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From particle mechanics to fields

Dynamics of a point particle is described by

$$S[q] = \int dt L(q, \dot{q}) \tag{1}$$

where $q = q(t)$ is a dynamical variable.
Field theory generalizes this dynamical variable to

$$q(t) \quad \longrightarrow \quad \phi_i(x^\mu)$$

whose value depends on points in spacetime x^μ. Then, the action for field becomes

$$S[\phi] = \int d^4x \mathcal{L}(\phi_i, \partial_\mu \phi_i). \tag{2}$$

$\mathcal{L}(\phi_i, \partial_\mu \phi_i)$ is Lagrangian density

The field $\phi_i(x)$ varies by theories

Theory	**Field**	**Lagrangian**
Klein-Gordon	ϕ	$-\frac{1}{2}(\partial\phi)^2 - V(\phi)$
Dirac	ψ^α	$\bar{\psi}(i\gamma \cdot \partial - m)\psi$
Maxwell	A_μ	$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$
Yang-Mills	A^a_μ	$-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}$
GR	$g_{\mu\nu}$	$\frac{M_P^2}{2}R$

Consider

$$S = \int_{\mathcal{M}} d^4x \mathcal{L}(\phi_i, \partial_\mu \phi_i). \tag{3}$$

Take a variation $\phi_i \to \phi_i + \delta\phi_i$.

$$\delta S = \int_{\mathcal{M}} d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi_i} \delta\phi_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \delta\partial_\mu \phi_i \right) \tag{4}$$

Use $\delta\partial_\mu\phi_i = \partial_\mu\delta\phi_i$ and apply integration by parts gives

$$\delta S = \underbrace{\int_{\mathcal{M}} d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right) \right] \delta\phi_i}_{\text{EOM}} + \underbrace{\int_{\mathcal{M}} d^4x \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \delta\phi_i \right)}_{\text{Boundary term}}. \tag{5}$$

Therefore, we have

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right) - \frac{\partial \mathcal{L}}{\partial \phi_i} = 0. \tag{6}$$

Example: Klein–Gordon Field

$$\mathcal{L} = -\frac{1}{2}\partial_\mu \phi \, \partial^\mu \phi - V(\phi)$$

where $V(\phi)$ is the scalar potential.

The Euler–Lagrange equation is

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0.$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = -\partial^\mu \phi, \qquad \frac{\partial \mathcal{L}}{\partial \phi} = -V'(\phi).$$

Therefore,

$$-\partial_\mu \partial^\mu \phi + V'(\phi) = 0.$$

Example: Maxwell Field

Consider the electromagnetic field

$$A_\mu(x).$$

The gauge-invariant field strength is

$$\boxed{F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.}$$

The Maxwell Lagrangian is

$$\boxed{\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J^\mu A_\mu.}$$

We now derive the equation of motion for A_μ.

Since

$$\delta F_{\mu\nu} = \partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu,$$

we have

$$\delta\mathcal{L} = -\frac{1}{2} F^{\mu\nu} \delta F_{\mu\nu} - J^\mu \delta A_\mu.$$

Using the antisymmetry

$$F^{\mu\nu} = -F^{\nu\mu},$$

this becomes

$$\delta\mathcal{L} = -F^{\mu\nu}\partial_\mu \delta A_\nu - J^\nu \delta A_\nu.$$

Integrating by parts,

$$-F^{\mu\nu}\partial_\mu \delta A_\nu = -\partial_\mu \left(F^{\mu\nu}\delta A_\nu\right) + (\partial_\mu F^{\mu\nu})\delta A_\nu.$$

Ignoring the boundary term,

$$\delta S = \int d^4x \left[\partial_\mu F^{\mu\nu} - J^\nu\right] \delta A_\nu.$$

Since δA_ν is arbitrary,

$$\boxed{\partial_\mu F^{\mu\nu} = J^\nu.}$$

This is the inhomogeneous Maxwell equation.

The other Maxwell equation follows from the identity

$$\partial_{[\mu} F_{\nu\rho]} = 0.$$

What is a Symmetry?

Consider a field theory described by

$$S[\phi] = \int_{\mathcal{M}} d^4x \, \mathcal{L}(\phi_i, \partial_\mu \phi_i).$$

A transformation

$$\phi_i \to \phi_i + \delta\phi_i$$

is a symmetry if the action is invariant:

$$\boxed{\delta S = 0.}$$

More generally, the Lagrangian may change by a total derivative:

$$\boxed{\delta \mathcal{L} = \partial_\mu K^\mu.}$$

Then

$$\delta S = \int_{\mathcal{M}} d^4x \, \partial_\mu K^\mu = \int_{\partial\mathcal{M}} d\Sigma_\mu K^\mu$$

which vanishes for suitable boundary conditions.

