By working out the matrix formed by Poisson's brackets of the constraints $\phi_1, \phi_2, \chi_1, \chi_2$, it can be shown [exercise] for example that there is no zero mode and hence all the constraints are of second-class.

The number of degrees of freedom remains the same. Since there are now 4 second-class constraints, the number of degrees of freedom is given by

$$\frac{2 \times 4 - 4}{2} = 2. \tag{2.18}$$

When there are only second-class constraints, one might solve them to obtain the **reduced phase space**. In our case, the solution of $(\phi_1, \phi_2, \chi_1, \chi_2) \approx (0, 0, 0, 0)$, is given by

$$A_0 = 0, \qquad A_i = A_i^T, \tag{2.19}$$

$$\pi^0 = 0, \qquad \pi_i = \pi_i^T, \tag{2.20}$$

where A_i^T and π_i^T satisfy $\partial_i A_i^T = 0 = \partial_i \pi_i^T$. By counting [exercise], there are 2 independent components of A_i^T and 2 of π_i^T. In the reduced phase space, there is no constraint. Therefore, the number of degrees of freedom is directly obtained from half of the dimensions of the reduced phase space.

3 ADM Decomposition of Einstein's gravity theory

In this section, let us analyse the gravity Einstein-Hilbert action. General relativity suggests that space and time are on equal footing. However, Hamiltonian analysis demands that time has a special role. The time evolution is described by Hamiltonian. It is therefore important to separate the spacetime of general relativity so that accommodates time evolution. The successful approach called ADM decomposition is invented by Arnowitt, Deser, and Misner in 1959.

In the ADM decomposition, 4-dimensional spacetime is sliced into constant-time hypersurfaces. The covariant prescription of metric $g_{\mu\nu}, \mu, \nu \in \{0, 1, 2, 3\}$ is thus not appropriate. Instead, the 10 independent components are expressed in terms of 10 independent variables which are

- The lapse function N. This is the normalisation of the normal dt to the surface.
- The shift vectors N^i. This is encodes the unalignment between the direction of the vector $g^{-1}(dt) = g^{0\mu}\partial_\mu$ and the tangent ∂_t to the curves with parameter t.
- The metric γ_{ij}, $i, j \in \{1, 2, 3\}$ on the hypersurface.

It can be worked out that the decomposition of spacetime metric is

$$ds^2 = -N^2 dt^2 + \gamma_{ij}(dx^i + N^i dt)(dx^j + N^j dt). \tag{3.1}$$

The Einstein-Hilbert action is given by

$$S = \int d^4x \sqrt{-g} R, \tag{3.2}$$

where $g = \det(g_{\mu\nu})$ and R is the Ricci scalar. The Lagrangian density

$$\mathcal{L} = \sqrt{-g} R \tag{3.3}$$

can be expressed in terms of the variables N, N^i, γ_{ij}. Let us denote the conjugate momenta of these variables as

$$\Pi_0, \qquad \Pi_i, \qquad \zeta^{ij}, \tag{3.4}$$

respectively. It can be shown that the Lagrangian density does not contain time derivatives on N and N^i. However, it contains time derivative on γ_{ij}. Therefore, we immediately see that

$$\Pi_0 \approx 0, \qquad \Pi_i \approx 0 \tag{3.5}$$

are primary constraints. From direct calculation, we obtain

$$\zeta^{ij} = \frac{\partial \mathcal{L}}{\partial \dot{\gamma}_{ij}} = -\sqrt{\gamma}([K]\gamma^{ij} - K^{ij}), \tag{3.6}$$

where

$$K_{ij} = \frac{1}{2N}\left(\dot{\gamma}_{ij} - N^k \partial_k \gamma_{ij} - 2\partial_{(i} N^k \gamma_{j)k}\right), \tag{3.7}$$

and $[K] \equiv \gamma^{ij} K_{ij}$. The tensor K_{ij} is called the **extrinsic curvature**.

From a lengthy direct calculation, we obtain the canonical Hamiltonian density (up to total derivative)

$$\begin{aligned}\mathcal{H} &= \zeta^{ij}\dot{\gamma}_{ij} - \mathcal{L} \\ &= N\mathcal{H}_0 + N^k\mathcal{H}_k.\end{aligned} \tag{3.8}$$

$$\mathcal{H}_0 \equiv -\sqrt{\gamma}^{(3)}R - \frac{1}{2\sqrt{\gamma}}[\zeta]^2 + \frac{1}{\sqrt{\gamma}}[\zeta^2], \qquad \mathcal{H}_k \equiv -2\zeta^i{}_{k|i}. \tag{3.9}$$

Here ${}^{(3)}R$ is the Ricci scalar on the 3D hypersurface, $[\zeta] \equiv \zeta^{ij}\gamma_{ij}, [\zeta^2] = \zeta^{ij}\gamma_{jk}\zeta^{kl}\gamma_{li}$, whereas the subscript vertical bar $_{|i}$ denotes covariant derivative that is compatible with the hypersurface metric γ_{ij}.

The total Hamiltonian is then

$$\mathcal{H} = N\mathcal{H}_0 + N^k\mathcal{H}_k + u^0\Pi_0 + u^i\Pi_i, \tag{3.10}$$

where u^0, u^i are Lagrange multipliers. Consistency requirements $\dot{\Pi}_0 \approx 0, \dot{\Pi}_i \approx 0$ suggest [exercise] that $\mathcal{H}_0$ and $\mathcal{H}_i$ are secondary constraints. We call $\mathcal{H}_0$ the **Hamiltonian constraint** and call $\mathcal{H}_i$ the **momentum constraints**. Consistency requirements $\dot{\mathcal{H}}_0 \approx 0, \dot{\mathcal{H}}_i \approx 0$ do not lead to new constraints nor fix Lagrange multipliers.

We may now classify the constraints $\Pi_0, \Pi_i, \mathcal{H}_0, \mathcal{H}_i$. Direct calculation suggests that Poisson bracket between any pair of these constraints weakly vanish. Therefore the system has only first-class constraints. There are 8 first-class constraints in total. Since the number of phase space dimensions is 2×10, the number of degrees of freedom is

$$\frac{2 \times 10 - 2 \times 8}{2} = 2, \tag{3.11}$$

as expected from the number of polarisations (the + and × modes) of gravity.