Spacetime Transformations

Consider a linear transformation of spacetime coordinates

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} + a^{\mu}.$$

Here

$$a^{\mu} = \text{translation}, \qquad \Lambda^{\mu}{}_{\nu} = \text{Lorentz transformation}.$$

A Lorentz transformation satisfies

$$\Lambda^{\rho}{}_{\mu} \Lambda^{\sigma}{}_{\nu} \eta_{\rho\sigma} = \eta_{\mu\nu}.$$

Equivalently,

$$\Lambda^{T} \eta \Lambda = \eta.$$

The Lorentz group is $SO(1,3)$. and the full Poincaré group combines

$$\text{translations} \ltimes SO(1,3).$$

How Do Fields Transform?

Under

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} + a^{\mu},$$

different tensor fields transform according to their tensorial nature.

Scalar

$$\phi'(x') = \phi(x)$$

Vector

$$V'^{\mu}(x') = \Lambda^{\mu}{}_{\nu} V^{\nu}(x)$$

Covector

$$V'_{\mu}(x') = (\Lambda^{-1})^{\nu}{}_{\mu} V_{\nu}(x)$$

Rank-2 tensor

$$T'^{\mu\nu}(x') = \Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\sigma} T^{\rho\sigma}(x)$$

For a covariant rank-2 tensor,

$$T'_{\mu\nu}(x') = (\Lambda^{-1})^{\rho}{}_{\mu} (\Lambda^{-1})^{\sigma}{}_{\nu} T_{\rho\sigma}(x).$$

Infinitesimal Lorentz Transformation

Write

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu},$$

where

$$\omega_{\mu\nu} = -\omega_{\nu\mu}.$$

Therefore

$$x'^{\mu} = x^{\mu} + \omega^{\mu}{}_{\nu} x^{\nu}.$$

For a scalar field,

$$\phi'(x') = \phi(x).$$

At the same spacetime point x,

$$\delta\phi(x) = \phi'(x) - \phi(x) = -\omega^{\mu}{}_{\nu} x^{\nu} \partial_{\mu}\phi.$$

For a vector field,

$$\delta V^\mu = -\omega^\rho{}_\sigma x^\sigma \partial_\rho V^\mu + \omega^\mu{}_\nu V^\nu.$$

For a co-vector field,

$$\delta A_\mu = -\omega^\rho{}_\sigma x^\sigma \partial_\rho A_\mu - \omega^\nu{}_\mu A^\nu.$$

For a rank-2 tensor,

$$\delta g_{\mu\nu} = -\omega^\rho{}_\sigma x^\sigma \partial_\rho g_{\mu\nu} - \omega^\rho{}_\mu g_{\rho\nu} - \omega^\rho{}_\nu g_{\mu\rho}.$$

For a translation,

$$x'^\mu = x^\mu + a^\mu,$$

For any tensor field,

$$\boxed{\delta\Phi = -a^\mu \partial_\mu \Phi}$$

where Φ is an arbitrary field.

Internal Symmetry

An internal transformation does not act on spacetime: $x^\mu \to x^\mu$.
Instead, it acts directly on the fields.
For a multiplet

$$\Psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_N \end{pmatrix},$$

consider

$$\Psi \to U\Psi.$$

For a continuous symmetry,

$$U = e^{i\alpha^a T^a}.$$

Infinitesimally,

$$\delta\Psi = i\alpha^a T^a \Psi.$$

Here T^a are the generators of the symmetry group.

Example: Global $U(1)$ Symmetry

Consider a complex scalar Lagrangian

$$\mathcal{L} = -\partial_\mu \phi^* \partial^\mu \phi - V(\phi^* \phi)$$

is invariant under

$$\phi \to e^{i\alpha}\phi, \qquad \alpha = \text{constant}.$$

Infinitesimally,

$$\delta\phi = i\alpha\phi, \qquad \delta\phi^* = -i\alpha\phi^*.$$

Since

$$\delta\mathcal{L} = 0,$$

this is a global continuous symmetry.

Noether's theorem will give a conserved current sartisfying

$$\partial_\mu J^\mu = 0.$$

$SU(N)$ Symmetry

Let Ψ transform in the fundamental representation:

$$\Psi \to U\Psi, \qquad U = e^{i\alpha^a T^a}.$$

The generators satisfy

$$[T^a, T^b] = if^{abc}T^c.$$

Infinitesimally,

$$\delta\Psi = i\alpha^a T^a \Psi.$$

For a global symmetry, according to the Noether's theorem, the theory has one conserved current for each generator:

$$\partial_\mu J^{\mu a} = 0, \qquad a = 1, \ldots, N^2 - 1.$$

Hence there are $N^2 - 1$ Noether charges.

Noether's Theorem

Under

$$\phi_i \to \phi_i + \delta\phi_i,$$

the Lagrangian density changes as

$$\delta\mathcal{L} = \left[\frac{\partial\mathcal{L}}{\partial\phi_i} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\right)\right]\delta\phi_i + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\delta\phi_i\right).$$

Being on-shell, the first term vanishes. For a symmetry,

$$\delta\mathcal{L} = \partial_\mu K^\mu.$$

Therefore, one can introduce a conserved current

$$J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\delta\phi_i - K^\mu.$$

satisfying

$$\partial_\mu J^\mu = 0,$$

From Current to Conserved Charge

The conservation equation is

$$\partial_\mu J^\mu = 0.$$

In $3+1$ dimensions,

$$\frac{\partial J^0}{\partial t} + \nabla \cdot \mathbf{J} = 0.$$

Define

$$\boxed{Q = \int d^3x\, J^0.}$$

Then

$$\frac{dQ}{dt} = -\int d^3x\, \nabla \cdot \mathbf{J} = -\oint_{\partial\Sigma} \mathbf{J} \cdot d\mathbf{S}.$$

For suitable boundary conditions,

$$\frac{dQ}{dt} = 0.$$

Examples of Noether's Theorem

Symmetry	**Current**	**Conserved quantity**
Time translation	$T^{\mu 0}$	Energy E
Spatial translation	$T^{\mu i}$	Momentum P^i
Lorentz transformation	$M^{\mu\rho\sigma}$	Angular momentum
Global $U(1)$	J^{μ}	Charge Q
$SU(N)$	$J^{\mu a}$	Q^a

$$\boxed{\text{Symmetry} \quad \longrightarrow \quad \text{Conservation law}}$$

Example: Noether Current for Translations

Consider a real scalar field

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - V(\phi).$$

Under an infinitesimal translation, $x^\mu \to x^\mu + a^\mu$, the scalar field transforms as

$$\delta\phi = -a^\nu\partial_\nu\phi.$$

$$\delta\mathcal{L} = -a^\nu\partial_\nu\mathcal{L} = -\partial_\mu(a^\mu\mathcal{L}).$$

Thus

$$K^\mu = -a^\mu\mathcal{L}.$$

Using

$$J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi - K^\mu,$$

we obtain

$$J^\mu = -a^\nu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi - \delta^\mu{}_\nu\mathcal{L}\right].$$

Since a^ν is an arbitrary constant parameter, define

$$J^\mu = -a^\nu T^\mu{}_\nu.$$

The canonical energy-momentum tensor is therefore

$$T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \partial_\nu \phi_i - \delta^\mu{}_\nu \mathcal{L}.$$

For

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - V(\phi),$$

we have

$$T^\mu{}_\nu = -\partial^\mu\phi\,\partial_\nu\phi - \delta^\mu{}_\nu \mathcal{L}.$$

Equivalently,

$$T_{\mu\nu} = -\partial_\mu\phi\,\partial_\nu\phi - g_{\mu\nu}\mathcal{L}$$

satisfying

$$\partial_\mu T^\mu{}_\nu = 0.$$

For the scalar field,

$$T_{\mu\nu} = -\partial_\mu \phi \, \partial_\nu \phi - g_{\mu\nu} \mathcal{L}.$$

Therefore,

$$T_{\mu\nu} = T_{\nu\mu}.$$

So the canonical energy-momentum tensor of a standard scalar field is already symmetric.

However, for a general field carrying spacetime indices, the canonical tensor

$$T^{\mu}{}_{\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \partial_\nu \Phi - \delta^{\mu}{}_{\nu} \mathcal{L}$$

need not be symmetric.

Examples include

Dirac field, vector field.

This motivates the **Belinfante–Rosenfeld improvement**.

Belinfante–Rosenfeld Energy-Momentum Tensor

One can construct an symmetrize the canonical energy-momentum tensor via

$$T^{\mu\nu}_{\mathrm{BR}} = T^{\mu\nu}_{\mathrm{can}} + \partial_\lambda B^{\lambda\mu\nu},$$

where $B^{\lambda\mu\nu}$ is anti-symmetric in the first two indices, i.e. $B^{\lambda\mu\nu} = -B^{\mu\lambda\nu}$. This guarantees that

$$\partial_\lambda \partial_\mu B^{\lambda\mu\nu} = 0.$$

Therefore, $T^{\mu\nu}_{\mathrm{BR}}$ would obey the continuity equation.
This tensor matches with the Hilbert stress-energy tensor defined as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_{\mathrm{m}}}{\delta g^{\mu\nu}}.$$

Notice that the tensor is indeed symmetric.

From Global to Local Symmetry

Consider a complex scalar field with global $U(1)$ symmetry:

$$\phi(x) \to e^{iq\alpha}\phi(x), \qquad \alpha = \text{constant}.$$

Now ask: What if the symmetry parameter can vary in spacetime?

$$\alpha \to \alpha(x)$$

Then

$$\partial_\mu \phi \to e^{iq\alpha(x)} \left[\partial_\mu \phi + iq(\partial_\mu \alpha)\phi\right].$$

Therefore,

$$\partial_\mu \phi$$

does not transform in the same way as ϕ. Ordinary derivative is not gauge covariant.
We need a new object.

Introduce the gauge field

$$A_\mu(x)$$

and define

$$D_\mu = \partial_\mu + iqA_\mu.$$

Demand that

$$D_\mu\phi \to e^{iq\alpha(x)} D_\mu\phi.$$

This requires

$$A_\mu \to A_\mu - \partial_\mu\alpha.$$

Indeed,

$$D'_\mu\phi' = (\partial_\mu + iqA'_\mu)e^{iq\alpha}\phi = e^{iq\alpha} D_\mu\phi.$$

Thus

$$D_\mu\phi \text{ transforms covariantly.}$$

The gauge field A_μ is the **connection** associated with local $U(1)$ symmetry.

The Maxwell Action

The covariant derivatives do not generally commute:

$$[D_\mu, D_\nu]\phi.$$

For $U(1)$,

$$[D_\mu, D_\nu] = iqF_{\mu\nu},$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

The field strength is gauge invariant.
Therefore, the simplest gauge-invariant action is

$$S_{\mathrm{EM}} = -\frac{1}{4}\int d^4x\, F_{\mu\nu}F^{\mu\nu}.$$

Noether's Second Theorem

Noether's First Theorem:

$$\text{continuous global symmetry} \quad \Longrightarrow \quad \text{conserved current.}$$

For a local symmetry, the transformation depends on arbitrary functions of spacetime:

$$\epsilon^a = \epsilon^a(x).$$

Suppose

$$\delta\phi_i = R_i{}^a \epsilon^a + R_i{}^{a\mu} \partial_\mu \epsilon^a + \cdots .$$

The variation of the action can be written as

$$\delta S = \int d^4x \, E_i \, \delta\phi_i + \text{boundary terms},$$

where

$$E_i = \frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right).$$

Since $\epsilon^a(x)$ is arbitrary,

$$E_i R_i^a - \partial_\mu \left(E_i R_i^{a\mu} \right) + \cdots \equiv 0.$$

These are **Noether identities**.
Consider the example:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}.$$

The local transformation is

$$\boxed{A_\mu \to A_\mu + \partial_\mu \alpha(x).}$$

Since

$$\delta A_\mu = \partial_\mu \alpha,$$

the variation of the action is

$$\delta S = \int d^4x \, \frac{\delta S}{\delta A_\mu} \partial_\mu \alpha.$$

Integrating by parts,

$$\delta S = -\int d^4x\, \alpha\, \partial_\mu \left(\frac{\delta S}{\delta A_\mu} \right).$$

Because $\alpha(x)$ is arbitrary,

$$\partial_\mu \left(\frac{\delta S}{\delta A_\mu} \right) \equiv 0.$$

For Maxwell theory,

$$\frac{\delta S}{\delta A_\mu} = \partial_\nu F^{\nu\mu},$$

so

$$\partial_\mu \partial_\nu F^{\nu\mu} \equiv 0.$$

Gauge Redundancy

The gauge transformation relates different descriptions of the **same physical configuration**. Thus

$$\boxed{\text{gauge symmetry} = \text{redundancy in the description.}}$$

Noether's Second Theorem implies that the field equations (Euler-Lagrange equations) are not independent. You cannot uniquely solve them for all components of the field.

This means the system contains unphysical, unobservable degrees of freedom (gauge freedom). For electromagnetism, the four components of A_μ do not represent four independent physical degrees of freedom.

$$A_\mu : \qquad 4 \text{ components} \quad \longrightarrow \quad 2 \text{ physical DOF.}$$

Gauge Theory $\longrightarrow$ Gravity

Gauge Theory	**Gravity**
A_μ	$\Gamma^\rho{}_{\mu\nu}$
Connection	Connection
D_μ	∇_μ
$[D_\mu, D_\nu]$	$[\nabla_\mu, \nabla_\nu]$
$F_{\mu\nu}$	$R^\rho{}_{\sigma\mu\nu}$
Gauge curvature	Spacetime curvature

Gauge theory:

$$\boxed{[D_\mu, D_\nu] \sim F_{\mu\nu}}$$

Gravity:

$$\boxed{[\nabla_\mu, \nabla_\nu] V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma}$$

Gravity as Field Theory

Consider a general coordinate transformations

$$x^\mu \to x'^\mu(x).$$

The basic strategy is analogous to gauge theory:

$$\begin{array}{ccl}
U(1) & \longrightarrow & \text{local } U(1) \\
 & & \downarrow \\
 & & A_\mu \\
\hline
\text{global coordinates transformations} & \longrightarrow & \text{local coordinate transformations} \\
 & & \downarrow \\
 & & g_{\mu\nu}
\end{array}$$

The metric plays the role of the gravitational field.

Under general coordinates translations, $x^\mu \to x'^\mu(x)$,
the a vector field transforms as,

$$V^\mu \to V'^\mu(x') = \frac{\partial x'^\mu}{\partial x^\nu} V^\nu(x).$$

The ordinary derivative does not transform as a tensor.
We therefore introduce the covariant derivative

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu{}_{\mu\rho} V^\rho.$$

The connection is chosen to ensure that $\nabla_\mu V^\nu$ transforms as a tensor.
For metric-compatible, torsion-free gravity,

$$\nabla_\alpha g_{\mu\nu} = 0, \qquad \Gamma^\rho{}_{\mu\nu} = \Gamma^\rho{}_{\nu\mu}.$$

These conditions determine

$$\Gamma^\rho{}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right).$$

Curvature as the Gravitational Field Strength

In gauge theory,

$$[D_\mu, D_\nu] \sim F_{\mu\nu}.$$

The same structure appears in gravity:

$$[\nabla_\mu, \nabla_\nu] V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma.$$

Thus

$$R^\rho{}_{\sigma\mu\nu}$$

is the gravitational analogue of the gauge-field strength.
Explicitly,

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho{}_{\nu\sigma} - \partial_\nu \Gamma^\rho{}_{\mu\sigma} + \Gamma^\rho{}_{\mu\lambda}\Gamma^\lambda{}_{\nu\sigma} - \Gamma^\rho{}_{\nu\lambda}\Gamma^\lambda{}_{\mu\sigma}.$$

From the Riemann tensor,

$$R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}, \qquad R = g^{\mu\nu} R_{\mu\nu}.$$

The Einstein–Hilbert Action

We now have the ingredients needed to construct a generally covariant field theory:

$$g_{\mu\nu}, \qquad \sqrt{-g}, \qquad R.$$

The simplest action containing the metric and up to two derivatives is

$$S_{\mathrm{EH}} = \frac{M_{\mathrm{Pl}}^2}{2} \int d^4x \sqrt{-g}\, R.$$

The full theory is

$$S = S_{\mathrm{EH}} + S_{\mathrm{matter}}.$$

For example,

$$S_{\mathrm{matter}} = -\int d^4x \sqrt{-g} \left[\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + V(\phi)\right].$$

Why Modify General Relativity?

General Relativity is defined by

$$S_{\mathrm{GR}} = \frac{M_{\mathrm{Pl}}^2}{2} \int d^4x \, \sqrt{-g} \, R + S_{\mathrm{matter}}.$$

But this does not uniquely determine the theory. We can ask:

What other generally covariant terms can be added?

For example,

$$R^2, \qquad R_{\mu\nu} R^{\mu\nu}, \qquad R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}, \qquad f(R),$$

or introduce exotic matter/energy fields–dark matters/energy.